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A new algorithm for anisotropic solutions

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Abstract. We establish a new algorithm that generates a new solution to the Einstein field equations, with an anisotropic matter distribution, from a seed isotropic solution. The new solution is expressed in terms of integrals of an isotropic gravitational potential; and the integration can be completed exactly for particular isotropic seed metrics. A good feature of our approach is that the anisotropic solutions necessarily have an isotropic limit. We find two examples of anisotropic solutions which generalise the isothermal sphere and the Schwarzschild interior sphere. Both examples are expressed in closed form involving elementary functions only.

Keywords. Exact solutions; anisotropic spheres; superdense matter.

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1. Introduction

Numerous models of static perfect fluid spheres, in the context of general relativity, have been constructed in the past because these are first approximations in building a realistic model for a star. Lists of exact solutions to the Einstein field equations modelling relativistic perfect fluid spheres are given in several treatments [1–3]. In these works it is assumed that the matter distribution is isotropic so that the radial pressure is the same as the transverse pressure. A strong case can be made to study matter distributions which are anisotropic in which the radial component of the pressure is not the same as the transverse component. Anisotropy should not be neglected when analysing the critical mass and the red-shift of highly compact bodies, and it is important in modelling boson stars and strange stars. Consequently, anisotropy has been studied extensively in recent years by a number of researchers within the framework of general relativity [4–11].

Most solutions of the Einstein field equations with isotropic matter have been obtained in an *ad hoc* approach. Recent investigations have attempted to generate isotropic models using a systematic and algorithmic approach [12–15]. However, there exist very few analogous results for generating anisotropic models. In this regard Maharaj and Chaisi [16] have established an algorithm that produces a new anisotropic solution from a given seed isotropic line element. Here we present a different algorithm that generates a new anisotropic solution. The present algorithm has a simpler form and involves fewer integrations, and it is consequently easier to apply. A desirable physical feature of our approach is that these solutions have an isotropic limit; we expect that gravity acts to eventually isotropize matter in the absence of other external forces. Note that many exact solutions found previously remain anisotropic with no possibility of regaining an isotropic matter distribution in a suitable limit [17–20].

The main objective of this paper is to demonstrate that it is possible to generate anisotropic solutions from a given isotropic solution. To achieve this we need to complete an integration which we show is possible for particular metrics. In §2 we provide the fundamental field equations for isotropic and anisotropic matter. The field equations are presented as a first-order system of differential equations. The algorithm that produces a new anisotropic solution is described in §3. As a first example we use the isothermal model to produce a new anisotropic solution in §4. As a second example we use the Schwarzschild interior model to produce a new anisotropic solution in §5. In both examples the solution can be given explicitly in terms of elementary functions which makes it possible to study the physical features. We briefly study the behaviour of the anisotropy factor in both examples.

2. Field equations

We utilise a form of the Einstein field equations in which only first-order derivatives appear. This representation assists in simplifying the integration process as pointed out by Chaisi and Maharaj [19] whose notation and conventions we follow. The line element for static spherically symmetric space-time is given by

$$\mathrm{d}s^2 = -e^{\nu}\mathrm{d}t^2 + e^{\lambda}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{1}$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary functions. The energy–momentum tensor, for nonradiating matter, for isotropic distributions has the form

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab}, \tag{2}$$

where μ is the energy density and p is the isotropic pressure. These are measured relative to the comoving four-velocity $u^a = e^{-\nu/2}\delta^a_0$. We define the mass function as

$$m(r) = \frac{1}{2}\int_0^r x^2\mu(x)\mathrm{d}x. \tag{3}$$

Consequently $M = m(R)$ is the total mass of a sphere of radius R. The Einstein field equations are equivalent to the system

$$e^{-\lambda} = 1 - \frac{2m}{r}, \tag{4}$$

$$r(r-2m)\nu' = pr^3 + 2m, \tag{5}$$

$$(\mu + p)\,\nu' + 2p' = 0, \tag{6}$$

where we have used (1)–(3).

