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Local shell-to-shell energy transfer via nonlocal interactions in fluid turbulence

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Abstract. In this paper we analytically compute the strength of nonlinear interactions in a triad, and the energy exchanges between wave-number shells in incompressible fluid turbulence. The computation has been done using first-order perturbative field theory. In three dimensions, magnitude of triad interactions is large for nonlocal triads, and small for local triads. However, the shell-to-shell energy transfer rate is found to be local and forward. This result is due to the fact that the nonlocal triads occupy much less Fourier space volume than the local ones. The analytical results on three-dimensional shell-to-shell energy transfer match with their numerical counterparts. In two-dimensional turbulence, the energy transfer rates to the nearby shells are forward, but to the distant shells are backward; the cumulative effect is an inverse cascade of energy.

Keywords. Homogeneous and isotropic turbulence; local transfer.

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1. Introduction

Many equations in physics, e.g., Schrödinger equation and diffusion equation, are local in real space. Here, to time-advance a variable at a point, we need the values of the variables and their finite-order derivatives at the same point. It is well-known that incompressible Navier–Stokes (NS) equation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u}, \quad (1)$$

$$\nabla\cdot\mathbf{u} = 0, \quad (2)$$

is *nonlocal in real space* [1,2]. Here $\mathbf{u}$ and p are the velocity and pressure fields respectively, and ν is the kinematic viscosity. The nonlocality is due to the pressure term of eq. (1), which is obtained by taking the divergence of incompressible Navier–Stokes (NS) equation [2]

$$\nabla^2 p = -\nabla \cdot \{\mathbf{u} \cdot \nabla \mathbf{u}\}\,. \tag{3}$$

Hence,

$$p(\mathbf{x},t) = -\int \frac{\nabla' \cdot \{\mathbf{u}(\mathbf{x}',t) \cdot \nabla' \mathbf{u}(\mathbf{x}',t)\}}{|\mathbf{x}-\mathbf{x}'|}, \tag{4}$$

which is nonlocal because $p(\mathbf{x},t)$ depends on the velocity field at $\mathbf{x}' \neq \mathbf{x}$.

In Fourier space, incompressible NS equation is

$$\frac{\partial u_i(\mathbf{k})}{\partial t} + \nu k^2 u_i(\mathbf{k}) = -\frac{i}{2} P_{ijm}(\mathbf{k}) \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d} u_j(\mathbf{p}) u_m(\mathbf{q}), \tag{5}$$

$$k_i u_i(\mathbf{k}) = 0, \tag{6}$$

where

$$P_{ijm}(\mathbf{k}) = k_j \left(\delta_{im} - \frac{k_i k_m}{k^2}\right) + k_m \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right),$$

$\mathbf{k} = \mathbf{p} + \mathbf{q}$, and d is the space dimensionality. Note that the factor $-k_i k_j k_m / k^2$ of $P_{ijm}(\mathbf{k})$ is due to the pressure term. To determine $\mathbf{u}(\mathbf{k}, t + \mathrm{d}t)$ we need the values of field $u_i(\mathbf{p})$ where $\mathbf{k} - \mathbf{p}$ could be quite large. Hence, incompressible NS equation is *nonlocal in Fourier space also* [2,3]. The basic unit of nonlinear interactions in turbulence, called triad interactions, involve three vectors $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ with $\mathbf{k} = \mathbf{p} + \mathbf{q}$. These triad interactions yield energy transfers among the participating modes. The strength of triad interaction is measured using energy exchanges in the triad.

The energy transfers between two wave-number shells can be computed using the triad interactions. One of the key ingredient of Kolmogorov's phenomenology of turbulence is 'local' shell-to-shell energy transfer. That is, maximum energy is transferred from a wave-number shell to the next wave-number shell. This observation has been verified in numerical simulations [4,5]. This result is surprising in view of nonlocal interactions in both real and Fourier space. In this paper, we compute the strength of triad interaction, and shell-to-shell energy transfer in incompressible fluid turbulence using field-theoretic technique.

There have been many attempts in the past to compute the strength of triad interactions and the energy transfers in fluid turbulence. Kraichnan [6] computed these quantities in both 2D and 3D turbulence using 'almost Markovian Galilean invariant' turbulence model. He showed that in 3D, 35% of the total energy transfer across a unit wave-number sphere involves triads in which the smallest wave number is more than one-half of the middle wave number. Hence, shell-to-shell energy transfer in 3D turbulence is local in wave-number space. Later, Domaradzki and Rogallo [5] numerically computed the above quantities and observed that energy transfers in shells were always local, but the triad interactions were nonlocal, i.e., triads having three wave numbers of very different magnitudes had large magnitudes. This is succinctly described by Domaradzki and Rogallo as 'nonlocal interactions and local energy transfer'. Domaradzki and Rogallo found their numerical results to be in excellent agreement with their own eddy-damped quasi-normal Markovian (EDQNM) calculation. They conjectured that the observed energy transfer was caused by triads with at least one wave number in the energy-containing range. Ohkitani and Kida [7] analyzed the triad interactions carefully and concluded that the nonlocal interaction was strong, but the energy exchange occurred predominantly between comparable scales. They claimed that the third mode of much larger scale was indifferent to the energy transfer as if it were a catalizer in a chemical reaction. Zhou [4] numerically computed the energy transfers using different wave-number summation scheme, and found the energy transfers to be local. Waleffe [8] did a similar analysis using a decomposition of the velocity field in terms of helical modes. Kishida *et al* [9] used wavelet basis to address the same problem and obtained similar results. For a review on this topic, refer to Zhou and Speziale [10]. In the present paper we re-look at some of the above conjectures.

