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Exact periodic waves and their interactions for the (2+1)-dimensional KdV equation

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Abstract. By means of the singular manifold method we obtain a general solution involving three arbitrary functions for the (2+1)-dimensional KdV equation. Diverse periodic wave solutions may be produced by appropriately selecting these arbitrary functions as the Jacobi elliptic functions. The interaction properties of the periodic waves are investigated numerically and found to be nonelastic. The long wave limit yields some new types of solitary wave solutions. Especially the dromion and the solitoff solutions obtained in this paper possess new types of solution structures which are quite different from the basic dromion and solitoff ones reported previously in the literature.

Keywords. The (2+1)-dimensional KdV equation; exact solutions; the singular manifold method.

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1. Introduction

Recently, searching for the exact periodic wave solutions to the nonlinear partial differential equations (PDEs) attracts considerable interest [1–8]. These periodic wave solutions consist primarily of the Jacobi elliptic functions because of their elegant properties. The Jacobi elliptic functions $sn\xi = sn(\xi|m)$, $cn\xi = cn(\xi|m)$, and $dn\xi = dn(\xi|m)$, where m $(0 < m < 1)$ is the modulus of the elliptic function, are double periodic and possess properties of trigonometric functions. Detailed explanations about Jacobi elliptic functions can be found in [9–11]. The methods used in the literature for obtaining the Jacobi elliptic wave solutions to the nonlinear PDE are mainly the Hirota bilinear and theta function method [4], Legendre polynomial and Lamé equation method [5,6] and the mapping method [7,8]. However, the interaction properties of the Jacobi elliptic waves have not yet been studied, partly because the mathematics is more involved. One of the goals of the present work is to demonstrate that the singular manifold method [12], a powerful method for obtaining the exact solutions to the nonlinear PDEs, can also be used to get exact periodic wave solutions to the equation under consideration. The other goal is to study the interaction properties between the periodic waves. Here, we take the (2+1)-dimensional KdV equation

$$\begin{aligned} &u_t + u_{xxx} - 3uv_x - 3u_x v = 0, \\ &u_x = v_y, \end{aligned} \tag{1}$$

as a simple illustration example. Equation (1) was derived by Boiti *et al* [13] using the idea of the weak Lax pair. Its Painleve property has been proved by Dorizzi *et al* [14]. And Lie algebraic structure and the infinite dimensional symmetries have been studied [15]. Some special forms of solitary wave solutions are also reported [16]. However, eq. (1) possesses many interesting periodic structures which have not yet been found. The organization of the paper is as follows. In §2, a general solution involving three arbitrary functions for the equation under consideration is obtained by means of the singular manifold method. Section 3 is devoted to getting the periodic wave solution from the general solution. The interaction properties between Jacobi elliptic waves are studied numerically in §4. The conclusion and discussion is given in the last section.

2. A general solution to eq. (1)

According to the singular manifold method [12], we take the truncated Laurent series

$$u = -2(\ln\varphi)_{xy} + u_2, \quad v = -2(\ln\varphi)_{xx} + v_2, \tag{2}$$

where $\{u_2,\ v_2\}$ is an arbitrary solution of eq. (1). For simplicity, we take the special solution as

$$u_2 = 0, \quad v_2 = v_2(x,t), \tag{3}$$

where $v_2(x,t)$ is an arbitrary function of the indicated variables. One can show that the following equation

$$\begin{aligned} &\varphi_t + \varphi_{xxx} - 3v_2\varphi_x = 0, \\ &\varphi_{xx}\varphi_{xy} - \varphi_x\varphi_{xxy} = 0, \end{aligned} \tag{4}$$

is sufficient for guaranteeing that eqs (2)–(4) solve eq. (1). Directly integrating the second equation of eq. (4) three times and then substituting the resultant into the first one, we obtain

$$\begin{aligned} &\varphi(x,y,t) = f(t,x)g(y) + h(y), \\ &v_2 = \frac{f_t + f_{xxx}}{3f_x}, \end{aligned} \tag{5}$$

where f, g and h are arbitrary functions of the indicated variables. Therefore, we obtain a general solution of eq. (1) as follows:

$$u = -\frac{2f_x(g_y h - g h_y)}{(fg+h)^2},$$
$$v = -\frac{2g[f_{xx}(fg+h) - f_x^2 g]}{(fg+h)^2} + \frac{f_t + f_{xxx}}{3f_x}, \tag{6}$$

where f, g and h are arbitrary functions of the variables t, x and y, respectively.

