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Field-theoretic calculation of kinetic helicity flux

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Abstract. In this paper we apply perturbative field-theoretic technique to helical turbulence. In the inertial range the kinetic helicity flux is found to be constant and forward. The universal constant K^{H} appearing in the spectrum of kinetic helicity was found to be 2.47.

Keywords. Kinetic helicity; helical turbulence.

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Kinetic helicity plays an important role in the dynamics of turbulence. Kinetic helicity is typically present in all rotating fluid systems, e.g., the Earth, the Sun, and the galaxies. In these astrophysical systems, kinetic helicity is important for the generation of magnetic field [1]. Recent numerical calculations of magnetohydrodynamic turbulence by Brandenburg [2] show that the kinetic helicity induces additional fluxes of energy and magnetic helicity. This event in turn causes a separation of positive and negative magnetic helicity, which plays a key role in dynamo process. Verma [4,3] obtained similar results in the field-theoretic calculation of energy and magnetic helicity fluxes.

Since the Earth is rotating, it creates kinetic helicity inside the Earth as well as in the atmosphere. Inside the Earth, it has an important role in the generation of geodynamo [5]. Outside the Earth, kinetic helicity is an important factor in the dynamics of atmosphere. In engineering applications, e.g. in turbines, the flow is typically helical. Given all these practical aspects, kinetic helicity is an important quantity to study in fluid turbulence.

Turbulence involves millions of interacting modes. It is very difficult to analyze these modes theoretically as well as numerically. In recent times, a new numerical procedure called 'large eddy simulations' (LES) has become quite popular [6]. In these simulations, the small scales are modelled using theories of inertial-range turbulence, which are not clearly understood. Here, some of the theoretical models become very handy. In LES, eddy viscosity or renormalized viscosity is fed at the cut-off scale [7,8]. Field-theoretical calculations of turbulence help us to compute renormalized viscosity, hence they have become important in recent times.

In this paper we will attempt to apply field-theoretic calculations to helical fluid turbulence.

One of the celebrated field-theoretic method, the renormalization group (RG) method, has been applied to fluid turbulence by Forster *et al* [9], Yakhot and Orszag [7], and McComb [10]. Later on it has been applied to many other forms of turbulence like scalar turbulence [7,11], MHD turbulence [3,12,13] etc. Zhou [14] applied RG calculations to helical fluid turbulence and computed the renormalized viscosity. He showed that kinetic helicity does not alter the renormalized viscosity.

We also apply the field-theoretic techniques to calculate turbulent cascade rates [4,3,15–17]. For helical turbulence, Lesieur [18] has analyzed the fluxes of kinetic energy and kinetic helicity using eddy-damped quasi-normal Markovian (EDQNM) approximation. He showed that at large wave numbers, kinetic helicity does not have any effect on the energy flux, and both kinetic energy and kinetic helicity have $k^{-5/3}$ energy spectrum in the inertial range. Using the same line of argument as in scalar turbulence, he showed that kinetic helicity flux is proportional to kinetic helicity. The EDQNM calculation also yields the proportionality constant as 2.25.

The EDQNM calculations have been very successful in turbulence calculations. However, they involve modeling of eddy-damped viscosity using an arbitrary constant of order one. So, field-theoretic calculations are superior because in field-theoretic calculations the constants can be determined from the first principles. It is needless to say that field-theoretic calculations involve certain assumptions, some of them apparently unrealistic. In spite of these limitations, field-theoretic calculations have provided major insights to understand turbulence (see Falkovich *et al* [19]). Due to this reason, we have performed field-theoretic calculation of helical turbulence, the result of which are presented below.

In this paper we will apply self-consistent one-loop calculation to helical turbulence and compute the fluxes of energy and kinetic helicity. The renormalized viscosity computed using RG procedure is used in the calculation. Contrast this with the arbitrary constant used in EDQNM calculation. In addition, the EDQNM calculations require numerical integration of energy equation, which is not required by our field-theoretic calculation. The calculation follows.