In all the above papers, turbulent interactions are measured using a function $S(k|p,q)$ (usually called transfer function) that denotes the sum of energy transfers from mode $\mathbf{p}$ and $\mathbf{q}$ to mode $\mathbf{k}$ [11]. Dar *et al* [12,13] pointed out that the energy transfer from one shell to another shell cannot be accurately computed using $S(k|p,q)$, essentially because the third mode of the interaction could lie outside both the shells under consideration. To overcome this difficulty, Dar *et al* [12] modified the above formulation. They used a new function $S(k|p|q)$, called *mode-to-mode energy transfer rate*, for the energy transfer from mode $\mathbf{p}$ to mode $\mathbf{k}$, with mode $\mathbf{q}$ acting as a mediator, and showed that the shell-to-shell energy transfer can be correctly computed by this formalism. The computation of $S(k|p|q)$ is done either numerically or using field-theoretic methods. Kishida *et al* [9] used similar formalism as Dar *et al* [12] for wavelets and numerically computed the shell-to-shell energy transfer rates. In the present paper we quantify the triad interactions using $S(k|p|q)$, and compute them using field-theoretic method. We also calculate the energy transfer rates between wave-number shells using a first-order perturbation theory. Our analytic arguments justify Domaradzki and Rogallo's [5], Zhou's [4], and Ohkitani and Kida's [7] numerical results that the turbulent interaction is nonlocal, but the shell-to-shell energy transfer is local.

In this paper we also compute mode-to-mode energy transfer rate $S(k|p|q)$ for space dimension other than 2. For $d = 2$, $S(k|p|q) < 0$ for most of $p < k$, unlike 3D case. This property of $S(k|p|q)$ is the reason for the inverse energy cascade. We find that the transition from backward to forward energy transfer takes place at $d_c = 2.25$.

Our paper is organized as follows: in §2 we compute $S(k|p|q)$ in the inertial range using first-order field theory. It is shown that nonlinear interactions in incompressible NS are nonlocal. The nature of $S(k|p|q)$ for 2D and 3D are contrasted. Section 3 contains estimates of the shell-to-shell energy transfer for neighbouring and distant shells; since the maximal energy transfer takes place between neighbouring shells, the shell-to-shell energy transfer is said to be local. In §§4 and 5 we compute shell-to-shell energy transfer rates in 3D and 2D respectively. Section 4 also contains a comparison of analytical results with their numerical counterparts. Section 6 contains an elementary discussion on the energy transfer rates in Burgers turbulence. Section 7 contains conclusions.

2. Nonlocal interactions in incompressible fluid turbulence

Kraichnan [6] has computed magnitudes of triad interactions using transfer function $S(k'|p,q)$ ($\mathbf{k}'+\mathbf{p}+\mathbf{q}=0$). In the following discussion we will compute the strength of triad interaction using Dar *et al*'s mode-to-mode energy transfer rate $S(k'|p|q)$ [12]

$$S(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\Im\left([\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})]\,[\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})]\right) \tag{7}$$

that represents energy transfer from mode $\mathbf{p}$ to mode $\mathbf{k}$, with mode $\mathbf{q}$ acting as a mediator. Here $\Im$ represents the imaginary part of the argument. Here we compute the ensemble average of S, $\langle S(\mathbf{k}'|\mathbf{p}|\mathbf{q})\rangle$, using the standard field-theoretic technique [11,14,15]. We expand the $\langle S(\mathbf{k}'|\mathbf{p}|\mathbf{q})\rangle$ (eq. (7)) to first order in perturbation (see [13,16,17] for details). We assume the flow to be homogeneous and isotropic.

Following the standard field-theoretic procedure, we perform average of $S(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ and obtain an expression for the energy transfer rate from mode $\mathbf{p}$ to mode $\mathbf{k}$, with mode $\mathbf{q}$ as a mediator

$$\langle S(k'|p|q)\rangle = \frac{T_1(k,p,q)C(p)C(q) + T_2(k,p,q)C(k)C(q) + T_3(k,p,q)C(k)C(p)}{\nu(k)k^2+\nu(p)p^2+\nu(q)q^2}, \tag{8}$$

where $C(k)$ is the equal-time correlation function, and $\nu(k)$ is the effective viscosity. The functions $T_i(k,p,q)$ are given by

$$T_1(k,p,q) = kp\left((d-3)z + (d-2)xy + 2z^3 + 2xyz^2 + x^2z\right), \tag{9}$$

$$T_2(k,p,q) = -kp\left((d-3)z + (d-2)xy + 2z^3 + 2xyz^2 + y^2z\right), \tag{10}$$

$$T_3(k,p,q) = -kq\left(xz - 2xy^2z - yz^2\right), \tag{11}$$

where d is the space dimensionality, and x, y, z are the cosines of angles between $(\mathbf{p},\mathbf{q})$, $(\mathbf{k},\mathbf{q})$, and $(\mathbf{k},\mathbf{p})$ respectively.