3. Exact periodic wave solutions and their interactions

Thanks to the arbitrariness of the functions $f(t,x)$, $g(y)$ and $h(y)$, one can obtain a variety of exact solutions to eq. (1) by choosing appropriately these arbitrary functions. In this paper, we are restricted to the periodic wave solutions in terms of the Jacobi elliptic functions. It is worth noticing that the Jacobi transformation $dn(\xi|m) = cn(\sqrt{m}\xi|m^{-1})$ [9] implies that any solution found by the dn function may be transformed into an equivalent one that can be obtained by cn function. Moreover, since other Jacobi elliptic functions have singularities, we consider only the periodic wave solutions in terms of sn and cn functions. In what follows, we discuss several cases, but only the expressions of the physical field u are given while the potential field v is omitted.

Case 1. $f = sn(kx - \omega t|m) \equiv sn(\xi)$, $g = sn(l_1 y|m_1) \equiv sn(\eta_1)$, $h = A + sn(l_2 y|m_2) \equiv A + sn(\eta_2)$.

It follows from eq. (6) that

$$u = \frac{2kcn(\xi)dn(\xi)[l_2 sn(\eta_1)cn(\eta_2)dn(\eta_2) - l_1 cn(\eta_1)dn(\eta_1)(A + sn(\eta_2))]}{[sn(\xi)sn(\eta_1) + sn(\eta_2) + A]^2}, \tag{7}$$

where k, l_1, l_2, ω and A are arbitrary constants, m_1 and m_2 the moduli of the elliptic function, and the constant A guarantees that the solution has no singularity (because sn and cn functions have zeros). These statements are valid throughout the paper, unless otherwise explained. Figure 1 illustrates eq. (7) with the parameters $k = 1$, $l_1 = 1$, $l_2 = 2$, $m_1 = 0.2$, $m_2 = 0.3$, $m = 0.5$, $A = 4$ and $t = 0$, which are valid throughout this paper, unless otherwise stated. As m, m_1 and $m_2 \to 1$, from eq. (7), one has

$$u = \frac{2k\,\mathrm{sech}^2(\xi)[l_2 \tanh(\eta_1)\,\mathrm{sech}^2(\eta_2) - l_1\,\mathrm{sech}^2(\eta_1)(A + \tanh(\eta_2))]}{[\tanh(\xi)\tanh(\eta_1) + \tanh(\eta_2) + A]^2}. \tag{8}$$

A typical spatial structure of solution (8), a new dromion solution to eq. (1), is shown in figure 2. It is worth mentioning that the new type of dromion solution eq. (8) with $A = 4$ is quite different from the basic dromion structure reported in the literature [17]. It is interesting to note that eq. (8) with $A = 2$ is a new solitoff solution to eq. (1), whose typical spatial structure is plotted in figure 3. The new type of solitoff solution eq. (8) with $A = 2$ is also different from the basic solitoff structure [18].

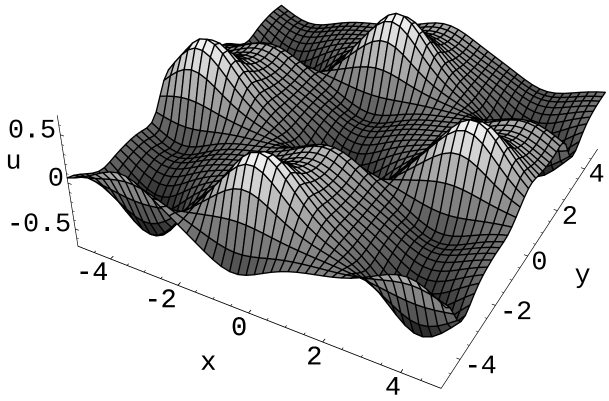

Figure 1. The typical spatial structure of solution (7).

