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Numerical consistency check between two approaches to radiative corrections for neutrino masses and mixings

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Abstract. We briefly outline the two popular approaches on radiative corrections to neutrino masses and mixing angles, and then carry out a detailed numerical analysis for a consistency check between them in MSSM. We find that the two approaches are nearly consistent with a discrepancy factor of 4.2% with running vacuum expectation value (VEV) (13% for scale-independent VEV) in mass eigenvalues at low-energy scale but the predictions on mixing angles are almost consistent. We check the stability of the three types of neutrino models, i.e., hierarchical, inverted hierarchical and degenerate models, under radiative corrections, using both approaches, and find consistent conclusions. The neutrino mass models which are found to be stable under radiative corrections in MSSM are the normal hierarchical model and the inverted hierarchical model with opposite CP parity. We also carry out numerical analysis on some important conjectures related to radiative corrections in the MSSM, viz., radiative magnification of solar and atmospheric mixings in the case of nearly degenerate model having same CP parity (MPR conjecture) and radiative generation of solar mass scale in exactly two-fold degenerate model with opposite CP parity and non-zero U_{e3} (JM conjecture). We observe certain exceptions to these conjectures. We find a new result that both solar mass scale and U_{e3} can be generated through radiative corrections at low energy scale. Finally the effect of scale-dependent vacuum expectation value in neutrino mass renormalisation is discussed.

Keywords. Radiative correction; renormalisation group equations; neutrino masses and mixing angles; conjecture.

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1. Introduction

Recent developments in the determination of neutrino masses and mixing angles from various oscillation experiments have strengthened our knowledge of neutrino physics [1]. In order to have a meaningful comparison of the theoretical predictions on neutrino masses and mixing angles within the framework of GUTs with or without supersymmetry, with the data from various neutrino oscillation experiments [2], the effects of radiative corrections are very essential [3]. Considerable progress has been achieved at this front and this can be mainly classified into two categories: (i) the evolution of the renormalisation group equations (RGEs) from high- to low-energy scale [4–7] and (ii) the low-energy threshold corrections [8]. In the case of running the RGEs in (i), the general underlying motivations are: to check the stability of the neutrino mass model under radiative corrections [9,10]; to generate solar mass scale and also reactor angle $|U_{e3}|$ from radiative corrections [11]; to generate correct radiative magnifications of solar and atmospheric mixing angles from CKM-like small values at high scale [12]; to get suitable deviations from the bimaximal solar and atmospheric mixings through radiative corrections [13,14], etc. One basic difference between the first one and the last three points is the very definition of the stability criteria under radiative corrections [15]. Radiative stability generally means that the effects of radiative corrections do not substantially alter the good predictions on neutrino masses and mixings already acquired through see-saw mechanism at high-scale M_R.

Within the framework of running the RGEs from high-energy scale to low-energy scale, we have again two different approaches so far employed in the literature. In the first approach (we call it Method A for simplicity) the running is carried out through the neutrino mass matrix m_{LL} as a whole, and at every energy scale one can extract neutrino masses and mixing angles through the diagonalisation of the neutrino mass matrix calculated at that particular energy scale [6,7,10]. In the second approach (Method B) the running of the RGEs is carried out in terms of neutrino mass eigenvalues and three mixing angles [16–18]. We confine our analysis to CP-conserving case, neglecting all CP phases in the neutrino mixing matrix.

In the present paper we carry out a detailed numerical analysis of these two approaches for a consistency check on numerical accuracy, and find out the stability criteria of the main three neutrino mass models [19]. We give all the zeroth-order as well as full textures of the left-handed neutrino mass matrices obtained from see-saw mechanism in Appendices A and B, and use these expressions for checking the stability criteria. In addition, we further study the validity of some existing conjectures based on radiative corrections. The effect of running vacuum expectation value to the evolution of neutrino masses is further examined in both approaches. The paper is organised as follows: In §2, we briefly outline the main points of the two approaches on renormalisation group analysis. The numerical analysis and main results are presented in §3. Section 4 concludes with a summary and discussion.

2. Renormalisation group analysis of neutrino masses and mixings

We present a very brief review on the two main approaches of taking quantum radiative corrections of neutrino masses and mixings in MSSM. Our main motivation is to have a numerical consistency check on the results of these two approaches, and apply again to check the validity on some existing conjectures related to radiative corrections.

2.1 *Method A: Evolution of neutrino mass matrix*

In this approach the quantum radiative corrections are taken on all the elements of the neutrino mass matrix m_{LL} where charged lepton mass matrix is diagonal. The diagonalisation of the neutrino mass matrix at any particular energy scale leads to the physical neutrino mass eigenvalues as well as three mixing angles. The neutrino mass matrix $m_{LL}(t)$ which is generally obtained from see-saw mechanism, is expressible in terms of $K(t)$, the coefficient of the dimension-five neutrino mass operator in a scale-dependent manner, $t = \ln(\mu/1 \text{ GeV})$,

$$m_{LL}(t) = v_u^2 K(t), \tag{1}$$

where the vacuum expectation value (VEV) is $v_u = v_0 \sin\beta$, and $v_0 = 174$ GeV in MSSM. The evolution equation of the coefficient $K(t)$ in the basis where the charged lepton mass matrix is diagonal, is given by [10]

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln K(t) = -\frac{1}{16\pi^2} \times \left[\frac{6}{5}g_1^2 + 6g_2^2 - 6h_t^2 - h_\tau^2\delta_{i3} - h_\tau^2\delta_{3j}\right]. \tag{2}$$

We replace $K(t)$ by $m_{LL}(t)$ in the above equation where we assume that VEV is scale-independent. Upon integration from high $(B-L)$ breaking scale $t_R(= \ln(M_R/1 \text{ GeV})$ to top-quark mass scale $t_0(= \ln(m_t/1 \text{ GeV}))$ where $t_0 \le t \le t_R$, we get a simple analytic solution as

$$m_{LL}(t_0) = [\mathrm{Diag}(\mathrm{e}^{-I_e}, \mathrm{e}^{-I_\mu}, \mathrm{e}^{-I_\tau}) \cdot m_{LL}(t_R) \cdot \mathrm{Diag}(\mathrm{e}^{-I_e}, \mathrm{e}^{-I_\mu}, \mathrm{e}^{-I_\tau})]R_0.$$