We take Kolmogorov's spectrum for the correlation function, i.e.,

$$C(k) = \frac{2\,(2\pi)^d}{S_d(d-1)}\frac{K_{\mathrm{Ko}}\,|\Pi|^{2/3}\,k^{-5/3}}{k^{d-1}}, \tag{12}$$

and renormalized viscosity of McComb and Watt [18] for $\nu(k)$ to be

$$\nu(k) = \sqrt{K_{\mathrm{Ko}}}\nu^*\,|\Pi|^{1/3}\,k^{-4/3}, \tag{13}$$

where K_{Ko} is the Kolmogorov's constant, Π is the energy flux, and ν^* is a constant related to the renormalized viscosity [18,19]. Note that in 2D fluid turbulence, Π is negative for the wave-number region with 5/3 spectral index. McComb and Watt [18] and Verma [19] have computed ν^* using renormalization technique. Here we take $K_{\mathrm{Ko}} = 1.6$, $\nu^* = 0.38$ for 3D, and $K_{\mathrm{Ko}} = 6.3$, $\nu^* = -0.6$ for 2D.

The interactions are self-similar in the inertial range, which is the region of our interest. Therefore, it is sufficient to analyze $S(k'|p|q)$ for triangles $(1, p/k, q/k) =$

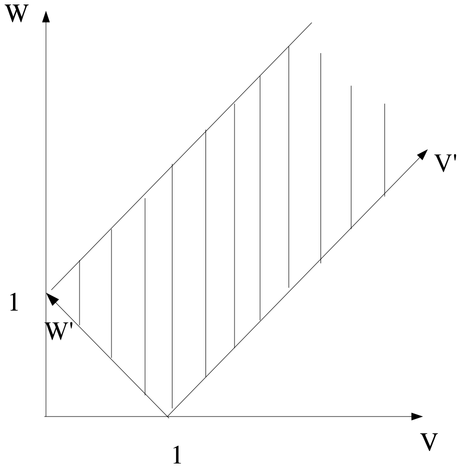

Figure 1. The interacting triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})/k = (1, v, w)$ under the condition $\mathbf{k} = \mathbf{p} + \mathbf{q}$ is represented by a point (v, w) in the hatched region. The axis (v', w') are inclined to axis (v, w) by 45°. Note that the local wave numbers are $v \approx 1, w \approx 1$ or $v' \approx w' \approx 1/\sqrt{2}$.

$(1, v, w)$. Since, $|k - p| \le q \le k + p$, $|1 - v| \le w \le 1 + v$, any interacting triad $(1, v, w)$ is represented by a point (v, w) in the hatched region of figure 1 [11].

The local wave numbers are $v \approx 1, w \approx 1$, while the rest are called nonlocal wave numbers. We substitute $C(k)$ and $\nu(k)$ in eq. (8), which yields

$$\langle S(v,w)\rangle = \left[\frac{4(2\pi)^{2d} K_{\mathrm{Ko}}^{-3/2}\Pi}{S_d^2 (d-1)^2 k^{2d}\nu^*}\right] * \frac{t_1(v,w)(vw)^{-5/3-(d-1)} + t_2(v,w)w^{-5/3-(d-1)} + t_3(v,w)v^{-5/3-(d-1)}}{1 + v^{2/3} + w^{2/3}} \tag{14}$$

where $t_i(v, w) = T_i(k, p, q)/k^2$. For convenience, $\langle S(v', w')\rangle$ are represented in terms of new variables (v', w') measured from the rotated axis shown in figure 1. It is easy to show that $v = 1 + (v' - w')/\sqrt{2}, w = (v' + w')/\sqrt{2}$.

Figure 2 illustrates density plots of $\langle S(v', w')\rangle$ without the bracketed factor. Figure 2a shows the plot for 3D, while figure 2b shows the one for 2D. Note that $\langle S(v'(v, w), w'(v, w))\rangle$ is the energy transferred from mode $p = v$ to mode $k = 1$. In the white region (positive), energy is transferred from mode p to mode k, while in the dark regions (negative), mode p receives energy from mode k. The value of S at $(v, w) = (1, 1)$, or $(v', w') = (1/\sqrt{2}, 1/\sqrt{2})$ is zero in both 2D and 3D.

In 3D the triads with $v \approx 0, w \to 1$ ($v' \approx 0, w' \approx \sqrt{2}$: the top-left corner in v'–w' plot) have large and positive $\langle S\rangle$, implying that the large wavelength modes give a large amount of energy to the modes near $k \approx 1$. These observations prove that the nonlinear interactions in incompressible NS *are nonlocal in Fourier space.* In 2D, the triad with $v \approx 0$ have large negative $\langle S\rangle$ implying that the large wavelength modes take energy from the modes near $k \approx 1$. These observations indicate that

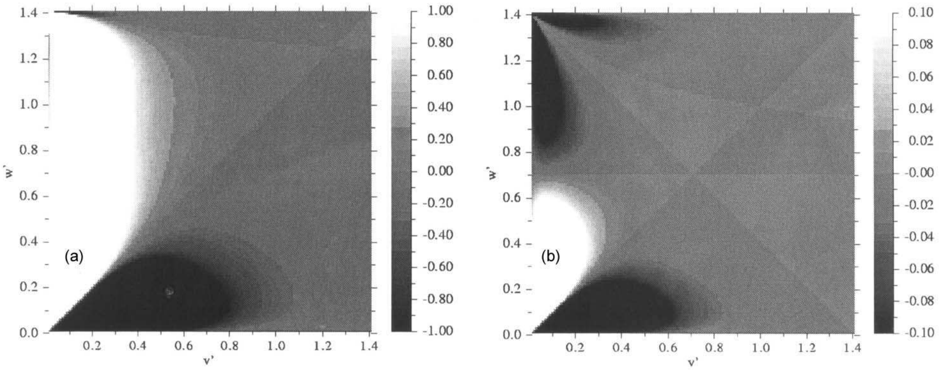

Figure 2. Density plot of $\langle S(v', w') \rangle$ of eq. (8) without the bracketed factor for (**a**) 3D and (**b**) 2D.

the interactions in 2D turbulence are nonlocal as well, but the large wavelength modes are the sink of energy.