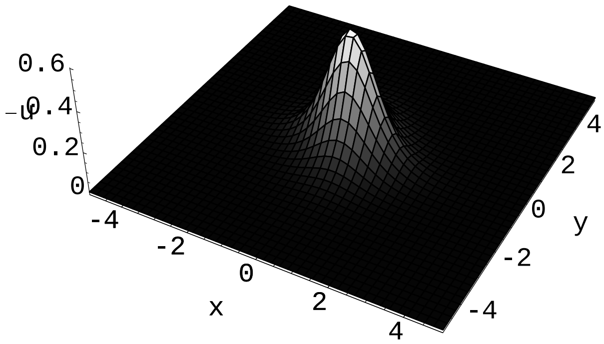

Figure 2. The typical spatial structure of solution (8) with $A = 4$.

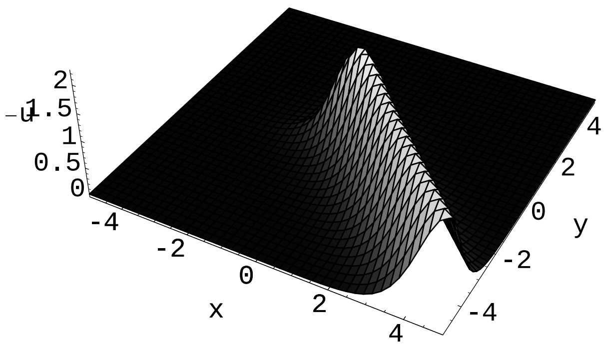

Figure 3. The typical spatial structure of solution (8) with $A = 2$.

Case 2. $f = cn(\xi)$, $g = cn(\eta_1)$, $h = A + cn(\eta_2)$.

From the first equation of eq. (6) we have another new type of periodic wave solution

$$u = \frac{2ksn(\xi)dn(\xi)[l_2 cn(\eta_1)sn(\eta_2)dn(\eta_2) - l_1 sn(\eta_1)dn(\eta_1)(A + cn(\eta_2))]}{[cn(\xi)cn(\eta_1) + cn(\eta_2) + A]^2}. \tag{9}$$

The long wave limit yields

$$u = \frac{2k\tanh(\xi)\text{sech}(\xi)[l_2\,\text{sech}(\eta_1)\tanh(\eta_2)\,\text{sech}(\eta_2) - l_1\tanh(\eta_1)\text{sech}(\eta_1)(A + \text{sech}(\eta_2))]}{[\text{sech}(\xi)\text{sech}(\eta_1) + \text{sech}(\eta_2) + A]^2}, \tag{10}$$

a new solitary wave solution to eq. (1), with two peaks and two pits for the same parameter values as figure 1. The functions f, g and h can certainly be taken as the elliptic ones of different types, which is left to readers. In what follows, we consider a case which can represent the interaction of waves.

Case 3. $f = sn(k_1 x - \omega_1 t|m_1) + sn(k_2 x - \omega_2 t|m_2) \equiv sn(\xi_1) + sn(\xi_2)$, $g = sn(l_1 y|M_1) \equiv sn(\eta_1)$, $h = A + sn(l_2 y|M_2) \equiv A + sn(\eta_2)$.

Hence, from eq. (6) we obtain the periodic wave solution of eq. (1)

$$u = \frac{2(\sum_{i=1}^{2} k_i cn(\xi_i)dn(\xi_i))[l_2 sn(\eta_1)cn(\eta_2)dn(\eta_2) - l_1 cn(\eta_1)dn(\eta_1)(A + sn(\eta_2))]}{[(sn(\xi_1) + sn(\xi_2))sn(\eta_1) + sn(\eta_2) + A]^2}. \tag{11}$$