This gives the following form:

$$m_{LL}(t_0) = \begin{pmatrix} m_{11}(t_R)\mathrm{e}^{-2I_e} & m_{12}(t_R)\mathrm{e}^{-(I_e+I_\mu)} & m_{13}(t_R)\mathrm{e}^{-(I_e+I_\tau)} \\ m_{21}(t_R)\mathrm{e}^{-(I_e+I_\mu)} & m_{22}(t_R)\mathrm{e}^{-2I_\mu} & m_{23}(t_R)\mathrm{e}^{-(I_\mu+I_\tau)} \\ m_{31}(t_R)\mathrm{e}^{-(I_e+I_\tau)} & m_{32}(t_R)\mathrm{e}^{-(I_\mu+I_\tau)} & m_{33}(t_R)\mathrm{e}^{-2I_\tau} \end{pmatrix} R_0.$$

After neglecting h_e^2 and h_μ^2 compared to h_τ^2, we have $\mathrm{e}^{-I_e} \simeq 1$ and $\mathrm{e}^{-I_\mu} \simeq 1$, and the above form of neutrino mass matrix can be simplified as

$$m_{LL}(t_0) \simeq \begin{pmatrix} m_{11}(t_R) & m_{12}(t_R) & m_{13}(t_R)\mathrm{e}^{-I_\tau} \\ m_{21}(t_R) & m_{22}(t_R) & m_{23}(t_R)\mathrm{e}^{-I_\tau} \\ m_{31}(t_R)\mathrm{e}^{-I_\tau} & m_{32}(t_R)\mathrm{e}^{-I_\tau} & m_{33}(t_R)\mathrm{e}^{-2I_\tau} \end{pmatrix} R_0. \tag{3}$$

As emphasised earlier, the above analytic solution of the neutrino mass matrix at low-energy scale is possible only where charged lepton mass matrix is diagonal. Here the overall factor R_0 which does not affect the mixing angles, is given by the expression

$$R_0 = \exp[(6/5)I_{g_1} + 6I_{g_2} - 6I_t] \tag{4}$$

which strongly depends on top-quark Yukawa coupling and gives very large contribution for small $\tan\beta$ value (=1.60). The integrals in the above expressions are defined as [7,10]

$$I_{g_i} = \frac{1}{16\pi^2}\int_{t_0}^{t_R} g_i^2(t)\mathrm{d}t \tag{5}$$

and

$$I_f = \frac{1}{16\pi^2}\int_{t_0}^{t_R} h_f^2(t)\mathrm{d}t, \tag{6}$$

where $i = 1, 2, 3$ and $f = t, b, \tau$ respectively. The numerical values of these integrals at different energy scales t where $t_0 \leq t \leq t_R$, can be calculated from the running of the RGEs for three gauge couplings (g_1, g_2, g_3) and the third family Yukawa couplings (h_t, h_b, h_τ) in MSSM through numerical approximation. This is another possible source of numerical errors in this method.

The mass eigenvalues (m_1, m_2, m_3) and the U_{MNS} mixing matrix [20] are estimated through diagonalisation of m_{LL} at every point in the energy scale t,

$$m_{LL}^{\mathrm{diag}} = \mathrm{Diag}(m_1, m_2, m_3) = V_{\nu L} m_{LL} V_{\nu L}^T \tag{7}$$

and $U_{\mathrm{MNS}} = V_{\nu L}^\dagger$ which is identified with U_{fi} in the neutrino oscillation relation from flavour state ν_f to mass eigenstate ν_i:

$$|\nu_f\rangle = U_{fi}|\nu_i\rangle, \tag{8}$$

where $f = \tau, \mu, e$ and $i = 1, 2, 3$. The MNS mixing matrix [20]

$$U_{\mathrm{MNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \tag{9}$$

is usually parametrised in terms of three rotations (neglecting CP-violating phases) by

$$U_{\mathrm{MNS}} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23} & c_{12}c_{23} - s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23} & -c_{12}s_{23} - c_{23}s_{13}s_{12} & c_{13}c_{23} \end{pmatrix}, \tag{10}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$ respectively. The unitarity conditions are also satisfied as

$$U_{e1}^2 + U_{e2}^2 + U_{e3}^2 = 1 = U_{e3}^2 + U_{\mu 3}^2 + U_{\tau 3}^2.$$

Hence we have $\tan\theta_{12} = |U_{e2}|/|U_{e1}|$ and $\tan\theta_{23} = |U_{\mu 3}|/|U_{\tau 3}|$ and $\sin\theta_{13} = |U_{e3}|$. The solar LMA MSW solution favours the 'light-side' described by the data $\tan\theta_{12} < 1$ [21–23] for the usual sign convention $|m_2| > |m_1|$ which is adopted in the present analysis. It is also possible to express mixing angles in terms of $\sin\theta_{ij}$ directly [24].

2.2 *Method B: Evolution of neutrino mass eigenvalues and mixing angles*

Here we follow the main expressions from [16] for the evolution of three neutrino mass eigenvalues and mixing angles. In this approach the RGEs for the eigenvalues of coefficient $K(t)$ can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}K_i = \frac{1}{16\pi^2}\sum_{f=e,\mu,\tau}\left[-\frac{6}{5}g_1^2 - 6g_2^2 + 6Tr(h_u^2) + 2h_f^2U_{fi}^2\right]K_i. \tag{11}$$

In the above expression we make further simplification of the term,

$$\sum_{f=e,\mu,\tau} h_f^2U_{fi}^2 = h_\tau^2U_{\tau i}^2 + h_\mu^2U_{\mu i}^2 + h_e^2U_{ei}^2 \simeq h_\tau^2U_{\tau i}^2$$

by neglecting h_μ^2 and h_e^2 compared to h_τ^2 as before but while doing this approximation we have also washed out some effects of MNS matrix elements $U_{\mu i}^2$ and U_{ei}^2. Taking the scale-independent VEV as before, we have the complete RGEs for three neutrino mass eigenvalues ($m_i = v_u^2K_i$) [16],

$$\frac{\mathrm{d}}{\mathrm{d}t}m_i = \frac{1}{16\pi^2}\left[\left(-\frac{6}{5}g_1^2 - 6g_2^2 + 6h_t^2\right) + 2h_\tau^2U_{\tau i}^2\right]m_i. \tag{12}$$

This is the key equation for running the neutrino mass eigenvalues along with the running of mixing angles in this approach. Since it is difficult to get an analytic solution of eq. (12), we solve this RGEs through numerical method. Thus the source of errors in this approximation is more in Method B than in Method A. For a demonstration we also find an approximate analytic solution of eq. (12) by neglecting the small effect due to the change of $U_{\tau i}^2$ within the integration range (i.e., for static mixing angles), as [15]

$$m_i(t_0) = m_i(t_R)\exp\left(\frac{6}{5}I_{g_1} + 6I_{g_2} - 6I_t\right)\exp(-2U_{\tau i}^2I_\tau). \tag{13}$$

These equations lead to the equations derived in ref. [11] (hence JM conjecture) and have interesting consequences. However, it should be emphasised that these equations are valid when the mixing angles are only static. We are not interested in the solution of static mixing angles in the present work. The present investigation is beyond JM conjecture with the inclusion of running mixing angles.