Another common behaviour in both the dimensions is for $w \to 0, v \approx 1$ ($v' \approx 0, w' \approx 0$, bottom-left corner in v'–w' plot). Here $\langle S \rangle \gg 0$ for $v < 1$, but $\langle S \rangle \ll 0$ for $v > 1$. This implies that for these types of triads, the p modes with magnitudes less than k always give energy to the k modes, while the p modes with $p > k$ always take energy from the k modes. When p, q are much larger than k ($v, w \to \infty$), $\langle S \rangle$ is small, implying that they interact weakly with $k \approx 1$. In 3D, $\langle S \rangle$ for most of these modes are negative implying that they receive energy from $k = 1$. In 2D, however, $\langle S \rangle$ for a large fraction of these triads are positive; hence they supply energy to $k = 1$.

The energy cascade is backward in 2D. This is due to the above-mentioned backward energy transfer from $k = 1$ mode to the smaller wave-number modes ($\langle S(v \approx 0, w \approx 1) \rangle < 0$), and backward energy transfer from large v, w modes to $k = 1$ mode ($\langle S(v, w \gg 1) \rangle > 0$). It is interesting to contrast this behaviour with 3D case where $\langle S(v, w) \rangle$ is somewhat opposite to 2D case.

The function $\langle S(v, w) \rangle$ in the region with $v \to 0$ is primarily positive for $d = 3$, but is negative for $d = 2$. The transition of negative $\langle S \rangle$ to positive $\langle S \rangle$ for the region with $v \to 0$ occurs near $d_c = 2.25$. Please refer to figure 3a for the illustration. It can be shown using field-theoretic calculation that the renormalized viscosity vanishes near $d_c = 2.25$, and the direction of energy cascade changes from negative to positive at $d = d_c$. Fournier and Frisch [20] report $d_c = 2.05$ which differs a bit from our d_c. The difference could be because of the fact that Fournier and Frisch [20] use combined energy transfer $S(k|p, q)$ for their EDQNM calculation. In figure 3b we also show the density plot of $\langle S(v, w) \rangle$ for $d = 100$. This is a representative illustration for large-space dimensions.

The function $\langle S(v, w) \rangle$ can be estimated in the limiting cases using the method given in Appendix of Leslie [11]. In the nonlocal region with $v \to 0$ (naturally, $w \approx 1$),

$$\langle S(v, w) \rangle \propto v^{\frac{4}{3} - d}. \tag{15}$$

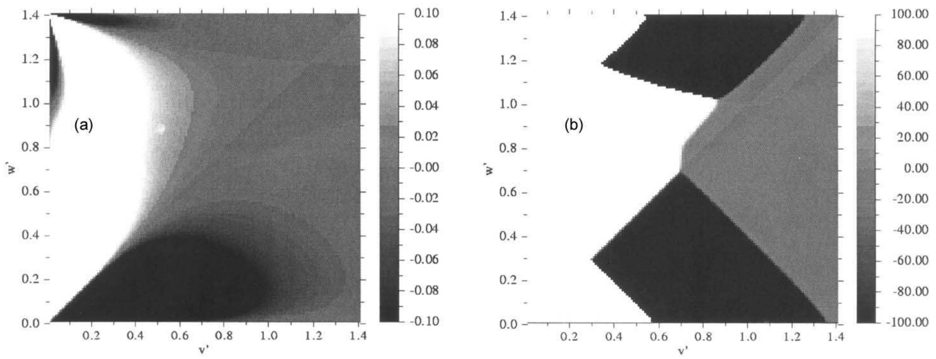

Figure 3. Density plot of $\langle S(v', w')\rangle$ of eq. (8) for (**a**) $d = 2.25$ and (**b**) $d = 100$.

Clearly, $S(v, w) \to \infty$ for both 2D and 3D. This observation is consistent with our earlier observation that interactions are nonlocal. However, when $v \approx w \approx 1$, we find that

$$\langle S(v,w)\rangle \propto \begin{cases} -(v-1) & \text{for } d=3 \\ \frac{8}{3}\left[(v-1)^2 - \frac{1}{2}(w-1)^2 - \frac{1}{2}(v-1)(w-1)\right] & \text{for } d=2 \end{cases}. \tag{16}$$

This result shows that the interactions within the local triads ($v \approx 1, w \approx 1$) are weak. In 3D, among these triads, the ones with $v < 1$ or $p < k$ have $\langle S\rangle > 0$, and hence the energy transfer is from mode p to mode k. The sign of $\langle S\rangle$, consequently the sign of energy transfer, is reversed for the modes with $v > 1$. However in 2D, S is somewhat complex in the neighbourhood of $v = w = 1$.

