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New uncertainties in QCD–QED rescaling factors using quadrature method

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Abstract. In this paper we briefly outline the quadrature method for estimating uncertainties in a function which depends on several variables, and apply it to estimate the numerical uncertainties in QCD–QED rescaling factors. We employ here the one-loop order in QED and three-loop order in QCD evolution equations of the fermion mass renormalisation. Our present calculation is found to be new and also reliable when compared to the earlier values employed by various authors.

Keywords. Fermion mass renormalisation; quadrature method; uncertainties; QCD–QED rescaling factors.

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1. Introduction

An important aspect of particle physics is the concept of mass of a fermion. The experimental values of the fermion masses are the input values that we always take in various calculations, and they are generally known as physical masses of the fermions. Mathematically such physical mass is the running mass $m_f(\mu)$ of the fermion defined at the scale equal to its own physical mass, i.e., $m_f(\mu = m_f) = m_f(m_f)$. This is true for heavier quarks $f = t, b, c$ which have $m_f(m_f) > 1$ GeV and for charged leptons (τ, μ, e). However for lighter quarks $(f = s, d, u)$, which have masses lesser than 1 GeV, the physical mass is defined as m_f (1 GeV). On the other hand, the fermion masses predicted in grand unified theories (GUTs) [1] are defined at the grand unification scale M_X. In order to express these masses at lower energy scales we need the renormalisation of fermion masses from high scale to low scale [2]. At low energy scale below the top-quark mass scale m_t, the fermion mass renormalisation is governed by the QCD–QED symmetry gauge group $SU(3)_{\rm C} \times U(1)_{\rm em}$. Thus the fermion mass renormalisation from top-quark mass scale down to the physical quark mass scale is parametrised by the QCD–QED rescaling factors η_f defined by $\eta_f = m_f(m_f)/m_f(m_t)$. This is an important parameter which appears in many expressions related for the predictions of GUTs at low-energy scale. The calculation of η_f involves the input values of e.m and strong gauge coupling constants α and α_3. The uncertainties associated with the experimental values of α and α_3 will propagate through several intermediate scales down to low energies in the definition of QCD–QED rescaling factors, making the uncertainties larger and larger. In this respect, a careful estimation of uncertainties in η_f using a reliable method which can control the magnifying tendency of the uncertainties in η_f, is highly desirable. The uncertainties in η_f so far reported in [3,4] do not match with each other, and the methods adopted by them are also not clearly specified. In this context, we find the quadrature method [5] quite satisfactory and appropriate for estimation of uncertainties of a function of many variables. It will be an important numerical exercise to estimate the uncertainties in η_f using the quadrature method and compare the results with the earlier values. We shall employ the one-loop order in QED and three-loop order in QCD evolution equations of the fermion mass renormalisation.

The paper is organised as follows. In §2, we outline the procedure for the quadrature method. Section 3 is devoted to a brief note on how to define QCD–QED rescaling factors and related quantities [3,6,7]. In §4, we present the numerical calculations of rescaling factors with uncertainties and analysis of results. The paper concludes with a summary in §5.

2. Estimation of uncertainties in quadrature method

We define the quadrature method for calculating uncertainties (uncertainties and errors both are used for the same meaning) in the following way. If a function F depends on several variables x_i, where $i = 1, 2, ..., n$, then the uncertainties in F resulting from the uncertainties of the independent variables x_i can be estimated by the following expression:

$$\delta F = \pm\sqrt{\sum \left(\frac{\partial F}{\partial x_i}\right)^2 (\delta x_i)^2}, \tag{1}$$

where $\pm\,(\partial F/\partial x_i)\,(\delta x_i)$ is the error in F due to the error in ith variable x_i. Here the uncertainties (or random errors of independent variables) add in quadrature [5] like orthogonal vectors, $\vec{a}$ and $\vec{b}$ whose resultant magnitude is given by $\sqrt{a^2+b^2}$.

We give here three simple properties which will be needed to estimate the uncertainties in different quantities leading to the calculations of rescaling factors and also other parameters which depend on it (e.g., neutrino masses and mixings).