In order to supplement eq. (12), the corresponding evolution equations for the MNS matrix elements U_{fi} are given by [16]

$$\frac{\mathrm{d}U_{fi}}{\mathrm{d}t} = -\frac{1}{16\pi^2}\sum_{k\neq i}\frac{m_k+m_i}{m_k-m_i}U_{fk}(U^TH_e^2U)_{ki}, \tag{14}$$

where $f = e,\mu,\tau$ and $i = 1,2,3$ respectively. Here H_e is the charged lepton Yukawa matrix in the diagonal basis. Neglecting h_μ^2 and h_e^2 as before, along with many elements of MNS matrix U_{fi}, one can simplify the terms,

$$(U^TH_e^2U)_{13} \simeq h_\tau^2(U_{1\tau}^TU_{\tau 3}),$$
$$(U^TH_e^2U)_{23} \simeq h_\tau^2(U_{2\tau}^TU_{\tau 3}),$$
$$(U^TH_e^2U)_{12} \simeq h_\tau^2(U_{1\tau}^TU_{\tau 2}).$$

Such approximation of neglecting the elements of mixing MNS matrix is not there in Method A. Denoting $A_{ki} = \frac{m_k+m_i}{m_k-m_i}$, we can write the RGEs for all the nine elements of MNS matrix. For example, we give here only three of them relevant to our requirement such as

$$\frac{\mathrm{d}U_{e2}}{\mathrm{d}t} \simeq -\frac{1}{16\pi^2}[U_{\tau 2}h_\tau^2(A_{32}U_{e3}U_{3\tau}^\dagger + A_{12}U_{e1}U_{1\tau}^\dagger)], \tag{15}$$

$$\frac{\mathrm{d}U_{e3}}{\mathrm{d}t} \simeq -\frac{1}{16\pi^2}[U_{\tau 3}h_\tau^2(A_{13}U_{e1}U_{1\tau}^\dagger + A_{23}U_{e2}U_{2\tau}^\dagger)], \tag{16}$$

$$\frac{\mathrm{d}U_{\mu 3}}{\mathrm{d}t} \simeq -\frac{1}{16\pi^2}[U_{\tau 3}h_\tau^2(A_{13}U_{\mu 1}U_{1\tau}^\dagger + A_{23}U_{\mu 2}U_{2\tau}^\dagger)]. \tag{17}$$

Using the MNS parametrisation in eq. (10), the above three expressions ((15)–(17)) simplify to [16]

$$\frac{\mathrm{d}s_{12}}{\mathrm{d}t} \simeq \frac{1}{16\pi^2}h_\tau^2 c_{12}[c_{23}s_{13}s_{12}U_{\tau 1}A_{31} - c_{23}s_{13}c_{13}U_{\tau 2}A_{32} + U_{\tau 1}U_{\tau 2}A_{21}], \tag{18}$$

$$\frac{\mathrm{d}s_{13}}{\mathrm{d}t} \simeq \frac{1}{16\pi^2}h_\tau^2 c_{23}c_{13}^2[c_{12}U_{\tau 1}A_{31} + s_{12}U_{\tau 2}A_{32}], \tag{19}$$

$$\frac{\mathrm{d}s_{23}}{\mathrm{d}t} \simeq \frac{1}{16\pi^2}h_\tau^2 c_{23}^2[-s_{12}U_{\tau 1}A_{31} + c_{12}U_{\tau 2}A_{32}]. \tag{20}$$

In this method, instead of getting an analytic solution in static mixings, we insist on numerical solution of neutrino mass eigenvalues in eq. (12) along with the running of mixing angles in eqs (18)–(20), via eq. (10).

The nature of approximations involved in Methods A and B are quite different. In addition to analytic form of solution in Method A and numerical solution in Method B, Method A is more accurate in the sense that while neglecting h_e^2 and h_μ^2 it does not wash out some of the MNS mixing matrix elements which also connect with the mass eigenvalues. However, the runnings of Yukawa couplings and gauge couplings are common for both. The calculation of integrals I_{gi} and I_f through numerical analysis, are present in Method A only. Thus the top-quark Yukawa coupling and gauge coupling effects enter in Methods A and B in different ways. In Method A, it enters in the form of exponentials of the integrals which involve more numerical approximations, whereas in Method B it enters as gauge and Yukawa couplings which involve lesser source of approximations. It is thus quite natural to expect discrepancies in the numerical output of these two methods.

2.3 *Effect of scale-dependent VEV*

In this section we modify the results of ref. [16] by considering the running of the VEV $v_u(t)$ through the neutrino mass formula $m_i(t) = v_u^2(t)K_i(t)$. This gives [15]

$$\frac{\mathrm{d}(\ln m_i)}{\mathrm{d}t} = \frac{\mathrm{d}(\ln K_i)}{\mathrm{d}t} + 2\frac{\mathrm{d}(\ln v_u)}{\mathrm{d}t}, \tag{21}$$

where the RGE for v_u in MSSM is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}v_u = \frac{1}{16\pi^2}\left[\frac{3}{20}g_1^2 + \frac{3}{4}g_2^2 - 3h_t^2\right]v_u. \tag{22}$$

The complete RGEs for neutrino mass eigenvalues are now given by [15]

$$\frac{\mathrm{d}}{\mathrm{d}t}m_i = \frac{1}{16\pi^2}\left[\left(-\frac{9}{10}g_1^2 - \frac{9}{2}g_2^2\right) + 2h_\tau^2 U_{\tau i}^2\right]m_i \tag{23}$$

leading to the approximate solution,

$$m_i(t_0) = m_i(t_R)\exp\left(\frac{9}{10}I_{g_1} + \frac{9}{2}I_{g_2}\right)\exp(-2U_{\tau i}^2 I_\tau). \tag{24}$$

Similarly, for Method A with the inclusion of scale-dependent VEV, R_0 in eq. (4) is now replaced by

$$R_0 = \exp\left(\frac{9}{10}I_{g_1} + \frac{9}{2}I_{g_2}\right). \tag{25}$$

Now the top-quark dependent term which imparts large contribution, has been cancelled out and this will certainly affect the overall magnitude of the neutrino masses, but not the mixing angles. It decreases the discrepancy of the results between two methods. With this modification the magnitudes of neutrino masses tend to increase with the decrease of energy scale. The enhancement factor in the magnitude of neutrino masses through RGEs is calculated as

$$f = \exp\left(6I_t - \frac{3}{10}I_{g_1} - \frac{3}{2}I_{g_2}\right) \tag{26}$$

which gives a positive numerical value greater than one even for large $\tan\beta$ values ($= 55$). We present the numerical analysis for both Methods A and B with the effect of scale-dependent VEV in eqs (23) and (25) respectively. Whenever necessary we compare these results with those without running VEV.