For $v \approx 1$ and $w \to 0$,

$$\langle S(v,w)\rangle \propto w^{\frac{1}{3}-d}, \tag{17}$$

which again diverges. When $v, w \to \infty$,

$$\langle S(v,w)\rangle \propto v^{-\frac{4}{3}-d}$$

implying that interactions with large wave-number modes are weak.

The above estimates are consistent with the graphical plots shown in figures 2 and 3. After this discussion, we move on to compute the shell-to-shell energy transfer rates in fluid turbulence.

3. Local shell-to shell energy transfer in incompressible fluid turbulence

The wave-number space is divided into shells $(k_0 s^n, k_0 s^{n+1})$, where $s > 1$, and n can take both positive and negative values. The energy transfer rate from mth shell $(k_0 s^m, k_0 s^{m+1})$ to nth shell $(k_0 s^n, k_0 s^{n+1})$ is given by [12]

$$T_{nm} = \sum_{k_0 s^n \le k \le k_0 s^{n+1}} \sum_{k_0 s^m \le p \le k_0 s^{m+1}} \langle S(k|p|q) \rangle . \quad (18)$$

If the shell-to-shell energy transfer rate is maximum for the nearest neighbours, and decreases monotonically with the increase of $|n-m|$, then the shell-to-shell energy transfer is said to be local.

If the amplitudes of the Fourier modes $\mathbf{u}(\mathbf{k})$ are available, either from experiments and/or from numerical simulations, then we can easily compute the shell-to-shell energy transfer rates using eqs (7) and (18). In this paper, we compute the energy transfer rates between the wave-number shells to first order in perturbation [11,13] that yields

$$T_{nm} = \int_{k_0 s^n \le k \le k_0 s^{n+1}} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \int_{k_0 s^m \le p \le k_0 s^{m+1}} \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d} \times \frac{T_1(k,p,q)C(p)C(q) + T_2(k,p,q)C(k)C(q) + T_3(k,p,q)C(k)C(p)}{\nu(k)k^2 + \nu(p)p^2 + \nu(q)q^2} . \quad (19)$$

We nondimensionalize the above equations using [11]

$$k = \frac{a}{u}; \quad p = \frac{a}{u}v; \quad q = \frac{a}{u}w, \quad (20)$$

where a is an arbitrary constant wave number. For our calculation we choose $a = k_0 s^{n-1}$. Three-dimensional integral under the constraint that $\mathbf{k}' + \mathbf{p} + \mathbf{q} = 0$ is given by [11]

$$\int_{\mathbf{p}+\mathbf{q}+\mathbf{k}=0} \mathrm{d}\mathbf{p} = S_{d-1} \int \mathrm{d}p \mathrm{d}q \left(\frac{pq}{k}\right)^{d-2} (\sin\alpha)^{d-3} . \quad (21)$$

Using these substitutions, we obtain

$$\frac{T_{nm}}{|\Pi|} = K_u^{3/2} \frac{4S_{d-1}}{(d-1)^2 S_d \nu^*} \int_{s^{-1}}^{1} \frac{\mathrm{d}u}{u} \int_{us^{m-n}}^{us^{m-n+1}} \mathrm{d}v \times \int_{|1-v|}^{1+v} \mathrm{d}w \, (vw)^{d-2} (\sin\alpha)^{d-3} F(v,w), \quad (22)$$

where $F(v,w)$ is given by

$$F(v,w) = \frac{t_1(v,w)(vw)^{-\frac{2}{3}-d} + t_2(v,w)w^{-\frac{2}{3}-d} + t_3(v,w)v^{-\frac{2}{3}-d}}{(1+v^{2/3}+w^{2/3})} \quad (23)$$

with $t_i(v,w) = T_i(k,kv,kw)/k^2$. Equation (22) provides us with the shell-to-shell energy transfer rates relative to energy flux Π. Clearly, T_{nm} depends only on $n-m$, or $T_{nm} = T_{n-i,m-i}$ where i is an integer. Hence, the shell-to-shell energy transfer is self-similar.

Now let us estimate the shell-to-shell energy transfer rates when $m \ll n$. The triads $(1,v,w)$ with $v \to 0$ participate in this energy transfer. As seen in the previous section, for these triads $S(v,w) \propto w^{(4/3)-d}$. Therefore,

$$T_{nm} \sim \int \frac{\mathrm{d}u}{u} \int \mathrm{d}v v^{d-2} v^{\frac{4}{3}-d} \int_{|1-v|}^{1+v} \mathrm{d}w w^{d-2}$$
$$\sim v_0^{4/3},$$

where v_0 is the outer radius of the shell m. Since $v_0 \to 0$, T_{nm} vanishes. Hence, we are able to show that shell-to-shell energy transfer between distant shells is negligible in spite of large interactions between distant wave numbers. This result is essentially due to the small volume of v or mth shell.

We can also compute the shell-to-shell energy transfer rates for close-by shells. Here the participating triads will satisfy $v \approx w \approx 1$. As shown in the earlier section, $S(v,w) \approx -(v-1)$ for these triads. Therefore,

$$T_{nm} \propto \int \frac{\mathrm{d}u}{u} \int \mathrm{d}v \int \mathrm{d}w (vw)(v-1), \tag{24}$$

with $n \approx m$. Since our bins are uniform in logarithmic scale, the volume of wave-number shells is of the order of 1 when $v \approx w \approx 1$. Also the range of $v-1$ is of the order of v, which is close to 1. Therefore T_{nm} will be finite. Hence, the shell-to-shell energy transfer rates between close-by shells is finite. This result is consistent with the local energy transfer assumption of Kolmogorov.