(i) If $F(x,y) = f(x,y) \times g(x,y)$, the uncertainties in F due to the random errors $\pm\delta x$ and $\pm\delta y$ in the independent variables x and y respectively, are given by

$$\delta F = \pm\sqrt{\left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\right)^2 (\delta x)^2 + \left(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right)^2 (\delta y)^2}. \tag{2}$$

(ii) If $F(x,y) = (f(x,y)/g(x,y))$, the uncertainties in F due to the random errors $\pm\delta x$ and $\pm\delta y$ in the independent variables x and y respectively, are given by

$$\delta F = \pm\sqrt{\left(\frac{1}{g}\frac{\partial f}{\partial x} - \frac{f}{g^2}\frac{\partial g}{\partial x}\right)^2 (\delta x)^2 + \left(\frac{1}{g}\frac{\partial f}{\partial y} - \frac{f}{g^2}\frac{\partial g}{\partial y}\right)^2 (\delta y)^2}. \tag{3}$$

(iii) If $F(x,y) = (f(x)/g(y))^p$, then the uncertainties in F due to the uncertainties $\pm\delta x$ and $\pm\delta y$ in the independent variables x and y respectively, are

$$\delta F = \pm Fp\sqrt{\frac{1}{f^2}\left(\frac{\partial f}{\partial x}\right)^2 (\delta x)^2 + \frac{1}{g^2}\left(\frac{\partial g}{\partial y}\right)^2 (\delta y)^2}. \tag{4}$$

3. QCD–QED rescaling factors

The QCD–QED rescaling factor η_f of the fermion f, which can take care of the fermion-mass-renormalisation from the top-quark mass scale down to the physical fermion mass scale $m_f(m_f)$, is defined as [8]

$$\eta_f = \frac{m_f(m_f)}{m_f(m_t)} = \left[\frac{m_f(m_f)}{m_f(m_t)}\right]_{\text{QED}} \left[\frac{m_f(m_f)}{m_f(m_t)}\right]_{\text{QCD}} \tag{5}$$

for $f = b, c$ quarks having $m_f(m_f) > 1$ GeV,

$$\eta_f = \frac{m_f(1\ \text{GeV})}{m_f(m_t)} = \left[\frac{m_f(1\ \text{GeV})}{m_f(m_t)}\right]_{\text{QED}} \left[\frac{m_f(1\ \text{GeV})}{m_f(m_t)}\right]_{\text{QCD}} \tag{6}$$

for $f = s, d, u$ quarks having $m_f(m_f) < 1$ GeV, and

$$\eta_f = \left[\frac{m_f(m_f)}{m_f(m_t)}\right]_{\text{QED}}, \quad f = \tau, \mu, e. \tag{7}$$

Here $m_f(m_t)$ is the running mass of the fermion f at the top-quark mass scale, $\mu = m_t$, and $m_f(m_f)$ corresponds to the physical mass of the fermion. For convenience we also define a quantity $R^f(\mu,\mu')$ which represents the inverse of the QCD–QED rescaling factor in the narrow range, $\mu' - \mu$, $(\mu > \mu')$,

$$\begin{aligned} R^f(\mu,\mu') &= \left[\frac{m_f(\mu)}{m_f(\mu')}\right]_{\text{QED}} \left[\frac{m_f(\mu)}{m_f(\mu')}\right]_{\text{QCD}} \\ &= R^f_{\text{QED}}(\mu,\mu') \times R^f_{\text{QCD}}(\mu,\mu'). \end{aligned} \tag{8}$$

