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Parity and the spin–statistics connection

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Abstract. A simple demonstration of the spin–statistics connection for general causal fields is obtained by using the parity operation to exchange spatial coordinates in the scalar product of a locally commuting field operator, evaluated at position $\mathbf{x}$, with the same field operator evaluated at $-\mathbf{x}$, at equal times.

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1. Introduction

Proofs of the spin–statistics theorem tend, broadly speaking, to fall into two classes. The first class, historically, depends upon analytic properties of field operator commutators [1–4]. The second class invokes topological arguments. Proofs in this latter class variously use homotopies in configuration space for identical particles [5–9] or arguments involving adiabatic exchange of particles carrying topological markers [10,11]. The proof by Schwinger [12] stands apart from both classes in exploiting the action of time-reversal on the Lagrangian density of a field.

The use, on the one hand, of identical particle exchange in the topological theorems, and, on the other, of a discrete symmetry applied to a scalar invariant in Schwinger's proof, suggests using another discrete symmetry, parity, to examine the effect of exchanging particle coordinates by passive transformations. This note presents a simple demonstration of the spin–statistics connection based upon that idea. The proof uses elementary parity and angular momentum properties of quantum fields only and is, in essence, algebraic.

2. Parity and causal fields

Irreducible representations of the Poincaré group are classified according to eigenvalues of two angular momentum-like infinitesimal generators $\mathbf{A}$ and $\mathbf{B}$ [1,4,13–15]. The (A, B) representation contains multiple spin angular momentum quantum numbers $|A-B| \le j \le A+B$. General fields are built up from the (A,B) representations. Familiar examples include the $(0,0)$ scalar field, and the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ Dirac field.

Let a spin j massive field $\Psi^{(AB)}$ be an element of a given (A,B) representation. The construction of this object is given in refs [14,16]. Applying the parity operation P gives [13]

$$P\Psi_{ab}^{(AB)}(\mathbf{x},t)P^{-1} = \eta_P(-1)^{A+B-j}\Psi_{ba}^{(BA)}(-\mathbf{x},t). \tag{1}$$

The intrinsic parity η_P of the field is ± 1. The action of P has no effect on spin, and assumes nothing regarding statistics.

3. Spin and statistics: Weinberg fields

Before considering the general case, the method of proof is worked out for the simpler case of $(j,0)$ representations, sometimes called Weinberg fields [17]. Define the field operator

$$\xi_\sigma \equiv \Psi_\sigma^{(j0)}, \tag{2}$$

where σ runs from $-j$ to j. The field $\xi_\sigma(x)$ annihilates a spin j particle (or creates an antiparticle) localized at space-time point x, with z-projection of angular momentum σ.

It will be shown that imposing local commutativity on ξ leads to the spin–statistics connection. Consider the field ξ evaluated at two points in space-time separated by space-like interval. A Lorentz frame exists in which the two points occur at equal time. So we may write the fields as $\xi(\mathbf{x},t)$ and $\xi(-\mathbf{x},t)$. The effect of P on their scalar product is, according to eq. (1) for $A=B=0$,

$$\begin{aligned} P\xi(\mathbf{x},t)\cdot\xi(-\mathbf{x},t)P^{-1} &= P\xi(\mathbf{x},t)P^{-1}\cdot P\xi(-\mathbf{x},t)P^{-1} \\ &= \xi(-\mathbf{x},t)\cdot\xi(\mathbf{x},t). \end{aligned} \tag{3}$$

The quantity in eq. (3) is the product of two terms with the same parity, and thus has even parity. Considered as a function of $\mathbf{x}$, an even parity scalar operator obeys $Pf(\mathbf{x})P^{-1} = f(\mathbf{x})$, and thus

$$\xi(\mathbf{x},t)\cdot\xi(-\mathbf{x},t) = \xi(-\mathbf{x},t)\cdot\xi(\mathbf{x},t). \tag{4}$$

The product on the right-hand side of eq. (4) is the scalar product of two irreducible spherical tensors of the same rank. It is given by [18,19]

$$\xi(-\mathbf{x},t)\cdot\xi(\mathbf{x},t) = \sum_\sigma (-1)^\sigma \xi_\sigma(-\mathbf{x},t)\xi_{-\sigma}(\mathbf{x},t). \tag{5}$$