For close-by shells, $p \approx k$, but q can take any value from $|k-p|$ to $k+p$. However, it can be easily shown that mode p in the triads with $q \ll k$ do not contribute significantly to the shell-to-shell energy transfer. Since, $S(v,w) \propto w^{\frac{1}{3}-d}$ for $w \to 0$,

$$T_{nm} \sim \int \frac{\mathrm{d}u}{u} \int \mathrm{d}\mu \mu \int \mathrm{d}w w^{d-2} w^{\frac{1}{3}-d} w \tag{25}$$
$$\sim w_0^{1/3}, \tag{26}$$

where $v = 1 + \mu w$, and $\mathrm{d}v = \mu \mathrm{d}w$. Clearly the above integral goes to zero. That is, the shell-to-shell energy transfer between close-by shells receive insignificant contribution from the triads satisfying $q \ll p \approx k$.

In this section, using the limiting values of $S(k|p|q)$, we showed that the shell-to-shell energy transfer is local in spite of nonlocal interactions among wave-number modes. However, T_{nm} can be easily computed for the shells in the inertial range. In the next section we will do these calculations.

4. Computation of shell-to-shell energy transfer in 3D

The shell-to-shell energy transfer rates have been calculated earlier by Ohkitani and Kida [7], and Zhou and Speziale [10] using numerical simulation and EDQNM approximation with $S(k|p,q)$. In this section we compute the shell-to-shell normalized energy transfer rates T_{nm}/Π in three dimensions using eq. (22). We take $s = 2^{1/4}$. The integration has been done numerically using Gauss-quadrature method. The constants $\nu^* = 0.38$ and $K = 1.6$ have been taken from McComb and Watt [18] and Verma [13,19]. The shell-to-shell energy transfer is self-similar, i.e., T_{nm} is function

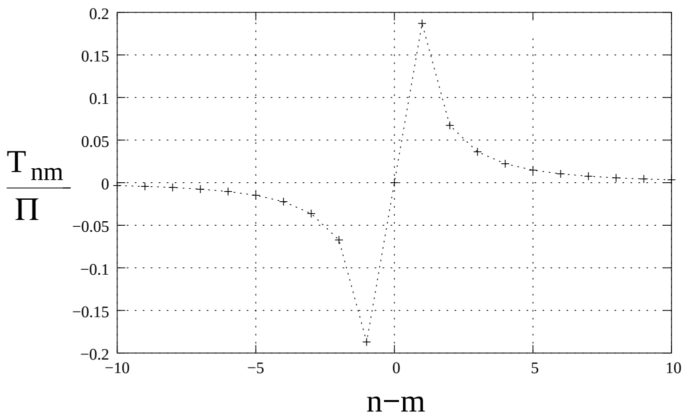

Figure 4. Plot of normalized shell-to-shell energy transfer T_{nm}/Π vs. $n-m$ for $d=3$. The nth shell is $(k_0 s^n, k_0 s^{n+1})$ with $s = 2^{1/4}$. The energy transfer is maximum for $n = m \pm 1$, hence the energy transfer is local. The energy transfer is also forward.

of $n-m$. Therefore we compute T_{nm}/Π for various $n-m$. Figure 4 contains this plot. Note that the shells m and n have been assumed to be inside the inertial range.

From figure 4 we can infer that the transfer rates T_{nm} in the inertial range are negative for $n < m$, and positive for $n > m$. Hence a shell gains energy from the smaller wave-number shells, and loses energy to the higher wave-number shells. This means that the energy cascades from the smaller wave numbers to the higher wave numbers (forward). The most significant energy transfer takes place from m to $m+1$. Hence, the shell-to-shell energy transfer is forward and local, which is consistent with Kolmogorov's picture of turbulence. Note that the energy transfer is local in spite of nonlocal triad interactions.

To validate our theoretical calculations, we have also computed the shell-to-shell energy transfer rates using the data from Direct Numerical Simulation on a 512^3 grid. The computation was performed when the turbulence was well-developed. The Reynold's number using Taylor's microscale was 64.8, and skewness was -0.54. We divide the wave-number space into 15 shells with boundaries at wave numbers 2,4,8,11.3,13.5,16,19,22.6,26.9,32,38.1,64,76.1,108,128,256. In the inertial range ($k \approx$ 10–35), the shell boundaries are $k_n = 2^{(n+11)/4}$. Please refer to Dar *et al* [12] for details on numerical procedure. Figure 5 shows the plots of T_{nm}/Π vs. $n-m$ for $m = 4, ..., 9$, which are in the inertial range shells. The plots show self-similarity, local, and forward energy transfer for the inertial range shells. The numerical and theoretical values are in close agreement.

For thicker shells $s = 2^{1/2}$, the ratio of the smallest to largest wave number of the triad is $2\sqrt{2}$. For these shells, energy transfer to the nearest neighbouring shell is close to 35%. These numbers are consistent with Kraichnan's [6] and Zhou's [4] results.

In the next section we will discuss shell-to-shell energy transfer in 2D turbulence.