The definitions of η_f and R^f in eqs (5)–(8) require only the contributions from the QED part for the charged leptons. In order to make use of eq. (8) we consider successive narrow mass ranges: m_b–m_t, m_τ–m_b, m_c–m_τ, 1 GeV–m_c, m_s–1 GeV, m_μ–m_s, m_d–m_μ, m_u–m_d, m_e–m_u, where m_f are the physical fermion mass scales between which evolution is being done. Then the QCD–QED rescaling factor η_f, defined in eqs (5)–(7), can be rewritten as

$$\eta_b = \eta_b(m_b, m_t) = \frac{1}{R^b(m_b, m_t)},$$

$$\eta_\tau = \eta_\tau(m_\tau, m_t) = \frac{1}{[R^l_{\rm QED}(m_\tau, m_b) R^l_{\rm QED}(m_b, m_t)]}, \quad l = \tau, \mu, e,$$

$$\eta_c = \eta_c(m_c, m_t) = \frac{1}{[R^c(m_c, m_\tau) R^c(m_\tau, m_b) R^c(m_b, m_t)]},$$

$$\eta_{s,d} = \eta_{s,d}(1\ {\rm GeV}, m_t) = \frac{\eta_b}{[R^{s,d}(1\ {\rm GeV}, m_c) R^{s,d}(m_c, m_\tau) R^{s,d}(m_\tau, m_b)]},$$

$$\eta_u = \eta_u(1\ {\rm GeV}, m_t) = \frac{\eta_c}{[R^u(1\ {\rm GeV}, m_c)]},$$

$$\eta_\mu = \eta_\mu(m_\mu, m_t)$$
$$= \frac{\eta_\tau}{[R^l_{\rm QED}(m_\mu, m_s) R^l_{\rm QED}(m_s, 1\ {\rm GeV}) R^l_{\rm QED}(1\ {\rm GeV}, m_c) R^l_{\rm QED}(m_c, m_\tau)]},$$

$$\eta_e = \eta_e(m_e, m_t) = \frac{\eta_\mu}{[R^l_{\rm QED}(m_e, m_u) R^l_{\rm QED}(m_u, m_d) R^l_{\rm QED}(m_d, m_\mu)]}. \quad (9)$$

We use one-loop order in QED and three-loop order in QCD evolution equations of fermion mass renormalisation for the evaluations of $R^f_{\rm QED}$ and $R^f_{\rm QCD}$ respectively. The contribution of one-loop QED, running from the scale μ' to μ, to the rescaling factors through eq. (8) is now given by [8]

$$R^f_{\rm QED}(\mu', \mu) = \left[\frac{\alpha(\mu)}{\alpha(\mu')}\right]^{r_f^{\rm QED}}, \quad \mu > \mu', \quad (10)$$

where

$$r_f^{\rm QED}(\mu', \mu) = \gamma_0^{\rm QED}/b_0^{\rm QED},$$

$$\gamma_0^{\rm QED} = -3Q_f^2,$$

$$b_0^{\rm QED} = \frac{4}{3}\left[3\sum Q_u^2 + 3\sum Q_d^2 + \sum Q_e^2\right]. \quad (11)$$

The summation in eqs (11) is over the active fermions at the relevant mass scale, and f is the specific fermion under consideration. We employ the one-loop RGE for the estimation of e.m. gauge couplings $\alpha(\mu)$ at successive renormalisation points,

$$\frac{1}{\alpha(\mu')} = \frac{1}{\alpha(\mu)} + \frac{b_0^{\rm QED}}{2\pi} \ln\left(\frac{\mu}{\mu'}\right). \tag{12}$$

The three-loop QCD running quark mass formula is given by [8,9]

$$m_q(\mu) = \widehat{m_q}\left[b_0\frac{\alpha_3(\mu)}{2\pi}\right]^{(\gamma_0/b_0)}\left(1 + A\frac{\alpha_3(\mu)}{4\pi} + B\left[\frac{\alpha_3(\mu)}{4\pi}\right]^2\right), \tag{13}$$

where $\widehat{m_q}$ is a common multiplicative factor of running quark mass, and

$$A = \frac{\gamma_1}{b_0} - \frac{b_1\gamma_0}{b_0^2},$$

$$B = \frac{1}{2}\left[A^2 + \frac{\gamma_2}{b_0} + \frac{b_1^2\gamma_0}{b_0^3} - \frac{b_1\gamma_1}{b_0^2} - \frac{b_2\gamma_0}{b_0^2}\right]. \tag{14}$$