The energy–momentum tensor for anisotropic matter which is not radiating has the form

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab} + \pi^{ab}. \tag{7}$$

The quantity $\pi^{ab} = \sqrt{3}S(r)\left(c^a c^b - \frac{1}{3}h^{ab}\right)$ is the anisotropic stress tensor; the space-like vector $c^a = e^{-\lambda/2}\delta^a_1$ is orthogonal to the fluid four-velocity $u^a = e^{-\nu/2}\delta^a_0$ and $|S(r)|$ is the magnitude of the stress tensor. The Einstein field equations, with the metric (1) and the matter content (7), can be written in the form

$$e^{-\lambda} = 1 - \frac{2m}{r}, \tag{8}$$

$$r(r-2m)\nu' = p_{\rm r} r^3 + 2m, \tag{9}$$

$$(\mu + p_{\rm r})\,\nu' + 2p'_{\rm r} = -\frac{4}{r}\,(p_{\rm r} - p_\perp) \tag{10}$$

for anisotropic matter distributions. The radial pressure $p_{\rm r}$ is distinct from the tangential pressure $p_\perp$. It is convenient to write $p_{\rm r}$ and $p_\perp$ in the form

$$p_{\rm r} = p + 2S/\sqrt{3}, \qquad p_\perp = p - S/\sqrt{3},$$

where S provides a measure of anisotropy. Note that for isotropic matter $p_{\rm r} = p_\perp = p$ and we regain (4)–(6).

3. The algorithm

In this section we establish a procedure for generating a new anisotropic solution of the Einstein field equations from a specified isotropic solution. We start by considering the Einstein field equations (4)–(6) with isotropic matter distribution. We assume that an explicit solution to (4)–(6) is known where

$$(\nu, \lambda, m, p) = (\nu_0, \lambda_0, m_0, p_0) \tag{11}$$

and functions ν_0, λ_0, m_0 and p_0 are explicitly given. Then eqs (4)–(6) are satisfied and we can write

$$e^{-\lambda_0} = 1 - \frac{2m_0}{r}, \tag{12}$$

$$r(r-2m_0)\nu'_0 = p_0 r^3 + 2m_0, \tag{13}$$

$$\left(\frac{2m'_0}{r^2} + p_0\right)\nu'_0 + 2p'_0 = 0. \tag{14}$$

We next consider the Einstein field equations (8)–(10) with anisotropic matter distribution and seek an explicit solution. To this end we propose the possible solution

$$(\nu, \lambda, m, p_{\rm r}, p_{\perp}) = \left(\nu_0,\ \lambda_0 + x(r),\ m_0 + y(r),\ p_0 + \alpha(r),\ p_0 - \frac{\alpha(r)}{2}\right), \tag{15}$$

where $(\nu_0, \lambda_0, m_0, p_0)$ are given by (11) and x, y and α are arbitrary functions, and we have set $\alpha = 2S/\sqrt{3}$ for convenience. Then the system (8)–(10) becomes

$$e^{-(\lambda_0+x)} = 1 - \frac{2m_0 + 2y}{r}, \tag{16}$$

$$r(r - 2m_0 - 2y)\nu_0' = p_0 r^3 + \alpha r^3 + 2m_0 + 2y, \tag{17}$$

$$\left(\frac{2m_0' + 2y'}{r^2} + p_0 + \alpha\right)\nu_0' + 2p_0' + 2\alpha' = -\frac{6}{r}\alpha. \tag{18}$$

The systems (12)–(14) and (16)–(18) lead to

$$x = -\ln\left\{1 - \frac{2y}{r}e^{\lambda_0}\right\}, \tag{19}$$

$$y = -\frac{\alpha r^3}{2(1 + r\nu_0')}, \tag{20}$$

$$\left(\frac{2y'}{r^2} + \alpha\right)\nu_0' + 2\alpha' = -\frac{6\alpha}{r} \tag{21}$$