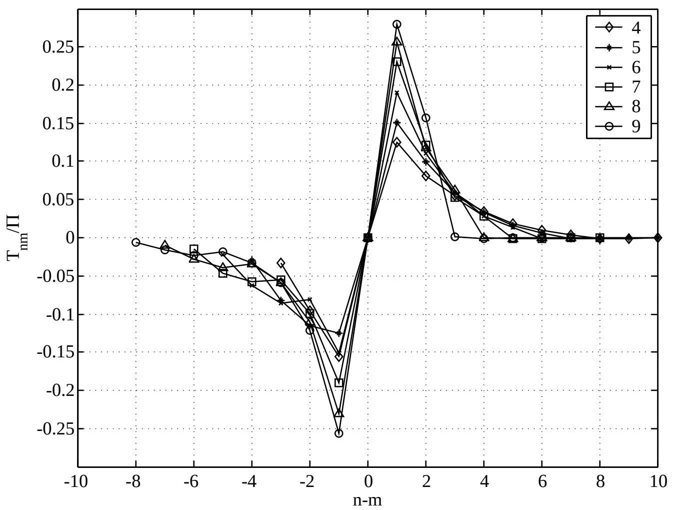

Figure 5. Plots of normalized shell-to-shell energy transfer T_{nm}/Π vs. $n-m$ for m from 4 to 9. The plots collapse on each other indicating self-similarity.

5. Shell-to-shell energy transfer in 2D fluid turbulence

We compute the shell-to-shell energy transfer in 2D following the same procedure as given above. The wave-number range considered is in the inverse cascade regime ($E(k) \propto k^{-5/3}$). We take $\nu_* = -0.6$ and $K = 6.3$ [6,13]. As shown in figure 6, the energy transfer rates from the shell m to the three neighbouring shells ($m+1, m+2, m+3$) are *forward*, and the transfers are negative for all shells $n > m+3$. The above result is very similar to Dar *et al* [12]'s numerical finding on 2D MHD turbulence (figure 10 of Dar *et al* [12]). The negative energy transfer from the distant shells are due to negative $\langle S(v,w)\rangle$ for $v \to 0$ (top-left of figure 2b), and positive $\langle S(v,w)\rangle$ for $v,w \gg 1$. The negative $\langle S(v,w)\rangle$ for $v \to 0$ indicate that large-wavelength modes receive energy from mode $k = 1$, and positive $\langle S(v,w)\rangle$ for $v,w \gg 1$ indicate that small-wavelength modes give energy to mode $k = 1$. A careful inspection of figure 2b indicates that the forward energy transfer to shells ($m+1, m+2, m+3$) is due to a narrow region near $v = w = 1$, or $v' = w' = 1/\sqrt{2}$, where energy transfer is from lower wave number to higher wave number.

The above results on shell-to-shell energy transfer is consistent with the energy flux picture. Note that

$$\Pi = \sum_{n=m+1}^{\infty} (n-m) T_{nm}.$$

When we perform $\sum_{n=m+1}^{\infty}(n-m)(T_{nm}/|\Pi|)$, we obtain -1, consistent with the inverse cascade of energy in 2D turbulence.

To summarize, in 2D fluid turbulence, the shell-to-shell energy transfer to the neighbouring shells is forward, but the energy transfer is backward for the distant shells. The above behaviour is due to forward local and backward nonlocal transfers described in §2.

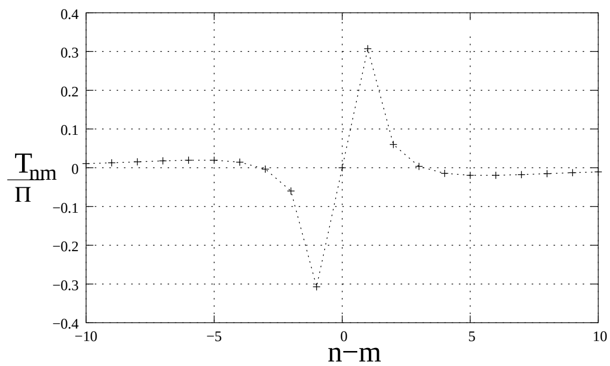

Figure 6. Plot of normalized shell-to-shell energy transfer $T_{nm}/|\Pi|$ vs. $n-m$ for $d = 2$ in the inertial range. The energy transfer rate from the shell m to the shells $m+1$, $m+2$, $m+3$ is forward, but $m+4$ onward it is negative. The net effect of all these transfer is the inverse energy flux Π.

In the next section we will contrast the energy transfers in incompressible fluid turbulence relative to Burgers turbulence (compressible limit).

6. Locality issues in Burgers turbulence

As discussed in the Introduction, Navier–Stokes equation is nonlocal in real space due to pressure. In Burgers equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\nabla^2\mathbf{u}.$$

The pressure term is dropped with an implicit assumption that the flow velocity is much greater than sound speed, or the sound speed is very small. This is the opposite limit of incompressible NS where the sound speed is infinite. Hence a very different behaviour is expected for Burgers equation [1]. Clearly, to time-advance the velocity field of Burgers equation at a point, we need local field, and its first and second derivatives. Hence, Burgers equation is *local in real space*, and nonlocal in Fourier space.

The formula for mode-to-mode energy transfer $S(k'|p|q)$ (eq. (7)) is not applicable for Burgers equation because it is compressible ($\nabla\cdot\mathbf{u} \neq 0$) [12,13]. Therefore, the shell-to-shell energy transfer cannot be computed accurately. Note however that energy flux can be computed for Burgers equation. The energy flux is multifractal, and $\Pi(k) \propto k^{-1/2}$. Therefore, $E(k) \propto \Pi^{2/3}k^{-5/3} = k^{-2}$ [1,21].