The QCD β-functions and anomalous dimensions are given as

$$\gamma_0 = 4,$$

$$\gamma_1 = \frac{202}{3} - \frac{20}{9}n_f,$$

$$\gamma_2 = \frac{3747}{3} - \left(\frac{160}{3}\xi(3) + \frac{2216}{27}\right)n_f - \frac{140}{81}n_f^2,$$

$$b_0 = 11 - \frac{2}{3}n_f,$$

$$b_1 = 102 - \frac{38}{3}n_f,$$

$$b_2 = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2, \tag{15}$$

where n_f is the number of quark flavours at the relevant mass scale and $\xi(3) = 1.202$. The QCD rescaling contribution to R^f in the relevant mass-range μ'–μ defined in eq. (8), can be obtained from eq. (13) as

$$R^f_{\rm QCD}(\mu,\mu') = \left[\frac{\alpha_3(\mu)}{\alpha_3(\mu')}\right]^{(\gamma_0/b_0)} \frac{1 + A\frac{\alpha_3(\mu)}{4\pi} + B\left[\frac{\alpha_3(\mu)}{4\pi}\right]^2}{1 + A\frac{\alpha_3(\mu')}{4\pi} + B\left[\frac{\alpha_3(\mu')}{4\pi}\right]^2} \tag{16}$$

since $\alpha_3(\mu)$ changes smoothly with μ, so values of the constants A and B remain the same for a given energy range including the limits.

Using eqs (10) and (14) via (8), the QCD–QED rescaling factors can be estimated. The values of $\alpha_3(\mu)$ in eq. (16) can be obtained by solving the three-loop QCD RGE with $\alpha_3 = g_3^2/4\pi$ [10],

$$\mu\frac{\mathrm{d}g_3(\mu)}{\mathrm{d}\mu} = \beta(g_3(\mu)) = -\frac{b_0}{16\pi^2}g_3^3 - \frac{b_1}{(16\pi^2)^2}g_3^5 - \frac{b_2}{(16\pi^2)^3}g_3^7. \tag{17}$$

The conventional solution of eq. (17) having a constant of integration called QCD dimensional parameter Λ – which provides a parametrisation of the μ dependence of $\alpha_3(\mu)$ [10], is given by

$$\alpha_3(\mu) = \frac{4\pi}{\beta_0\ln(\mu^2/\Lambda^2)}\left[1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln[\ln(\mu^2/\Lambda^2)]}{\ln(\mu^2/\Lambda^2)} + \frac{4\beta_1^2}{\beta_0^4\ln^2(\mu^2/\Lambda^2)}\right.$$
$$\left.\times\left(\left(\ln[\ln(\mu^2/\Lambda^2)] - \frac{1}{2}\right)^2 + \frac{\beta_2\beta_0}{8\beta_1^2} - \frac{5}{4}\right)\right], \tag{18}$$

where

$$\beta_0 = b_0, \quad \beta_1 = b_1/2, \quad \beta_2 = 2b_2. \tag{19}$$

Since β-function coefficients change by discrete amount as flavour thresholds are crossed while integrating the differential equation (17) for $\alpha_3(\mu)$, Λ also changes to take care of the validity of eq. (18) for all values of μ. This leads to the concept of different $\Lambda^{(n_f)}$ for each range of μ corresponding to an effective number of quark flavours n_f. In the $\overline{MS}$ scheme, one finds the relation among different $\Lambda^{(n_f)}$ [10] as