3. Numerical analysis and results

For a complete numerical analysis of the RGEs for both Methods A and B presented in the previous section, we follow here two steps: (a) bottom-up running [7] and (b) top-down running [10]. In step (a), the running of the RGEs for the third family Yukawa couplings (h_t, h_b, h_τ) and three gauge couplings (g_1, g_2, g_3) in MSSM is carried out from top-quark mass scale (t_0) at low-energy end to high-energy scale M_R where $B-L$ symmetry breaks down [7,15]. In the present analysis we consider the high-scale $M_R = 10^{13}$ GeV and take the large $\tan\beta$ input value ($\tan\beta = 55$). For simplicity of the calculation, we assume here the supersymmetric breaking scale

Table 1. Values of the Yakawa couplings and gauge couplings at $t_0 = 5.159$ and $t_R = 29.954$.

$t_0 = 5.159$	$t_R = 29.954$	$t_R = 29.954$
$h_t = 1.000165276$	$h_t = 0.849373996$	$I_t = 0.109299302$
$h_b = 0.866235097$	$h_b = 0.641425729$	$I_b = 0.0759593919$
$h_\tau = 0.555977506$	$h_\tau = 0.679428339$	$I_\tau = 0.0620016195$
$g_1 = 0.463751$	$g_1 = 0.626455069$	$I_{g_1} = 0.0544101298$
$g_2 = 0.6513289$	$g_2 = 0.708234191$	$I_{g_2} = 0.0803824514$
$g_3 = 1.1891966$	$g_3 = 0.784410238$	$I_{g_3} = 0.141634345$

at the top-quark mass scale $t_0 = \ln m_t$ [7]. We adopt the standard procedure to get the values of gauge couplings at top-quark mass scale from the experimental CERN-LEP measurements at M_Z, using one-loop RGEs, assuming the existence of a one-light Higgs doublet and five quark flavours below m_t scale [15]. Similarly, the Yukawa couplings are also evaluated at top-quark mass scale using QCD-QED rescaling factors in the standard fashion [15]. We present the values of the Yukawa couplings and gauge couplings at two scales $t_0 = 5.159$ and $t_R = 29.954$ in table 1.

The values of the integrals (I_f, I_{g_i}) are estimated between the two limits (t_0, t_R). In step (b), the running of three neutrino masses (m_1, m_2, m_3) and mixing angles (s_{12}, s_{13}, s_{23}) is carried out together with the running of the Yukawa and gauge couplings, from the high scale t_R to low scale t_0. In this case we use the values of Yukawa and gauge couplings evaluated earlier at the scale t_R from the first stage running of RGEs. In principle one can evaluate neutrino masses and mixing angles at every point in the energy scale. To start with, in Method A the neutrino mass matrix $m_{LL}(t_R)$ at high scale is obtained from the given input neutrino masses and mixing angles through the inverse of eq. (7). In Method B the neutrino masses and mixing angles at high scale are extracted through the diagonalisation on $m_{LL}(t_R)$. Our results are based on the equations with scale-dependent VEV effects if otherwise stated.

As emphasised earlier, we follow the equations which include the effects of running VEV given in eqs (3) and (25) for Method A and eqs (23) and (18)–(20) for Method B. We present the results of our numerical analysis in tables 2–5. First we check the stability of the three neutrino mass models (patterns) under radiative corrections in MSSM. Tables 2a and b give the values of neutrino masses and mixing angles at high and low scales for three neutrino mass models – hierarchical, inverted hierarchical and degenerate models (see Appendices A and B). For a check on numerical consistency, we evaluate the quantities for both Methods A and B outlined in the previous section. Both methods give nearly consistent results to about a factor of 4.2% discrepancy in neutrino mass eigenvalues. This discrepancy is increased to about 13% in case where there is no running of VEV. However, the predictions on mixing angles are almost consistent. As explained in the previous section, the origin of such discrepancy in neutrino mass eigenvalues stems partly from the analytical versus approximate numerical methods adopted in these two approaches respectively. The other source is the use of integrals in Method A

Table 2a. Running of neutrino masses and mixing angles from high-scale $M_R = 10^{13}$ GeV to top-quark mass scale $m_t = 175$ GeV in MSSM for hierarchical (Type III) and inverted hierarchical (Type II) models (m_{LL} collected from Appendix B). Methods A and B are explained in the text.

Type	Item	$\mu = M_R$	(A): $\mu = m_t$	(B): $\mu = m_t$
III	m_1	0.00336 eV	0.0049321 eV	0.00477285 eV
	m_2	0.007357 eV	0.010731 eV	0.010196 eV
	m_3	0.057013 eV	0.0805056 eV	0.07691779 eV
	s_{23}	0.65630	0.6632	0.68441
	s_{13}	0.07358	0.08141	0.07911
	s_{12}	0.5838	0.58807	0.608521
	Δm^2_{12}	4.28×10^{-5} eV^2	9.088×10^{-5} eV^2	8.117×10^{-5} eV^2
	Δm^2_{23}	3.20×10^{-3} eV^2	6.209×10^{-3} eV^2	5.82×10^{-3} eV^2
IIB	m_1	−0.070445 eV	−0.103077 eV	−0.0986569 eV
	m_2	0.070977 eV	0.1038356 eV	0.0993825 eV
	m_3	0.0005324 eV	0.000753 eV	0.0007211 eV
	s_{23}	0.7071	0.68486	0.685182
	s_{13}	0.0	0.00045	0.000445
	s_{12}	0.7057745	0.70581	0.7057745
	Δm^2_{12}	7.52×10^{-5} eV^2	15.69×10^{-5} eV^2	15.36×10^{-5} eV^2
	Δm^2_{23}	4.96×10^{-3} eV^2	10.61×10^{-3} eV^2	9.72×10^{-3} eV^2
IIA	m_1	0.0497257 eV	0.0709716 eV	0.0679571 eV
	m_2	0.0500693 eV	0.0750938 eV	0.07184342 eV
	m_3	0.000005 eV	0.00000672 eV	0.00000677 eV
	s_{23}	0.707107	0.68486	0.68518
	s_{13}	0.0	0.0	0.0
	s_{12}	0.465	0.99874	0.998697
	Δm^2_{12}	3.43×10^{-5} eV^2	59.64×10^{-5} eV^2	54.34×10^{-5} eV^2
	Δm^2_{23}	2.47×10^{-3} eV^2	5.64×10^{-3} eV^2	5.17×10^{-3} eV^2