We need to integrate (21) to find the function α. The remaining functions x and y are defined in terms of α. Two cases arise: $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ then eqs (19)–(21) have the solution

$$(\alpha, x, y) = (0, 0, 0). \tag{22}$$

The trivial solution (22) corresponds to the isotropic case. Thus this algorithm regains the isotropic solution in the appropriate limit. If $\alpha \neq 0$ then we can eliminate y from (21) to get

$$\frac{2\nu_0'}{r^2}\left\{-\frac{\alpha' r^3}{2(1 + r\nu_0')} - \frac{3\alpha r^2}{2(1 + r\nu_0')} + \frac{\alpha r^3}{2}\frac{\nu_0' + r\nu_0''}{(1 + r\nu_0')^2}\right\}$$
$$+\alpha\nu_0' + 2\alpha' = -\frac{6\alpha}{r}.$$

This differential equation can be written as

$$\frac{\alpha'}{\alpha} - \frac{\nu_0'}{2 + r\nu_0'}\left\{3 - r\frac{\nu_0' + r\nu_0''}{1 + r\nu_0'}\right\} = -\left(\nu_0' + \frac{6}{r}\right)\left(\frac{1 + r\nu_0'}{2 + r\nu_0'}\right) \tag{23}$$

after some simplification. On integration (23) leads to

$$\ln \alpha = J_\alpha + \ln k, \tag{24}$$

where $\ln k$ is a constant of integration, $k \neq 0$, and we have set

$$J_\alpha = \int \left\{ \frac{\nu_0'}{2 + r\nu_0'} \left(\frac{3 + 2r\nu_0' - r^2\nu_0''}{1 + r\nu_0'} \right) - \left(\nu_0' + \frac{6}{r} \right) \left(\frac{1 + r\nu_0'}{2 + r\nu_0'} \right) \right\} \mathrm{d}r.$$

We can write (24) in the compact form

$$\alpha = k e^{J_\alpha}. \tag{25}$$

Equations (19), (20) and (25) correspond to anisotropic matter.

Thus if given a known isotropic solution (11) we can generate a new anisotropic solution (15) where

$$\alpha = k e^{J_\alpha}, \tag{26}$$

$$x = -\ln \left\{ 1 - \frac{2y}{r} e^{\lambda_0} \right\}, \tag{27}$$

$$y = -\frac{\alpha r^3}{2(1 + r\nu_0')} \tag{28}$$

and the integral J_α is given by

$$J_\alpha = \int \left\{ \frac{\nu_0'}{2 + r\nu_0'} \left(\frac{3 + 2r\nu_0' - r^2\nu_0''}{1 + r\nu_0'} \right) - \left(\nu_0' + \frac{6}{r} \right) \left(\frac{1 + r\nu_0'}{2 + r\nu_0'} \right) \right\} \mathrm{d}r.$$

The integration in J_α can be explicitly performed as ν_0 is specified in the isotropic solution (11). Note that eqs (26)–(28) apply to both cases $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ we can set $k = 0$ and regain the isotropic result (22). When $\alpha \neq 0$ then $k \neq 0$ and we regain the anisotropic equations (19), (20) and (25).

It is remarkable that our simple ansatz leads to a new anisotropic solution of the Einstein field equations. This is subject to completing the integration in J_α; clearly this is possible for particular choices of the isotropic function ν_0. We demonstrate two examples of anisotropic solutions for familiar choices of ν_0 in the next two sections. The algorithm that we have generated in this paper is easy to apply as there is only a single integration to be performed unlike the earlier algorithm of Maharaj and Chaisi [16] which is more complicated and involves further integrations. We believe that new anisotropic solutions that arise from our procedure are likely to produce realistic anisotropic stellar models. We emphasise that a desirable feature of our approach is that our models contain an isotropic limit which is often not the case in other approaches.