$$\beta_0^{(n_f-1)}\ln\left(\frac{\Lambda^{(n_f)}}{\Lambda^{(n_f-1)}}\right)^2 = \left(\beta_0^{(n_f)} - \beta_0^{(n_f-1)}\right)\ln\left(\frac{\mu}{\Lambda^{(n_f)}}\right)^2$$
$$+2\left(\frac{\beta_1^{(n_f)}}{\beta_0^{(n_f)}} - \frac{\beta_1^{(n_f-1)}}{\beta_0^{(n_f-1)}}\right)\ln\left[\ln\left(\frac{\mu}{\Lambda^{(n_f)}}\right)^2\right]$$
$$-\frac{2\beta_1^{(n_f-1)}}{\beta_0^{(n_f-1)}}\ln\left(\frac{\beta_0^{(n_f)}}{\beta_0^{(n_f-1)}}\right)$$
$$+\frac{4\frac{\beta_1^{(n_f)}}{\left(\beta_0^{(n_f)}\right)^2}\left(\frac{\beta_1^{(n_f)}}{\beta_0^{(n_f)}} - \frac{\beta_1^{(n_f-1)}}{\beta_0^{(n_f-1)}}\right)\ln\left[\ln\left(\frac{\mu}{\Lambda^{(n_f)}}\right)^2\right]}{\ln\left(\frac{\mu}{\Lambda^{(n_f)}}\right)^2}$$
$$+\frac{\frac{1}{\beta_0^{(n_f)}}\left[\left(2\frac{\beta_1^{(n_f)}}{\beta_0^{(n_f)}}\right)^2 - \left(2\frac{\beta_1^{(n_f-1)}}{\beta_0^{(n_f-1)}}\right)^2 - \frac{\beta_2^{(n_f)}}{2\beta_0^{(n_f)}} + \frac{\beta_2^{(n_f-1)}}{2\beta_0^{(n_f-1)}} - \frac{22}{9}\right]}{\ln\left(\frac{\mu}{\Lambda^{(n_f)}}\right)^2}. \tag{20}$$

The QCD parameter $\Lambda^{(5)}$ which corresponds to $n_f = 5$ can be calculated by using eq. (18) with the input experimental value of $\alpha_3(M_Z)$ through a computer search program and uncertainties induced from $\alpha_3(M_Z)$ can be estimated using quadrature method. Then using the value of $\Lambda^{(5)}$ in eq. (20), $\Lambda^{(4)}$ can be evaluated. $\Lambda^{(3)}$ can be obtained from eq. (20) by applying the value of $\Lambda^{(4)}$. Their error bars can be estimated using quadrature method. Subsequently, $\alpha_3(\mu)$ and $R^f_{\rm QCD}$ can be calculated for the energy, $\mu < M_Z$.

If we consider SUSY above M_Z, assuming the existence of one-light Higgs doublet (N_H) and five quark flavours ($n_f = 5$) in the energy range m_t–M_Z, the strong gauge coupling at the scale m_t are evaluated using RGE solution with one-loop order

$$\frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(M_Z)} + \frac{3}{2\pi}\ln\frac{\mu}{M_Z} \tag{21}$$

for $\mu > M_Z$.

4. Numerical calculations and analysis of results

The most recent experimental values of fermion masses [11] are

$$\begin{aligned}
&m_t = (174.3 \pm 5.1)\ \text{GeV}; \quad m_b = (4.1\text{–}4.4)\ \text{GeV};\\
&m_\tau = 1.7769^{+0.00024}_{-0.00026}\ \text{GeV};\\
&m_c = (1.15\text{–}1.35)\ \text{GeV}; \quad m_s = (0.080\text{–}0.130)\ \text{GeV};\\
&m_\mu = 0.1056\ \text{GeV}; \quad m_d = (0.0040\text{–}0.0080)\ \text{GeV};\\
&m_u = (0.0015\text{–}0.0040)\ \text{GeV}; \quad m_e = 0.00051\ \text{GeV}.
\end{aligned} \tag{22}$$

We are using $m_t = 175$ GeV, $m_b = 4.25$ GeV, $m_\tau = 1.777$ GeV and $m_c = 1.25$ GeV in our present calculation. The CERN-LEP data [12] of $\alpha^{-1}(M_Z) = 127.9 \pm 0.1$ and strong coupling constant $\alpha_3(M_Z) = 0.1172 \pm 0.002$ at $M_Z = 91.187$ GeV, referred to as Case I. For comparison, we also estimate the uncertainties using values of $\alpha_3(M_Z) = 0.120 \pm 0.0028$ referred to as Case II and $\alpha_3(M_Z) = 0.118 \pm 0.007$ used for calculation of uncertainties in [4], now referred to as Case III.