whereas use of gauge and Yukawa couplings in Method B. More importantly, in Method B considerable effects of MNS mixing elements have been washed out in the process of approximations.

On the question of radiative stability of the neutrino mass models, only the hierarchical model (Type III) and the inverted hierarchical model (Type IIB) with opposite CP parity, appear to be stable under RG analysis in MSSM. In fact, both Δm^2_{21} and Δm^2_{23} are slightly bigger, whereas the mixing angles are slightly increased with the decrease in energy scale. But there is no substantial change in both parameters, which may spoil the good predictions already achieved at the high-energy scale. In the case of inverted hierarchical model with the same CP parity (IIA), the solar angle (s_{12}) is not stable under radiative corrections. Similarly, in all the three degenerate models (IA,B,C) both solar and atmospheric mass scales as well as solar and atmospheric angles, are not stable under radiative corrections.

Table 2b. Running of neutrino masses and mixing angles from high-scale $M_R = 10^{13}$ GeV to top-quark mass scale $m_t = 175$ GeV in MSSM for degenerate models (m_{LL} collected from Appendix B). Methods A and B are explained in the text.

Type	Item	$\mu = M_R$	(A): $\mu = m_t$	(B): $\mu = m_t$
IA	m_1	0.396484 eV	0.547682 eV	0.524893 eV
	m_2	-0.396532 eV	-0.579721 eV	-0.554875 eV
	m_3	0.4 eV	0.5996609 eV	0.573812 eV
	s_{23}	0.707107	0.99885	0.998810
	s_{13}	0.0	0.556	0.558
	s_{12}	0.70931	0.853267	0.85202
	Δm^2_{12}	3.81×10^{-5} eV^2	36.13×10^{-3} eV^2	32.36×10^{-3} eV^2
	Δm^2_{23}	2.76×10^{-3} eV^2	23.52×10^{-3} eV^2	21.37×10^{-3} eV^2
IB	m_1	0.396841 eV	0.530599 eV	0.5086461 eV
	m_2	0.396891 eV	0.598332 eV	0.572544 eV
	m_3	0.4 eV	0.600798 eV	0.574904 eV
	s_{23}	0.707107	0.99950	0.99949
	s_{13}	0.0	0.00854	0.011254
	s_{12}	0.459701	0.99999	0.99999
	Δm^2_{12}	3.97×10^{-5} eV^2	76.46×10^{-3} eV^2	69.10×10^{-3} eV^2
	Δm^2_{23}	2.48×10^{-3} eV^2	2.96×10^{-3} eV^2	2.69×10^{-3} eV^2
IC	m_1	0.396841 eV	0.5623985 eV	0.538642 eV
	m_2	0.396891 eV	0.598332 eV	0.572544 eV
	m_3	-0.4 eV	-0.566827 eV	-0.542889 eV
	s_{23}	0.707107	0.7071	0.70722
	s_{13}	0.0	0.0	0.0
	s_{12}	0.4597	1	0.999999
	Δm^2_{12}	3.97×10^{-5} eV^2	41.72×10^{-3} eV^2	37.68×10^{-3} eV^2
	Δm^2_{23}	2.48×10^{-3} eV^2	36.72×10^{-3} eV^2	33.09×10^{-3} eV^2

In table 3 we analyse the results of [12] (hence MPR conjecture) which states that quark and lepton mixing angles are identical at high-energy scale, and large solar and atmospheric neutrino mixing angles together with the small reactor angle, can be understood purely as a result of RG evolution provided the three neutrino masses are quasi-degenerate and have the same CP parity. In the present numerical analysis in Method B, both Yukawa and gauge couplings are running together with neutrino masses and mixings. It appears that in MPR paper [12] only neutrino masses and mixings are running, but both Yukawa couplings and gauge couplings entered in the equations are static at low-scale values. We present here three sets of readings (Cases (i), (ii), (iii)) which give good radiative magnifications. However, in Case (iv) we use the same input values quoted in MPR paper but we observe less magnification contrary to MPR paper [12]. This discrepancy may be due to the running of gauge and Yukawa couplings along with neutrino masses and mixings in the form of coupled equations in the present analysis. In short, MPR

Table 3. Analysis on MPR conjecture [12] related to radiative magnification on solar and atmospheric mixing angles at low scale. The parameters $(m^0_{1,2,3}, s^0_{23,13,12})$ are defined at high scale $M_R = 10^{13}$ GeV and others are defined at low scale $m_t = 175$ GeV. Cases (i)–(iv) include different sets of arbitrary input parameters (based on Method B).

Case (i)	Case (ii)	Case (iii)	Case (iv)
$m^0_1 = 0.5288446$ eV	0.5288446 eV	0.5288446 eV	0.3682 eV
$m^0_2 = 0.5309554$ eV	0.5309554 eV	0.5309554 eV	0.370 eV
$m^0_3 = 0.6$ eV	0.6 eV	0.6 eV	0.421 eV
$s^0_{23} = 0.0311$	0.03	0.031	0.038
$s^0_{13} = 0.005$	0.0	0.004	0.0025
$s^0_{12} = 0.22$	0.2	0.215	0.22
$m_1 = 0.7628872$ eV	0.762681596 eV	0.75705427 eV	0.53116755 eV
$m_2 = 0.763179655$ eV	0.7635522 eV	0.75745328 eV	0.53255123 eV
$m_3 = 0.768991035$ eV	0.76882465 eV	0.76312792 eV	0.53882401 eV
$s_{23} = 0.709379$	0.71158975	0.71025485	0.467712
$s_{13} = 0.135225$	0.071834	0.12237	0.05389
$s_{12} = 0.656099$	0.6637173	0.670588	0.32458
$\Delta m^2_{12} = 44.63 \times 10^{-5}$ eV^2	132.9×10^{-5} eV^2	60.43×10^{-5} eV^2	147.2×10^{-5} eV^2
$\Delta m^2_{23} = 8.92 \times 10^{-3}$ eV^2	8.08×10^{-3} eV^2	8.63×10^{-3} eV^2	6.74×10^{-3} eV^2

conjecture [12] is verified well within this discrepancy. The analysis presented in table 3 is the results of Method B, but we find that both Methods A and B give almost consistent results.