4. Example 1

As a first example we demonstrate the applicability of the algorithm in §3 by generating anisotropic isothermal spheres. The line element for the isothermal model [21] has the form

$$\mathrm{d}s^2 = -r^{4c/(1+c)}\mathrm{d}t^2 + \left(1 + \frac{4c}{(1+c)^2}\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \quad (29)$$

where c is a constant. The corresponding isotropic functions for (29) are given by

$$(\nu_0, \lambda_0, m_0, p_0) = \left(\frac{4c}{1+c}\ln r,\ \ln\left\{1 + \frac{4c}{1+c}\right\}, \frac{2cr}{4c+(1+c)^2}, \frac{4c^2}{4c+(1+c)^2}\left(\frac{1}{r^2}\right)\right). \quad (30)$$

The energy density function has the form

$$\mu_0 = \frac{4c}{4c+(1+c)^2}\left(\frac{1}{r^2}\right). \quad (31)$$

Hence (30) and (31) imply that

$$p_0 = c\mu_0 \quad (32)$$

which is a linear barotropic equation of state. Isothermal spheres with the density profile $\mu \propto r^{-2}$ and the equation of state (32) appear in a variety of models for both Newtonian and relativistic stars [19,20]. They have been studied extensively in astrophysics as an equilibrium approximation to more complicated systems which are close to a dynamically relaxed state [22].

With the isotropic functions (30) we can evaluate the integral J_α in (26) and we find

$$J_\alpha = -\int\left(\frac{3+14c+19c^2}{1+4c+3c^2}\right)\frac{\mathrm{d}r}{r} = -\frac{3+14c+19c^2}{1+4c+3c^2}\ln r + \ln k$$

which leads to the expressions

$$\alpha = kr^{-\frac{3+14c+19c^2}{1+4c+3c^2}},$$
$$x = -\ln\left\{1 + k\frac{4c+\left(1+c^2\right)}{(1+c)(1+5c)}r^{-\frac{1+6c+13c^2}{1+4c+3c^2}}\right\},$$
$$y = -\frac{k}{2}\left(\frac{1+c}{1+5c}\right)r^{-\frac{2c+10c^2}{1+4c+3c^2}}.$$

Consequently, we obtain the new line element in the form

$$\mathrm{d}s^2 = -r^{\frac{4c}{1+c}}\mathrm{d}t^2 + \left(1 + \frac{4c}{(1+c)^2}\right) \times\left(1 + k\frac{4c+(1+c)^2}{(1+c)(1+5c)}r^{-\frac{1+6c+13c^2}{1+4c+3c^2}}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \quad (33)$$

and the matter variables have the analytic representation

$$m = \frac{2cr}{4c + (1+c)^2} - \frac{k}{2}\left(\frac{1+c}{1+5c}\right) r^{-\frac{2c+10c^2}{1+4c+3c^2}}, \tag{34}$$

$$p_{\mathrm{r}} = \frac{1}{r^2}\frac{4c^2}{4c + (1+c)^2} + kr^{-\frac{3+14c+19c^2}{1+4c+3c^2}}, \tag{35}$$

$$p_{\perp} = \frac{1}{r^2}\frac{4c^2}{4c + (1+c)^2} - \frac{k}{2}r^{-\frac{3+14c+19c^2}{1+4c+3c^2}}. \tag{36}$$

The isotropic isothermal sphere model (29) produces the anisotropic isothermal sphere model (33) utilizing our algorithm. With the parameter values $k = 0$, we regain the conventional isothermal sphere.

The degree of anisotropy is

$$S = \frac{k}{2}\sqrt{3}r^{-\frac{3+14c+19c^2}{1+4c+3c^2}}. \tag{37}$$

Mathematica [23] was used to graph the anisotropy factor (37). The plots are as shown in figures 1 and 2 for particular values of the parameters shown. The anisotropy factor S is plotted against the radial distance on the interval $0 < r \leq 1$. There is a singularity at $r = 0$ that has been carried over from the other dynamical and metric functions. However, because the constants k and c can be picked arbitrarily, the pair can be chosen such that $S(r)$ is monotonically decreasing or increasing. The physical considerations of a problem may lead to the choice of one profile as opposed to the other; for example the $S(r)$ profile in figure 1 may be preferable where a stellar body with vanishing anisotropy as one moves from the center of the body to the boundary is considered. The $S(r)$ profile in figure 2 could be chosen over the one in figure 1 for boson star models as proposed by Dev and Gleiser [4]. The fairly simple behaviour of $S(r)$ in these plots shows that a more extensive physical analysis of the solutions is possible, which will be carried out in future work.