By hypothesis, commutation relations of a causal field ($-$ for Bose, $+$ for Fermi) vanish outside the light cone; in particular [20]

$$[\xi_\sigma(\mathbf{x},t), \xi_\lambda(-\mathbf{x},t)]_\mp = 0. \tag{6}$$

Therefore,

$$\xi(-\mathbf{x},t)\cdot\xi(\mathbf{x},t)=\pm\sum_{\sigma}(-1)^{\sigma}\xi_{-\sigma}(\mathbf{x},t)\xi_{\sigma}(-\mathbf{x},t), \tag{7}$$

as the fields are Bose or Fermi. Upon inverting the order of summation by replacing σ with $-\sigma'$,

$$\xi(-\mathbf{x},t)\cdot\xi(\mathbf{x},t)=\pm\sum_{\sigma'}(-1)^{-\sigma'}\xi_{\sigma'}(\mathbf{x},t)\xi_{-\sigma'}(-\mathbf{x},t), \tag{8}$$

and noting

$$(-1)^{-\sigma'}=\begin{cases}(-1)^{\sigma'} & \text{integer } j\\ -(-1)^{\sigma'} & \text{half-integer } j\end{cases}=(-1)^{2j}(-1)^{\sigma'}, \tag{9}$$

we obtain for eq. (4)

$$\xi(\mathbf{x},t)\cdot\xi(-\mathbf{x},t)=\pm(-1)^{2j}\xi(\mathbf{x},t)\cdot\xi(-\mathbf{x},t). \tag{10}$$

Take the matrix element of both sides of eq. (10) between the vacuum and a state with one quantum of the field ξ localized at $\mathbf{x}$ with z-value of its spin equal to ρ and one quantum at $-\mathbf{x}$, with spin z-value $-\rho$. Equation (10) becomes

$$\begin{aligned}&\langle VAC|\xi_{\rho}(\mathbf{x},t)\xi_{-\rho}(-\mathbf{x},t)|(+\mathbf{x},t;+\rho)(-\mathbf{x},t;-\rho)\rangle\\ &=\pm(-1)^{2j}\langle VAC|\xi_{\rho}(\mathbf{x},t)\xi_{-\rho}(-\mathbf{x},t)|(+\mathbf{x},t;+\rho)(-\mathbf{x},t;-\rho)\rangle.\end{aligned} \tag{11}$$

By hypothesis, a value of ρ exists for which the matrix element is nonvanishing, allowing us to conclude

$$1=\pm(-1)^{2j}, \tag{12}$$

which is the connection between spin and statistics.

4. Spin and statistics: General fields

The argument just given is readily extended to the case of the general (A,B) representation. The field $\xi_{mn}^{(AB)}$ now carries two indices $-A\le m\le A$ and $-B\le n\le B$, and the scalar product in eq. (5) is replaced by an expresson that couples two (A,B) spherical tensors to a $(0,0)$ scalar, in an extension of Racah's [19] original derivation of eq. (5), which now becomes (retaining the dot product notation)

$$\begin{aligned}&\sum_{m,n}\begin{pmatrix}A & A & 0\\ -m & m & 0\end{pmatrix}\begin{pmatrix}B & B & 0\\ -n & n & 0\end{pmatrix}\xi_{mn}(-\mathbf{x},t)\xi_{-m-n}(\mathbf{x},t)\\ &\qquad\propto\sum_{m,n}(-1)^{\sigma}\xi_{mn}(-\mathbf{x},t)\xi_{-m-n}(\mathbf{x},t)\\ &\qquad\equiv\xi(-\mathbf{x},t)\cdot\xi(\mathbf{x},t),\end{aligned} \tag{13}$$

where $\sigma = m + n$, and the objects in parentheses are Wigner 3j symbols. By eq. (1) for the (0,0) representation, the result of applying P to (13) once again gives eq. (4). Both the spin j and summation index σ are half-integral if and only if one of A and B is half-integral. Therefore, eq. (10) holds for the general (A, B) representation, and taking the matrix element of eq. (10) between the vacuum and a suitable state $|(\mathbf{x}, t; \mu, \nu)(-\mathbf{x}, t; -\mu, -\nu)\rangle$ gives, again, the proper spin–statistics connection.

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