In table 4 we supply some new results connected to two more conjectures on radiative corrections in MSSM. JM conjecture [11] specifies that radiative corrections can generate the neutrino mass-squared difference required for the large mixing angle (LMA) MSW solution to the solar neutrino problem if two of the three neutrino masses are assumed to be exactly degenerate $(m, -m, m')$ at high-energy scale, and also if U_{e3} at high scale is non-zero (see Cases (i) and (ii) of table 4). We have shown in table 4 that for a limited range of non-zero values of m', it is also possible to generate LMA MSW solution even if U_{e3} is zero at high scale. If both m' and U_{e3} are zeros at high scale, then it is not possible to generate LMA MSW solution. Case (ii) in table 4 is interesting in the sense that it gives exception to JM conjecture by taking $U_{e3} = 0$ and $m' = 0.07$ eV. This is the new result of the present work. At low energy we get $\Delta m^2_{21} = 19.04 \times 10^{-5}$ eV^2 and $\sin\theta_{12} = 0.7042421$ which is high but lies in the light side of the data ($\tan\theta_{12} < 1$) [21–23]. Based on the data of Case (ii) in table 4, figure 1 shows the evolution of the three mixing angles, and the CHOOZ angle $\sin\theta_{13}$ is generated through radiative corrections. Figure 2 presents the evolution of the three neutrino masses, and the solar mass scale generated through radiative corrections at low scale, is then demonstrated in figure 3. It is in general to tone down the solar mixing angle through further fine tuning. As an example, using $s_{12} = 0.7$ at high scale, one can get $s_{12} = 0.697$ at lower scale. We also discuss another conjecture in ref. [14] (hence MST conjecture) which states that starting from bimaximal mixings at high scale, radiative corrections due to the τ-Yukawa coupling leads to solar angle towards the dark side

Table 4. Analysis on JM conjecture [11] related to radiative magnification on solar mass scale at low scale. The parameters $(m^0_{1,2,3}, s^0_{23,13,12})$ are defined at high scale $M_R = 10^{13}$ GeV and others are defined at low scale $m_t = 175$ GeV. Cases (i)–(iv) include different sets of arbitrary input parameters (based on Method B).

Case (i)	Case (ii)	Case (iii)	Case (iv)
$m^0_1 = -0.095$ eV	-0.095 eV	-0.08 eV	0.08 eV
$m^0_2 = 0.095$ eV	0.095 eV	-0.08 eV	-0.08 eV
$m^0_3 = 0.0$ eV	0.07 eV	0.04 eV	0.0 eV
$s^0_{23} = 0.707107$	0.707107	0.707107	0.707107
$s^0_{13} = 0.1$	0.0	0.1	0.0
$s^0_{12} = 0.707107$	0.707107	0.707107	0.707107
$m_1 = -0.132232$ eV	-0.1329549 eV	-0.11127132 eV	-0.11204921 eV
$m_2 = 0.1338083$ eV	0.1336695 eV	0.112837402 eV	0.11204921 eV
$m_3 = 0.0$ eV	0.09446706 eV	0.05418472 eV	0.0 eV
$s_{23} = 0.6852$	0.638701	0.6745	0.6852
$s_{13} = 0.0969$	0.087023	0.1329	0.0
$s_{12} = 0.705008$	0.7042421	0.70341	0.707107
$\Delta m^2_{12} = 41.94 \times 10^{-5}$ eV^2	19.04×10^{-5} eV^2	35.09×10^{-5} eV^2	0.0 eV^2
$\Delta m^2_{23} = 17.91 \times 10^{-3}$ eV^2	8.96×10^{-3} eV^2	9.79×10^{-3} eV^2	12.56×10^{-3} eV^2

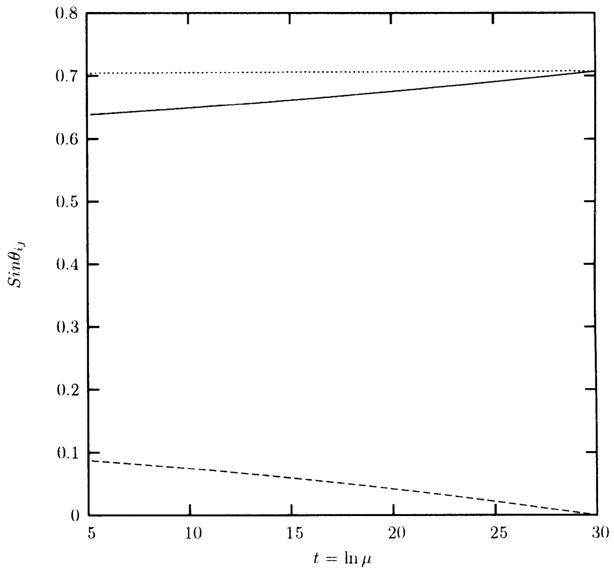

Figure 1. Evolution of the three mixing angles with energy scale in JM conjecture [11]. $\sin\theta_{23}$, $\sin\theta_{13}$ and $\sin\theta_{12}$ are represented by solid line, dashed line and dotted line respectively.

Table 5. Running of neutrino mass at low scale (I) with and (II) without the effect of scale-dependent VEV v_u^2 (based on Method B).