5. Example 2

As a second example we demonstrate the applicability of the algorithm in §3 by generating anisotropic Schwarzschild spheres. The line element for the interior

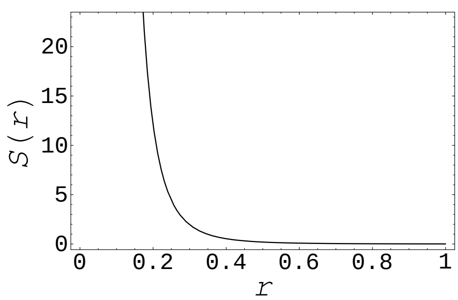

Figure 1. Decreasing $S(r)$ for anisotropic isothermal sphere; $c = 1$, $k = 0.01$.

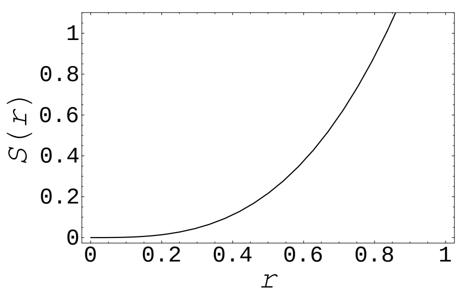

Figure 2. Increasing $S(r)$ for anisotropic isothermal sphere; $c = -0.5$, $k = 2$.

Schwarzschild model [1] is

$$\mathrm{d}s^2 = -\left(A - B\Delta\right)^2 \mathrm{d}t^2 + \Delta^{-2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{38}$$

where $\Delta = \sqrt{1 - r^2/R^2}$, A and B are constants. The corresponding isotropic functions for (38) are given by

$$(\nu_0, \lambda_0, m_0, p_0) = \left(2\ln\left\{A - B\Delta\right\},\ -\ln\Delta^2, \frac{r^3}{2R^2},\ -\frac{1}{R^2}\left[\frac{A - 3B\Delta}{A - B\Delta}\right]\right). \tag{39}$$

The energy density function has the form $\mu_0 = 3/R^2$. We therefore have

$$\mu_0 = \text{constant} \tag{40}$$

for incompressible matter. This is a reasonable approximation in particular situations as the interior of dense neutron stars and superdense relativistic stars are of near uniform density [24,25]. Consequently the assumption (40) of uniform energy density is often used to build prototypes of realistic stars in the modelling process [4,17,26].

The integral J_α in (26) takes the form

$$\begin{aligned} J_\alpha = \int \Bigg\{ & \Big[2Br\Big(3 - r^2\Big(\frac{2Br^2}{R^4\Delta^3(A + B\Delta)} - \frac{2B^2r^2}{R^4\Delta^2(A + B\Delta)^2} \\ & + \frac{2B}{R^2\Delta(A + B\Delta)}\Big) + \frac{4Br^2}{R^2\Delta(A + B\Delta)}\Big)\Big] \\ & \times\Big[R^2\Delta(A + B\Delta)\Big(1 + \frac{2Br^2}{R^2\Delta(A + B\Delta)}\Big)\Big(2 + \frac{2Br^2}{R^2\Delta(A + B\Delta)}\Big)\Big]^{-1} \\ & - \Big(\frac{6}{r} + \frac{2Br}{R^2\Delta(A - B\Delta)}\Big)\Big(1 + \frac{2Br^2}{R^2\Delta(A - B\Delta)}\Big) \\ & \times\Big(2 + \frac{2Br^2}{R^2\Delta(A - B\Delta)}\Big)^{-1}\Bigg\}\mathrm{d}r. \end{aligned} \tag{41}$$