The equation for incompressible Navier–Stokes in three dimensions is given by

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k},t) = -\frac{i}{2} P^{+}_{ijm}(\mathbf{k}) \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} u_j(\mathbf{p},t) u_m(\mathbf{k}-\mathbf{p},t), \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where

$$P_{ijm}(\mathbf{k}) = k_j P_{im}(\mathbf{k}) - k_m P_{ij}(\mathbf{k}),$$

$$P_{im}(\mathbf{k}) = \delta_{im} - \frac{k_i k_m}{k^2}.$$

Kinetic helicity, which is assumed to be nonzero for our calculation, is given by

$$H_{\mathrm{K}} = \frac{1}{2} \langle \mathbf{u} \cdot \omega \rangle .$$

We can define kinetic helicity spectrum using

$$\langle u_i(\mathbf{k},t)\omega_j(\mathbf{k}',t)\rangle = P_{ij}(\mathbf{k})H_{\mathrm{K}}(\mathbf{k})(2\pi)^3\delta(\mathbf{k}+\mathbf{k}'). \tag{3}$$

Note that

$$H_{\mathrm{K}} = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} H_{\mathrm{K}}(\mathbf{k}).$$

Using the definition $\omega = \nabla \times \mathbf{u}$, we obtain

$$\langle u_i(\mathbf{k})u_j(\mathbf{k}',t)\rangle = \left[P_{ij}(\mathbf{k})C^{uu}(\mathbf{k}) - i\epsilon_{ijl}k_l\frac{H_{\mathrm{K}}(\mathbf{k})}{k^2}\right](2\pi)^3\delta(\mathbf{k}+\mathbf{k}'). \tag{4}$$

From the above definition it is clear that kinetic helicity breaks mirror symmetry due to the presence of ϵ_{ijk}.

It is known that both kinetic energy and kinetic helicity are conserved in three dimensions [18]. However, we can obtain interesting pattern in the energy transfers among Fourier modes if we focus on a single triad. For a single interacting triad $(\mathbf{p},\mathbf{q},\mathbf{k})$ with $\mathbf{p}+\mathbf{q}=\mathbf{k}$, the evolution equation of kinetic helicity in a triad interaction is given by

$$\begin{aligned}\frac{\partial}{\partial t}H_{\mathrm{K}}(\mathbf{k}) &= \frac{1}{2}\Re\left[\omega^*(\mathbf{k})\cdot\frac{\partial \mathbf{u}(\mathbf{k})}{\partial t} + \mathbf{u}^*(\mathbf{k})\cdot\frac{\partial \omega(\mathbf{k})}{\partial t}\right] \\ &= S^{H_{\mathrm{K}}}(\mathbf{k}|\mathbf{p}|\mathbf{q}) + S^{H_{\mathrm{K}}}(\mathbf{k}|\mathbf{q}|\mathbf{p}),\end{aligned} \tag{5}$$

where

$$\begin{aligned}S^{H_{\mathrm{K}}}(\mathbf{k}|\mathbf{p}|\mathbf{q}) &= \frac{1}{2}\Im[(\mathbf{k}\cdot\mathbf{u}(\mathbf{q}))\{\omega^*(\mathbf{k})\cdot\mathbf{u}(\mathbf{p}) + \omega(\mathbf{p})\cdot\mathbf{u}^*(\mathbf{k})\} \\ &\quad -(\mathbf{k}\cdot\omega(\mathbf{q}))(\mathbf{u}^*(\mathbf{k})\cdot\mathbf{u}(\mathbf{p}))], \\ &= \frac{1}{2}\Re[\epsilon_{jlm}k_ik_lu_m^*(\mathbf{k})u_j(\mathbf{p})u_i(\mathbf{q}) \\ &\quad -\epsilon_{jlm}k_ip_lu_j^*(\mathbf{k})u_m(\mathbf{p})u_i(\mathbf{q}) \\ &\quad +\epsilon_{ilm}k_iq_lu_j^*(\mathbf{k})u_j(\mathbf{p})u_m(\mathbf{q})].\end{aligned} \tag{6}$$