Now using the above data, we evaluate the gauge couplings $\alpha(\mu)$ and $\alpha_3(\mu)$ at various renormalisation points starting from m_t down to the individual fermion mass (for quark it stops at 1 GeV) from their corresponding eqs (12), (18) and (21) and values are presented in tables 1 and 5 respectively. The coefficients of β-functions and anomalous dimensions in RGEs for QED and QCD with constants A, B relevant in different energy ranges are estimated and shown in tables 1–3. We also calculate the inverse of the fermion mass renormalisation factors $R_f^{\rm QED}(\mu,\mu')$ for charged leptons, up-quarks and down-quarks shown in tables 1, 2 and $R_f^{\rm QCD}(\mu,\mu')$ in three Cases I, II, III in table 5. The values of QCD dimensional parameters $\Lambda^{(n_f)}$ for estimating $\alpha_3(\mu)$ at various energy scales are presented in table 4.

We present in table 6 the numerical values of QCD–QED rescaling factors η_f ($f = b, c, s, d, u$) with uncertainties for three different values of $\alpha_3(M_Z)$ as input. The rescaling factors for charged leptons are estimated as $\eta_\tau = 1.017 \pm 0.0007$, $\eta_\mu = 1.027 \pm 0.0038$, $\eta_e = 1.046 \pm 0.0099$ for all cases. In order to analyse the uncertainties in the rescaling factors, we calculate the percentage errors for three different Cases I, II, III, starting from input values of $\alpha_3(M_Z)$ to η_f ($f = b, c, s, d, u$)

Table 1. Coefficients of β-functions in the RGEs for QED, the values of r_f^{QED} in different energy ranges (μ–μ'), $\mu > \mu'$, the inverse of the gauge coupling $\alpha^{-1}(\mu)$ and inverse of the rescaling factors $R_f^{\text{QED}}(\mu, \mu')$ for charged leptons (e, μ, τ) with input value of $\alpha^{-1}(M_Z) = 127.9 \pm 0.1$.

Energy range (μ–μ')	b_0^{QED}	r_f^{QED} ($f = e, \mu, \tau$)	$\alpha^{-1}(\mu)$	R^f_{QED} ($f = e, \mu, \tau$)
m_t–m_b	80/9	−0.3375	126.98 ± 0.1	0.9865 ± 0.0002
m_b–m_τ	79/9	−0.3553	132.24 ± 0.1	0.9968 ± 0.0005
m_τ–m_c	64/9	−0.4219	133.40 ± 0.1	0.9987 ± 0.0006
m_c–1 GeV	16/3	−0.5625	133.81 ± 0.1	0.9990 ± 0.0008
1 GeV–m_s	16/3	−0.5625	133.99 ± 0.1	0.9939 ± 0.0007
m_s–m_μ	44/9	−0.6136	135.45 ± 0.1	0.9981 ± 0.0089
m_μ–m_d	32/9	−0.8438	135.87 ± 0.1	0.9914 ± 0.0011
m_d–m_u	29/9	−0.9643	137.26 ± 0.1	0.9979 ± 0.0014
m_u–m_e	4/3	−2.2500	137.55 ± 0.1	0.9921 ± 0.0032
m_e–0	–	–	138.04 ± 0.1	–

Table 2. Coefficients of β-functions in the RGEs for QED, the values of r_f^{QED} in different energy ranges (μ–μ'), $\mu > \mu'$, and inverse of the rescaling factors $R_f^{\text{QED}}(\mu, \mu')$ for up- and down-quarks with renormalisation scale stopping at 1 GeV.