With the substitution $u = \Delta = \sqrt{1 - r^2/R^2}$, (41) becomes

$$\begin{aligned} J_\alpha = \int \Bigg(& \frac{\left(B + 3Au - 4Bu^2\right)\left(2B + Au - 3Bu^2\right)}{\left(1 - u^2\right)\left(A - Bu\right)\left(B + Au - 2Bu^2\right)} \\ & - \frac{3Bu^2\left(A - Bu\right)}{\left(2B + Au - 3Bu^2\right)\left(B + Au - 2Bu^2\right)} \\ & + \frac{2B^2\left(1 - u^2\right)\left(A - 2Bu - 2Au^2 + 3Bu^3\right)}{u\left(A - Bu\right)\left(2B + Au - 3Bu^2\right)\left(B + Au - 2Bu^2\right)}\Bigg)\mathrm{d}u. \end{aligned}$$

The above integral can be simplified with the help of partial fractions. We obtain

$$
\begin{aligned}
J_\alpha &= \int \left(\frac{1}{u} + \frac{3}{2\,(1-u)} - \frac{3}{2\,(1+u)} + \frac{B}{A-Bu} + \frac{A-6Bu}{2B+Au-3Bu^2} \right. \\
&\quad \left. - \frac{2A-7Bu}{B+Au-2Bu^2} \right) \mathrm{d}u \\
&= \ln u - \frac{3}{2}\ln\{1-u\} - \frac{3}{2}\ln\{1+u\} \\
&\quad - \ln\{A-Bu\} + \ln\left\{2B+Au-3Bu^2\right\} \\
&\quad - \int \left(\frac{A/B}{\frac{A^2+8B^2}{16B^2} - \left(u-\frac{A}{4B}\right)^2} - \frac{(7/2)\left(u-\frac{A}{4B}\right)}{\frac{A^2+8B^2}{16B^2} - \left(u-\frac{A}{4B}\right)^2} \right) \mathrm{d}u \\
&= \ln\left\{ \frac{u\left(2B+Au-3Bu^2\right)}{(A-Bu)\left(1-u^2\right)^{3/2}\left(B+Au-2Bu^2\right)^{7/4}} \right\} \\
&\quad + \frac{A}{4\sqrt{A^2+8B^2}} \ln\left\{ \frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(u-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(u-\frac{A}{4B}\right)} \right\}.
\end{aligned}
$$

We have completed the integration and obtained J_α in terms of the intermediate variable u. In terms of the original variable r (in Δ) we can write J_α as

$$
\begin{aligned}
J_\alpha &= \ln\left\{ \frac{\Delta\left(2B+A\Delta-3B\Delta^2\right)}{(A-B\Delta)\left(1-\Delta^2\right)^{3/2}\left(B+A\Delta-2B\Delta^2\right)^{7/4}} \right\} \\
&\quad + \frac{A}{4\sqrt{A^2+8B^2}} \ln\left\{ \frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta-\frac{A}{4B}\right)} \right\}.
\end{aligned}
$$

Then the function α in (26) becomes

$$
\begin{aligned}
\alpha &= \frac{kR^3\Delta\left(2B+A\Delta-3B\Delta^2\right)}{r^3\,(A-B\Delta)\left(B+A\Delta-2B\Delta^2\right)^{7/4}} \\
&\quad \times \left(\frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta-\frac{A}{4B}\right)} \right)^{\frac{A}{4\sqrt{A^2+8B^2}}}
\end{aligned} \tag{42}
$$

and (27) and (28) respectively lead to

$$
\begin{aligned}
x &= -\ln\left\{ 1 + \frac{kR^3\left(2B+A\Delta-3B\Delta^2\right)}{r\Delta\,(A-B\Delta)\left(B+A\Delta-2B\Delta^2\right)^{7/4}} \right. \\
&\quad \times \left(1+\frac{2Br^2}{R^2\Delta\,(A-B\Delta)}\right)^{-1} \\
&\quad \left. \times \left(\frac{1-\frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta-\frac{A}{4B}\right)}{1+\frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta-\frac{A}{4B}\right)} \right)^{\frac{A}{4\sqrt{A^2+8B^2}}} \right\},
\end{aligned} \tag{43}
$$

$$y = -\frac{kR^3\Delta\left(2B + A\Delta - 3B\Delta^2\right)}{2\left(A - B\Delta\right)\left(B + A\Delta - 2B\Delta^2\right)^{7/4}}\left(1 + \frac{2Br^2}{R^2\Delta\left(A - B\Delta\right)}\right)^{-1}$$
$$\times\left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}. \tag{44}$$