The above formula stands for the mode-to-mode kinetic helicity transfer from mode $\mathbf{p}$ to mode $\mathbf{k}'$ with mode $\mathbf{q}$ acting as a mediator [3]. It is also interesting to observe that the sum of transfer rates of kinetic helicity in a triad is zero, i.e.,

$$\begin{aligned}&S^{H_{\mathrm{K}}}(\mathbf{k}|\mathbf{p}|\mathbf{q}) + S^{H_{\mathrm{K}}}(\mathbf{k}|\mathbf{q}|\mathbf{p}) + S^{H_{\mathrm{K}}}(\mathbf{p}|\mathbf{k}|\mathbf{q}) \\ &\quad + S^{H_{\mathrm{K}}}(\mathbf{p}|\mathbf{q}|\mathbf{k}) + S^{H_{\mathrm{K}}}(\mathbf{q}|\mathbf{k}|\mathbf{p}) + S^{H_{\mathrm{K}}}(\mathbf{q}|\mathbf{p}|\mathbf{k}) = 0.\end{aligned} \tag{7}$$

This result implies that kinetic helicity is conserved in a triad, which is also referred to as detailed conservation of kinetic helicity in a triad interaction. Similar result has been derived for kinetic energy [18].

Using the mode-to-mode energy transfer rates, we can write down the expressions for the fluxes of kinetic energy and kinetic helicity as follows [3]:

$$\Pi(k_0) = \frac{1}{(2\pi)^3\delta(-\mathbf{k}+\mathbf{p}+\mathbf{q})}\int_{k'>k_0}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\int_{p<k_0}\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\left\langle S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q})\right\rangle, \tag{8}$$

$$\Pi_{H_{\rm K}}(k_0) = \frac{1}{(2\pi)^3\delta(-\mathbf{k}+\mathbf{p}+\mathbf{q})} \times \int_{k'>k_0} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \int_{p<k_0} \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \left\langle S^{H_{\rm K}}(\mathbf{k}|\mathbf{p}|\mathbf{q})\right\rangle , \tag{9}$$

where

$$S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \Im([\mathbf{k}\cdot\mathbf{u}(\mathbf{q})][\mathbf{u}^*(\mathbf{k})\cdot\mathbf{u}(\mathbf{p})]). \tag{10}$$

Now we calculate the above energy fluxes in the inertial range to the leading order in perturbation series. It is assumed that $\mathbf{u}(\mathbf{k})$ is quasi-Gaussian as in EDQNM approximation. Under this assumption, the triple correlation $\langle uuu\rangle$ is zero to zeroth order, but nonzero to first order. To first order $\langle uuu\rangle$ is written in terms of fourth-order correlations $\langle uuuu\rangle$, which is replaced by its Gaussian value, a sum of products of second-order correlations $\langle uu\rangle$. Hence, we can express $\left\langle S^{uu,H_{\rm K}}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\right\rangle$ in terms of second-order correlation functions (essentially energy spectrum). To first order, the Feynman diagrams for $\left\langle S^{uu,H_{\rm K}}\right\rangle$ are given below.

The functions $\left\langle S^{uu,H_{\rm K}}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\right\rangle$ are given below in terms of Green's functions and correlation functions:

$$\begin{aligned}\langle S^{uu}(k'|p|q)\rangle &= \int_{-\infty}^{t} \mathrm{d}t'(2\pi)^3 \\ &\times\Big[T_1(k,p,q)G^{uu}(k,t-t')C^{uu}(p,t,t')C^{uu}(q,t,t') \\ &+T_1'(k,p,q)G^{uu}(k,t-t')\frac{H_{\rm K}(p,t,t')}{p^2}\frac{H_{\rm K}(q,t,t')}{q^2} \\ &+T_5(k,p,q)G^{uu}(p,t-t')C^{uu}(k,t,t')C^{uu}(q,t,t') \\ &+T_5'(k,p,q)G^{uu}(p,t-t')\frac{H_{\rm K}(k,t,t')}{k^2}\frac{H_{\rm K}(q,t,t')}{q^2} \\ &+T_9(k,p,q)G^{uu}(q,t-t')C^{uu}(k,t,t')C^{uu}(p,t,t') \\ &+T_9'(k,p,q)G^{uu}(q,t-t')\frac{H_{\rm K}(k,t,t')}{k^2}\frac{H_{\rm K}(p,t,t')}{p^2}\Big] \\ &\times\delta(\mathbf{k}'+\mathbf{p}+\mathbf{q})\end{aligned} \tag{11}$$

$$\begin{aligned}\left\langle S^{H_{\rm K}}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\right\rangle &= \int_{-\infty}^{t} \mathrm{d}t'(2\pi)^3 \\ &\times\Big[T_{31}(k,p,q)G^{uu}(k,t-t')\frac{H_{\rm K}(p,t-t')}{p^2}C^{uu}(q,t-t') \\ &+T_{32}(k,p,q)G^{uu}(k,t-t')C^{uu}(p,t-t')\frac{H_{\rm K}(q,t-t')}{q^2} \\ &+T_{33}(k,p,q)G^{uu}(p,t-t')\frac{H_{\rm K}(k,t-t')}{k^2}C^{bb}(q,t-t') \\ &+T_{34}(k,p,q)G^{uu}(p,t-t')C^{uu}(k,t-t')\frac{H_{\rm K}(q,t-t')}{q^2}\end{aligned}$$

$$
\begin{aligned}
&+T_{35}(k,p,q)G^{uu}(q,t-t')\frac{H_{\mathrm{K}}(k,t-t')}{k^2}C^{uu}(p,t-t')\\
&+T_{36}(k,p,q)G^{uu}(q,t-t')C^{uu}(k,t-t')\frac{H_{\mathrm{K}}(p,t-t')}{p^2}\\
&+T_{37}(k,p,q)G^{uu}(k,t-t')C^{uu}(q,t-t')\frac{H_{\mathrm{K}}(p,t-t')}{p^2}\\
&+T_{38}(k,p,q)G^{uu}(k,t-t')\frac{H_{\mathrm{K}}(q,t-t')}{q^2}C^{uu}(p,t-t')\\
&+T_{39}(k,p,q)G^{uu}(p,t-t')\frac{H_{\mathrm{K}}(k,t-t')}{k^2}C^{uu}(q,t-t')\\
&+T_{40}(k,p,q)G^{uu}(p,t-t')C^{uu}(k,t-t')\frac{H_{\mathrm{K}}(q,t-t')}{q^2}\\
&+T_{41}(k,p,q)G^{uu}(q,t-t')\frac{H_{\mathrm{K}}(k,t-t')}{k^2}C^{uu}(p,t-t')\\
&+T_{42}(k,p,q)G^{uu}(q,t-t')C^{uu}(k,t-t')\frac{H_{\mathrm{K}}(p,t-t')}{p^2}\\
&+T_{43}(k,p,q)G^{uu}(k,t-t')\frac{H_{\mathrm{K}}(p,t-t')}{p^2}C^{uu}(q,t-t')\\
&+T_{44}(k,p,q)G^{uu}(k,t-t')C^{uu}(p,t-t')\frac{H_{\mathrm{K}}(q,t-t')}{q^2}\\
&+T_{45}(k,p,q)G^{uu}(p,t-t')\frac{H_{\mathrm{K}}(k,t-t')}{k^2}C^{uu}(q,t-t')\\
&+T_{46}(k,p,q)G^{uu}(p,t-t')C^{uu}(k,t-t')\frac{H_{\mathrm{K}}(q,t-t')}{q^2}\\
&+T_{47}(k,p,q)G^{uu}(q,t-t')\frac{H_{\mathrm{K}}(k,t-t')}{k^2}C^{uu}(p,t-t')\\
&+T_{48}(k,p,q)G^{uu}(q,t-t')C^{uu}(k,t-t')\frac{H_{\mathrm{K}}(p,t-t')}{p^2}\Bigg]\\
&\times\delta(\mathbf{k}'+\mathbf{p}+\mathbf{q}). \qquad (12)
\end{aligned}
$$