Item	$\mu = M_R$	$\mu = m_t$ (II)	$\mu = m_t$ (I)
m_1	0.528844 eV	0.399836034 eV	0.762722015 eV
m_2	0.5309554 eV	0.399989337 eV	0.763014257 eV
m_3	0.6 eV	0.403035134 eV	0.768824458 eV
s_{23}	0.0311	0.709379	0.709379
s_{13}	0.005	0.135225	0.135225
s_{12}	0.22	0.656099	0.656099
Δm_{12}^2	2.24×10^{-3} eV^2	12.26×10^{-5} eV^2	44.588×10^{-5} eV^2
Δm_{23}^2	7.81×10^{-2} eV^2	2.45×10^{-3} eV^2	8.90×10^{-3} eV^2

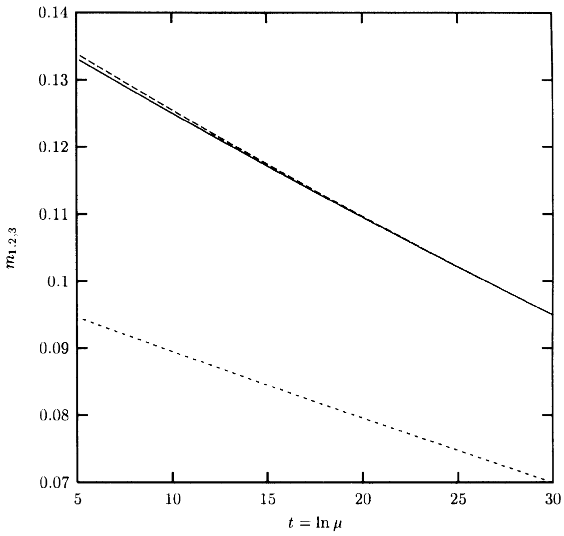

Figure 2. Evolution of the three neutrino mass eigenvalues in JM conjecture [11]. m_1, m_2 and m_3 are represented by dashed line, solid line and dotted line respectively.

at low scale ($\tan\theta_{12} > 1$). The results presented in table 4 also show that MST conjecture [14] is not always valid for Cases (i)–(iv) discussed here.

Finally, we give a few comments on the effect of scale-dependent VEV in RGEs. Without this effect the neutrino mass decreases with the decrease in energy scale. However, the effect of running VEV in neutrino mass formula leads to the increase of neutrino masses with the decrease of energy scale [15]. This feature is common to other fermion masses (i.e., quarks and charged leptons). In table 5, following Method B, we calculate the neutrino masses and mixings (I) with or (II) without the effect of running the VEV, at low-energy scale. There is a factor of about 1.91

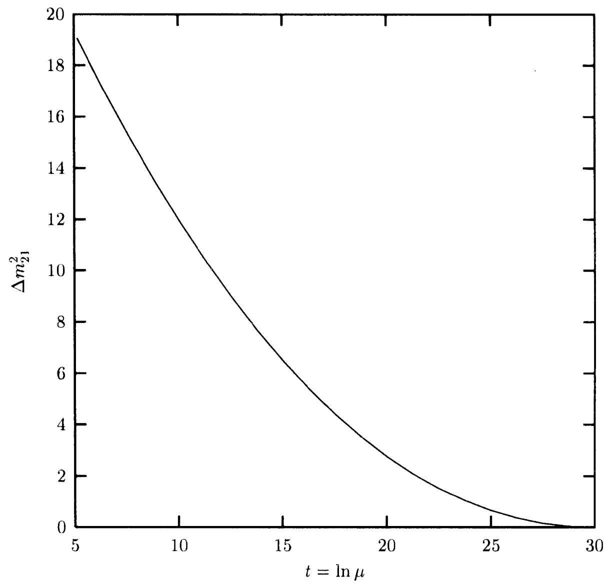

Figure 3. Evolution of the Δm^2_{21} in JM conjecture [11] with energy scale. Its value at high-energy scale is zero.

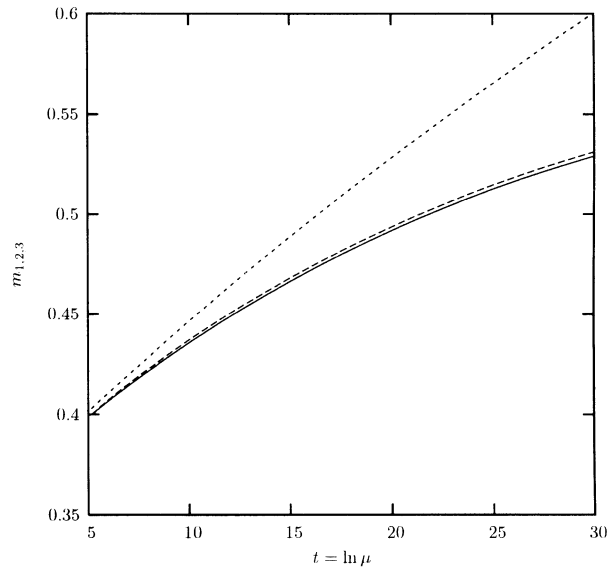

Figure 4. Evolution of the three nearly degenerate neutrino masses in MPR conjecture [12] (without the effect of scale-dependent VEV in II). m_3, m_2 and m_1 are represented by solid line, dashed line and dotted line respectively.

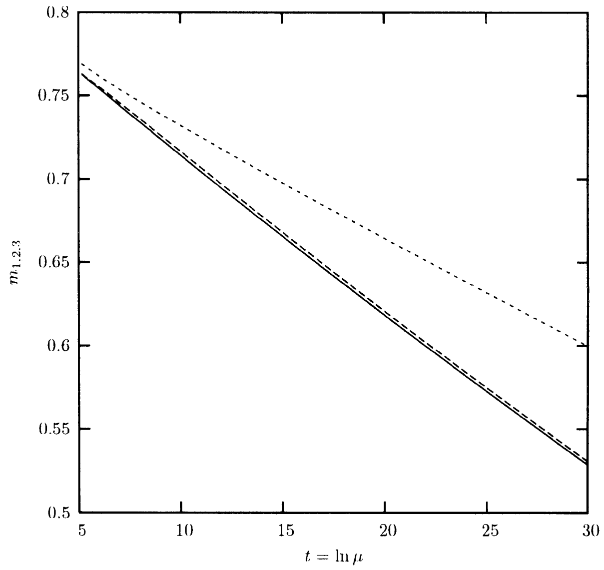

Figure 5. Evolution of the three nearly degenerate neutrino masses in MPR conjecture [12] (with the effect of scale-dependent VEV in I). m_3, m_2 and m_1 are represented by solid line, dashed line and dotted line respectively.

higher in the case of low-energy neutrino mass values obtained with the running effect of VEV (I) compared to (II) in Method B. This factor is only 1.68 in Method A and this discrepancy is inherent to different nature of approximations. Figures 4 and 5 show the evolution of neutrino masses for the above two cases (II) and (I) respectively.