Energy range (μ–μ')	b_0^{QED}	r_f^{QED} ($f = u, c, t$)	r_f^{QED} ($f = s, d, b$)	R^f_{QED} ($f = u, c, t$)	R^f_{QED} ($f = s, d, b$)
m_t–m_b	80/9	−0.1500	−0.0375	0.9839 ± 0.0002	0.9985 ± 0.0001
m_b–m_τ	79/9	−0.1579	−0.0395	0.9986 ± 0.0002	0.9997 ± 0.0001
m_τ–m_c	64/9	−0.1875	−0.0469	0.9994 ± 0.0003	0.9999 ± 0.0001
m_c–1 GeV	16/3	−0.2500	−0.0625	0.9996 ± 0.0004	0.9999 ± 0.0001

through the intermediate steps $\Lambda^{(5)}$, $\Lambda^{(4)}$, $\Lambda^{(3)}$, $\alpha_3(\mu)$ and $R^f_{\text{QCD}}(\mu - \mu')$. The percentage errors in the QCD–QED rescaling factors for three different input values are presented in table 7. For Case I, the percentage error in $\alpha_3(M_Z)$ is 1.7% $(= \frac{0.002}{0.1172} \times 100\%)$ and those of η_b, η_c, $\eta_{s,d}$, η_u are 2.3%, 6.6%, 9.3% and 9.3% respectively. This shows that percentage error increases at lower energy scales where thresholds may cause nonlinear changes and hence the enhancement of the errors. This is also apparent for Cases II and III. We also calculate the numbers put in square bracket in Cases II and III in table 7, that are obtained by taking the ratio of the percentage errors of corresponding parameters to those of Case I. For example, in Case II these numbers for $\alpha_3(M_Z)$, η_b, η_c, $\eta_{s,d}$, η_u are respectively 1.4, 1.4, 1.5, 1.6 and 1.6. They are almost same indicating that the quadrature method can keep the variation of percentage errors in respective rescaling factors in tune with those of input percentage errors. Similar correlation is also found in Case III. Thus the error propagations are consistent in different cases in this method.

Table 3. Coefficients of β functions in RGEs, anomalous dimensions for QCD and values of the constants A and B with $\gamma_0 = 4$ throughout the computation.

Energy range (μ–μ')	γ_1	γ_2	b_0	b_1	b_2	A	B	n_f
m_t–m_b	506/9	474.89	23/3	116/3	9769/54	4.7020	24.0123	5
m_b–m_τ	526/9	636.62	25/3	154/3	21943/54	4.0563	22.2280	4
m_τ–m_c	526/9	636.62	25/3	154/3	21943/54	4.0563	22.2280	4
m_c–1 GeV	546/9	794.90	9	64	34767/54	3.5802	21.9434	3

Table 4. The values of QCD dimensional parameters $\Lambda^{(n_f)}$ with uncertainties estimated for three different values of $\alpha_3(\mu)$ for Cases I, II, III.

Case	$\Lambda^{(5)}$ GeV	$\Lambda^{(4)}$ GeV	$\Lambda^{(3)}$ GeV
I	0.1995 ± 0.0227	0.2789 ± 0.0284	0.3213 ± 0.0295
II	0.2329 ± 0.0354	0.3205 ± 0.0436	0.3642 ± 0.0446
III	0.2087 ± 0.0820	0.2905 ± 0.1022	0.3333 ± 0.1057

Here it will be relevant to discuss the uncertainties estimated by Deshpande and Keith [4] using the value of $\alpha_3(M_Z) = 0.118 \pm 0.007$ which is the same as our Case III. They found small uncertainties as follows.

$$\eta_b = 1.53^{+0.07}_{-0.06}, \quad \eta_c = 2.09^{+0.27}_{-0.19}, \quad \eta_{s,d} = 2.36^{+0.53}_{-0.29}, \quad \eta_u = 2.38^{+0.52}_{-0.30}.$$

These low uncertainties may have resulted from an apparent mistake in their estimation of uncertainties in $\alpha_3^{-1}(m_t) = 9.30^{-0.047}_{+0.054}$; instead, it should be corrected as $\alpha_3^{-1}(m_t) = 9.30^{-0.47}_{+0.54}$. In this case one can expect larger uncertainties in η_f though the method of error estimation is not mentioned specifically.