7. Conclusions

It is known that the nonlinear interactions in *incompressible* Navier–Stokes equation is nonlocal in real space due to the pressure term. In this paper we investigated locality in Fourier space by computing the strength of triad interactions using the formula for the mode-to-mode energy transfer. Our calculation is based on first-order field-theoretic technique. We take Kolmogorov's 5/3 power-law for the energy spectrum, and the renormalized viscosity for the effective viscosity. It has been shown that the magnitudes of interactions for the nonlocal triads $k \approx p \gg q$ and $k \approx q \gg p$ are large, while the interactions are small for the local triads $k \approx p \approx q$. This result shows that nonlinear interactions in *incompressible* fluid turbulence is *nonlocal in Fourier space as well.*

The shell-to-shell energy transfer rates have been investigated by many researchers and ourselves. It is a common wisdom that the shell-to-shell energy transfer is local, that is, maximum energy transfer takes place between nearest shells. We find that local shell-to-shell energy transfer is compatible with the nonlocal triad interactions because the local triads occupy more Fourier space volume as compared to nonlocal ($k \approx q \gg p$) ones. The local shell-to-shell energy transfer via nonlocal triad interactions is consistent, as seen by Domaradzki and Rogallo [5], Zhou [4], Ohkitani and Kida [7], and Zhou and Speziale [10] in their numerical simulations and EDQNM calculations. We have ourselves computed shell-to-shell energy transfer numerically; our theoretical results match with numerical results very well. In this paper we show this behaviour analytically. The role of the smallest wave-number mode in the triad is somewhat confusing in earlier papers. We have resolved some of these issues.

We observe interesting behaviour in two dimensions. The shell-to-shell energy transfer rates to the nearby shells are forward, whereas the transfer rates to the far off shells are backward. The net effect is a negative energy flux. This theoretical result is consistent with Dar *et al*'s numerical finding [12]. The inverse cascade of energy is consistent with the backward nonlocal energy transfer in mode-to-mode picture $[S(k|p|q)]$. We also show that the transition from backward energy transfer to forward energy transfer takes place at $d_c \approx 2.25$.

The contribution of local triads to the effective or renormalized viscosity has been debated in turbulence literature. In Yakhot and Orszag's [22] renormalization theory, the renormalized viscosity gets contribution from highly nonlocal wave-number triads. Kraichnan [23,24] first raised the above objection, and proposed some alternatives. The local energy transfer and nonlocal triad interaction results discussed in this paper could be of relevance for this issue; this aspect needs further investigation.

To conclude, an application of field-theoretic techniques to turbulence yields interesting results regarding triad interactions and shell-to-shell energy transfers. The method described here has also been applied to magnetohydrodynamic turbulence, and the results are presented in Verma *et al* [25]. Further investigations of locality in compressible turbulence, and other areas of turbulence will provide us useful clues in furthering our understanding of turbulence.

References

[1] U Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995)
[2] L D Landau and E M Lifshitz, *Fluid mechanics* (Pergamon Press, Oxford, 1987)
[3] S A Orszag, in *Fluid dynamics, Les Houches 1973 Summer School of theoretical physics* (Gordon and Breach, Berlin, 1977) p. 273

[4] Y Zhou, *Phys. Fluids* **5**, 1092 (1993)
[5] J A Domaradzki and R S Rogallo, *Phys. Fluids* **A2**, 413 (1990)
[6] R H Kraichnan, *J. Fluid Mech.* **47**, 525 (1971)
[7] K Ohkitani and S Kida, *Phys. Fluids* **A4**, 794 (1992)
[8] F Waleffe, *Phys. Fluids* **A4**, 350 (1992)
[9] K Kishida, K Araki, S Kishiba and K Suzuki, *Phys. Rev. Lett.* **83**, 5487 (1999)
[10] Y Zhou and C G Speziale, *Appl. Mech. Rev.* **51**, 267 (1998)
[11] D C Leslie, *Development in the theory of turbulence* (Oxford University Press, Clarendon, 1973)
[12] G Dar, M K Verma and V Eswaran, *Physica* **D157**, 207 (2001)
[13] M K Verma, *Phys. Rep.* **401**, 229 (2004)
[14] R H Kraichnan, *Phys. Fluids* **8**, 1385 (1965)
[15] W D McComb, *The physics of fluid turbulence* (Oxford University Press, Clarendon, 1990)
[16] M K Verma, *Phys. Rev.* **E64**, 26305 (2001)
[17] M K Verma, *Pramana – J. Phys.* **61**, 577 (2003)
[18] W D McComb and A G Watt, *Phys. Rev.* **A46**, 4797 (1992)
[19] M K Verma, *Phys. Plasmas* **8**, 3945 (2001)
[20] J D Fournier and U Frisch, *Phys. Rev.* **A17**, 747 (1979)
[21] M K Verma, *Physica* **A277**, 359 (2000)
[22] V Yakhot and S A Orszag, *J. Sci. Comput.* **1**, 3 (1986)
[23] R H Kraichnan, *Phys. Fluids* **30**, 1583 (1987)
[24] R H Kraichnan, *Phys. Fluids* **30**, 2400 (1987)
[25] M K Verma, A Ayyer and A Chandra, *Phys. Plasmas* (2005) to appear