We write Green's functions in terms of 'effective' or 'renormalized' viscosity $\nu(k)$ computed by McComb and Watt [20], Zhou and Vahala [11], Verma [13] and Leslie [15]

$$G^{uu}(k,t-t') = \theta(t-t')\exp\big(-\nu(k)k^2(t-t')\big) \qquad (13)$$

with

$$\nu(k) = (K)^{1/2}\Pi^{1/3}k^{-4/3}\nu^*, \qquad (14)$$

where $\nu^* = 0.38$. Zhou and Vahala [11] showed that the value of renormalized viscosity is unaffected by the introduction of kinetic helicity [11]. The relaxation time for $C^{uu,}(k,t,t')$ is assumed to be the same as that of $G^{uu}(k,t,t')$. Therefore, the time dependence of the unequal-time correlation functions will be [15]

$$C^{uu}(k,t,t') = \theta(t-t')\exp\big(-\nu(k)k^2(t-t')\big)\,C^{uu}(k,t,t), \qquad (15)$$

with

$$C^{uu,bb}(k,t,t) = \frac{(2\pi)^3}{4\pi} K \Pi^{2/3} k^{-11/3}, \tag{16}$$

where K is the Kolmogorov's constant and Π is the total energy flux. Please note that the forcing is assumed to be present at the large scale, which maintains Kolmogorov's spectrum for the energy spectrum.

Equation (12) reveals that $\langle S^{H_K} \rangle$ is linear in H_K. Now using eq. (9) we obtain

$$\Pi_{H_K} \propto H_K.$$

After this we apply dimensional analysis, which yields [18]

$$H_K(k) = K^H \Pi_{H_K} \Pi^{-1/3} k^{-5/3}. \tag{17}$$

We substitute the above forms of Green's and correlation functions in the expression of $\langle S^{uu,H_K} \rangle$, perform the t' integral, and nondimensionalize the equations by substituting [15]

$$k = \frac{k_0}{u}; \quad p = \frac{k_0}{u} v; \quad q = \frac{k_0}{u} w. \tag{18}$$

As a result we obtain

$$\Pi = K^{3/2} \Pi I_1 + \frac{1}{k^2} \frac{(K^H)^2}{K^{1/2}} \frac{(\Pi_{H_K})^2}{\Pi} I_2, \tag{19}$$

and

$$\Pi_{H_K} = \Pi_{H_K} K^H K^{1/2} I_3, \tag{20}$$

where I_{1-3} are nondimensional numbers obtained by performing integrals [3,17]. An inspection of eq. (19) reveals that the second term on the right-hand side, which comes from helicity, will become negligible for large wave numbers. This observation shows that kinetic helicity has negligible effect on the large wave numbers. The computation of integrals yields $I_1 = 0.53$ and $I_3 = 0.65$. From the value of I_1 we find that $K = 1.58$, which is consistent with earlier results. By substituting the values of I_3 and K in eq. (20) we obtain K^H =2.47. This value of K^H is quite close to $K^H = 2.25$ obtained using EDQNM [18] calculation. Thus we derived the universal constant using field-theoretic method.

In this paper we applied a field-theoretic technique to compute the kinetic helicity flux. The computation was done for the inertial range. The calculation shows that the kinetic helicity flux is constant and forward (from smaller wave numbers to larger wave numbers). The universal constant appearing in the spectrum of kinetic helicity was found to be $K^H = 2.47$.

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