A few comments on the analysis of estimating uncertainties are in order. In the first place, the uncertainties are symmetric in accordance with the quadrature method outlined in §2. In the second place, we investigate consistency of error propagation in different cases using this method.

5. Summary and conclusion

To summarise, we have outlined the procedure for estimating the uncertainties using the quadrature method. In particular, we employ this method to estimate the numerical symmetric uncertainties in QCD–QED rescaling factors η_f acquired from the uncertainties in input values of $\alpha_3(M_Z)$ and $\alpha(M_Z)$ while running the energy scale from high to low. We have used the three-loop order in QCD and one-loop order in QED evolution equations to calculate the rescaling factors while their uncertainties estimated by the quadrature method are found to be new in comparison to earlier estimates in literature. The present estimation of uncertainties in η_f using quadrature method is very convincing and also it regulates the propagation of uncertainties while running the energy scale from high to low. These uncertainties in rescaling factors can reliably be used in other low-energy scale predictions

Table 5. The values of $\alpha_3(\mu)$ and $R^f_{\text{QCD}}(\mu\text{–}\mu')$, $\mu > \mu'$ with uncertainties for three different input experimental values of $\alpha_3(M_Z)$. First and second rows of each case represent the values of $\alpha_3(\mu)$ and $R^f_{\text{QCD}}(\mu\text{–}\mu')$ respectively.

Case	m_t–m_b	m_b–m_τ	m_τ–m_c	m_c–1 GeV	< 1 GeV
I	0.1131 ± 0.0019	0.2202 ± 0.0074	0.3108 ± 0.0156	0.3825 ± 0.0254	0.4593 ± 0.0396
	0.6769 ± 0.0153	0.8201 ± 0.0289	0.8813 ± 0.0450	0.8974 ± 0.0583	
II	0.1157 ± 0.0026	0.2308 ± 0.0109	0.3335 ± 0.0243	0.4214 ± 0.0427	0.5229 ± 0.0732
	0.6662 ± 0.0211	0.8072 ± 0.0415	0.8647 ± 0.0678	0.8762 ± 0.0939	
III	0.1138 ± 0.0065	0.2232 ± 0.0262	0.3171 ± 0.0562	0.3930 ± 0.0934	0.4759 ± 0.1495
	0.6736 ± 0.0531	0.8165 ± 0.1017	0.8769 ± 0.1605	0.8920 ± 0.2117	

Table 6. QCD–QED rescaling factors with uncertainties for quarks for three different input values of $\alpha_3(M_Z)$.

Case	$\alpha_3(M_Z)$	η_b	η_c	$\eta_{s,d}$	η_u
I	0.1172 ± 0.002	1.4795 ± 0.0334	2.0816 ± 0.1374	2.2857 ± 0.2114	2.3205 ± 0.2149
II	0.1200 ± 0.0028	1.5033 ± 0.0476	2.1901 ± 0.2167	2.4593 ± 0.3587	2.5005 ± 0.3648
III	0.1180 ± 0.007	1.4868 ± 0.1172	2.1116 ± 0.4962	2.3290 ± 0.7779	2.3682 ± 0.7909

Table 7. Percentage errors propagating from input values of $\alpha_3(M_Z)$ to rescaling factors η_f. The numbers in square brackets represent the ratios of percentage errors of the parameters to those of the respective parameters in Case I.

Case	$\alpha_3(M_Z)$ (%)	η_b (%)	η_c (%)	$\eta_{s,d}$ (%)	η_u(%)
I	1.7	2.3	6.6	9.3	9.3
II	2.3 [1.4]	3.1 [1.4]	9.9 [1.5]	14.6 [1.6]	14.6 [1.6]
III	5.9 [3.5]	7.9 [3.5]	23.5 [3.6]	33.4 [3.6]	33.4 [3.6]

such as in calculation of neutrino masses and mixings from those obtained in see-saw mechanism [13]. We emphasise that the quadrature method is an important mathematical tool for computing reliable errors.

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