Hence the new line element has the form

$$\begin{aligned} \mathrm{d}s^2 = &-\left(A - B\Delta\right)^2\mathrm{d}t^2 \\ &+\frac{1}{\Delta^2}\left[1 + \frac{kR^3\left(2B + A\Delta - 3B\Delta^2\right)}{r\Delta\left(A - B\Delta\right)\left(B + A\Delta - 2B\Delta^2\right)^{7/4}}\right. \\ &\times\left(1 + \frac{2Br^2}{R^2\Delta\left(A - B\Delta\right)}\right)^{-1} \\ &\left.\times\left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}\right]^{-1} \\ &\times\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \end{aligned} \tag{45}$$

and the matter variables have the form

$$\begin{aligned} m = &\frac{r^3}{2R^2} - \frac{kR^3\Delta\left(2B + A\Delta - 3B\Delta^2\right)}{2\left(A - B\Delta\right)\left(B + A\Delta - 2B\Delta^2\right)^{7/4}} \\ &\times\left(1 + \frac{2Br^2}{R^2\Delta\left(A - B\Delta\right)}\right)^{-1} \\ &\times\left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}, \end{aligned} \tag{46}$$

$$\begin{aligned} p_{\mathrm{r}} = &-\frac{A - 3B\Delta}{R^2\left(A - B\Delta\right)} + \frac{kR^3\Delta\left(2B + A\Delta - 3B\Delta^2\right)}{r^3\left(A - B\Delta\right)\left(B + A\Delta - 2B\Delta^2\right)^{7/4}} \\ &\times\left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}, \end{aligned} \tag{47}$$

$$\begin{aligned} p_{\perp} = &-\frac{A - 3B\Delta}{R^2\left(A - B\Delta\right)} - \frac{kR^3\Delta\left(2B + A\Delta - 3B\Delta^2\right)}{2r^3\left(A - B\Delta\right)\left(B + A\Delta - 2B\Delta^2\right)^{7/4}} \\ &\times\left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}. \end{aligned} \tag{48}$$

The isotropic Schwarzschild sphere model (38) generates the anisotropic Schwarzschild sphere model (45)–(48). With the parameter value $k = 0$ we regain the original interior Schwarzschild sphere.

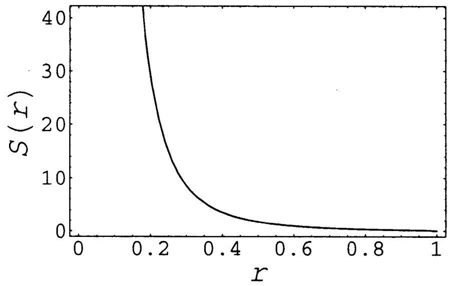

Figure 3. $S(r)$ for anisotropic Schwarzschild sphere; $A = 3$, $B = 1$, $k = 1$ and $R = 1$.

The degree of anisotropy has the form

$$S = \frac{\sqrt{3}kR^3\Delta\left(2B + A\Delta - 3B\Delta^2\right)}{2r^3\left(A - B\Delta\right)\left(B + A\Delta - 2B\Delta^2\right)^{7/4}} \times \left(\frac{1 - \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}{1 + \frac{4B}{\sqrt{A^2+8B^2}}\left(\Delta - \frac{A}{4B}\right)}\right)^{\frac{A}{4\sqrt{A^2+8B^2}}}. \tag{49}$$

Mathematica [23] was once again used to plot the anisotropic factor (49). The resulting plot is shown in figure 3 for chosen particular values of the parameters A, B, k and R. Other choices of these parameters may produce a different behaviour for S. The plot of S against r is in the interval $0 < r \leq 1$. The fact that the anisotropic factor is in closed form and the profile as shown in figure 3 shows that physical analysis of this model can be investigated, which will be pursued at a later stage.

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