4. Summary and discussion

We summarise the main points in this work. First we briefly review the main points of the formalism based on two approaches on the evolution of RGEs of neutrino masses and mixings. We include the effect of scale-dependent VEV. The first one (A) deals with the running of the whole neutrino mass matrix from which one can extract mass eigenvalues and mixings at any particular energy scale, whereas in the second approach (B) the three neutrino mass eigenvalues and the three mixing angles are running directly. Detailed numerical analysis shows that both approaches agree up to a discrepancy factor of 4.2% in mass eigenvalues (but 13% without running VEV). The predictions on mixing angles are almost consistent in these two approaches. The main origin of such discrepancy is the different nature of approximations involved in these two approaches. Method A leads to analytic solution in a compact way whereas Method B comprises of numerical solution of many simultaneous equations. Using both approaches we show that hierarchical model (III) and inverted hierarchical model with opposite CP parity (IIB) are almost stable under radiative corrections in MSSM. The evolution of $\sin\theta_{12}$ is very fast in the inverted hierarchical model with the same CP parity (IIA), and hence the model is not stable. We also verify the MPR conjecture [12] in which radiative magnification of solar and atmospheric mixings are possible in the case of nearly degenerate model with the same CP parity. We find that runnings of masses and mixings with and without the running of gauge and Yukawa couplings, give reasonably different magnifications. However, such radiative amplification generally involves a delicate fine-tuning of the initial conditions which are to some extent unnatural [17]. However, such problems are not there in the theory of neutrino masses derived from Kahler potential in supersymmetric model, and neutrino mixing angles can easily be driven to large values at low energy as they approach infrared pseudo-fixed points at large mixing [17].

We also study JM conjecture [11] which specifies the radiative generation of solar scale in exactly two-fold degenerate model having opposite CP parity $(m, -m, m_3)$ and non-zero values of U_{e3}. Our numerical analysis shows that the same radiative generation of solar mass scale is also possible with the conditions $U_{e3} = 0.0$ and non-zero value of m_3. We also discuss the MST conjecture [14] which states that starting from bimaximal mixings at high scale, radiative corrections lead to the solar angle towards the dark side of the data at low-energy scale. We show that MST conjecture is not always valid. We suggest further generalisation of JM conjecture and this will be reported in subsequent communication [25]. Finally, the effect of running the VEV in neutrino mass renormalisation is discussed and it is observed that neutrino mass increases with the decrease of energy scale when we include the running of VEV. This gives magnification of a factor of 1.91 in neutrino masses at low-energy scale, compared to the values calculated without the running VEV effect (in Method B). However this enhancement factor is calculated as 1.68 in Method A. Numerical analysis in three generations with arbitrary CP violating phases is of great interest [18] and there has been a possibility that within a restricted range of the physical parameters including phases, the degenerate models are found to be stable under radiative corrections in MSSM [26]. As emphasised before, oversimplifications of analytic expressions and a departure from the simultaneous running of Yukawa and gauge couplings along with neutrino masses and mixings, may have the danger of getting misleading conclusions.

Appendix A

The zeroth-order left-handed Majorana neutrino mass matrices with texture zeros, m_{LL}, corresponding to three models of neutrinos, viz., degenerate (Type I), inverted hierarchical (Type II) and normal hierarchical (Type III). These mass matrices are compatible with the LMA MSW solution as well as maximal atmospheric mixings [19].

Type	m_{LL}	$m_{LL}^{\rm diag}$
IA	$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} m_0$	$\mathrm{Diag}(1, -1, 1) m_0$

Type	m_{LL}	m_{LL}^{diag}
IB	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_0$	$\text{Diag}(1,1,1)m_0$
IC	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} m_0$	$\text{Diag}(1,1,-1)m_0$
IIA	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} m_0$	$\text{Diag}(1,1,0)m_0$
IIB	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} m_0$	$\text{Diag}(1,-1,0)m_0$
III	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} m_0$	$\text{Diag}(0,0,1)m_0$

Appendix B

The left-handed Majorana mass matrix m_{LL} for three different models of neutrinos presented in Appendix A. These results are collected from ref. [19].

Degenerate model (*Type IA*)

$$m_{LL} = \begin{pmatrix} (-2\delta_1 + 2\delta_2) & (\frac{1}{\sqrt{2}} - \delta_1) & (\frac{1}{\sqrt{2}} - \delta_1) \\ (\frac{1}{\sqrt{2}} - \delta_1) & (\frac{1}{2} + \delta_2) & (-\frac{1}{2} + \delta_2) \\ (\frac{1}{\sqrt{2}} - \delta_1) & (-\frac{1}{2} + \delta_2) & (\frac{1}{2} + \delta_2) \end{pmatrix} 0.4.$$

Degenerate model (*Type IB*)

$$m_{LL} = \begin{pmatrix} (1 - 2\delta_1 - 2\delta_2) & -\delta_1 & -\delta_1 \\ -\delta_1 & (1 - \delta_2) & -\delta_2 \\ -\delta_1 & -\delta_2 & (1 - \delta_2) \end{pmatrix} 0.4.$$

Degenerate model (*Type IC*)

$$m_{LL} = \begin{pmatrix} (1 - 2\delta_1 - 2\delta_2) & -\delta_1 & -\delta_1 \\ -\delta_1 & -\delta_2 & (1 - \delta_2) \\ -\delta_1 & (1 - \delta_2) & -\delta_2 \end{pmatrix} 0.4.$$

Inverted hierarchical model (Type IIA)

$$m_{LL} = \begin{pmatrix} (1-2\epsilon) & -\epsilon & -\epsilon \\ -\epsilon & \frac{1}{2} & (\frac{1}{2}-\eta) \\ -\epsilon & (\frac{1}{2}-\eta) & \frac{1}{2} \end{pmatrix} 0.05.$$

Inverted hierarchical model (Type IIB)

$$m_{LL} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & \lambda^3 & 0 \\ 1 & 0 & \lambda^3 \end{pmatrix} 0.05.$$

Hierarchical model (Type III)

$$m_{LL} = \begin{pmatrix} -\lambda^4 & \lambda & \lambda^3 \\ \lambda & 1-\lambda & -1 \\ \lambda^3 & -1 & 1-\lambda^3 \end{pmatrix} 0.03.$$

The values of the parameters used are: Type IA: $\delta_1 = 0.0061875$, $\delta_2 = 0.0030625$, Types IB and IC: $\delta_1 = 3.6\times10^{-5}$, $\delta_2 = 3.9\times10^{-3}$, Type IIA: $\eta = 0.0001$, $\epsilon = 0.002$, and Types IIB and III: $\lambda = 0.22$. All neutrino masses are in eV.

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