We can easily draw by MATHEMATICA the evolution figures of eq. (11) with the parameters $k_1 = 1$, $k_2 = 2$, $l_1 = 1$, $l_2 = 2$, $\omega_1 = 1$, $\omega_2 = -1$, $m_1 = 0.2$, $m_2 = 0.3$, $M_1 = 0.5$, $M_2 = 0.7$ and $t = -5, 0, 5$, respectively and from the figures we can find that the interaction of the periodic wave solution (11) is nonelastic. As in Cases 1 and 2, the long wave limit of periodic waves will typically generate solitary wave forms. In fact, such limit for the present family of doubly periodic waves is especially rich, since one can proceed with the long wave limit in one direction only, which will be done in this case. The limit m_1 and $m_2 \to 1$ yields

$$u = \frac{2(\sum_{i=1}^{2} k_i\,\text{sech}^2(\xi_i))[l_2 sn(\eta_1)cn(\eta_2)dn(\eta_2) - l_1 cn(\eta_1)dn(\eta_1)(A + sn(\eta_2))]}{[(\tanh(\xi_1) + \tanh(\xi_2))sn(\eta_1) + sn(\eta_2) + A]^2}, \tag{12}$$

which is doubly periodic in y and decays exponentially in the propagating direction x and therefore called y-doubly periodic solitons. By graphic method one can easily find that the interaction of the y-doubly periodic solitons (12) is nonelastic. As M_1 and $M_2 \to 1$, it follows from eq. (11) that

$$u = \frac{2(\sum_{i=1}^{2} k_i cn(\xi_i)dn(\xi_i))[l_2\tanh(\eta_1)\text{sech}^2(\eta_2) - l_1\,\text{sech}^2(\eta_1)(A + \tanh(\eta_2))]}{[(sn(\xi_1) + sn(\xi_2))\tanh(\eta_1) + \tanh(\eta_2) + A]^2}, \tag{13}$$

which is doubly periodic in the propagating direction x and exponentially decays in y, and thus we call it x-doubly periodic solitons. The evolution of eq. (13) indicates that the interaction between x-doubly periodic solitons is also nonelastic. The limit m_1, m_2, M_1 and $M_2 \to 1$ yields

$$u = \frac{2(\sum_{i=1}^{2} k_i \operatorname{sech}^2(\xi_i))[l_2 \tanh(\eta_1)\operatorname{sech}^2(\eta_2) - l_1 \operatorname{sech}^2(\eta_1)(A + \tanh(\eta_2))]}{[(\tanh(\xi_1) + \tanh(\xi_2))\tanh(\eta_1) + \tanh(\eta_2) + A]^2}, \tag{14}$$

which is a two-dromion solution to eq. (1). From eq. (14), we can see that our dromion structure is quite different from the basic dromion structure reported in [17]. By graphic method we can see that the interaction of the two dromions is also nonelastic. It is interesting to notice that eq. (14) with $A = 3$ is a dromion–solitoff solution to eq. (1) and its evolution indicates that the interaction between dromion and solitoff is nonelastic. The limit m_1 and $m_2 \to 1$ while M_1 and $M_2 \to 0$ yields

$$u = \frac{2(\sum_{i=1}^{2} k_i \operatorname{sech}^2(\xi_i))[l_2 \sin(\eta_1)\cos(\eta_2) - l_1 \cos(\eta_1)(A + \sin(\eta_2))]}{[(\tanh(\xi_1) + \tanh(\xi_2))\sin(\eta_1) + \sin(\eta_2) + A]^2}, \tag{15}$$

which is periodic in y and decays exponentially in the propagating direction x and therefore called y-periodic solitons. The evolution of eq. (15) is very similar to that of eq. (12) and so the interaction of the y-periodic solitons is nonelastic. As m_1 and $m_2 \to 0$ while M_1 and $M_2 \to 1$, from eq. (11) one has

$$u = \frac{2(\sum_{i=1}^{2} k_i \cos(\xi_i)[l_2 \tanh(\eta_1)\operatorname{sech}^2(\eta_2) - l_1 \operatorname{sech}^2(\eta_1)(A + \tanh(\eta_2))]}{[(\sin(\xi_1) + \sin(\xi_2))\tanh(\eta_1) + \tanh(\eta_2) + A]^2}, \tag{16}$$

which is periodic in the propagating direction x and exponentially decays in y, and thus we call it x-periodic solitons. The evolution of eq. (16) is quite similar to eq. (13) and this indicates that the interaction between x-periodic solitons is also nonelastic.

4. Conclusion and discussion

We have obtained a general solution involving three arbitrary functions for the (2+1)-dimensional KdV equation by means of the singular manifold method. On its basis, a series of doubly periodic wave solutions in terms of rational functions of the Jacobi elliptic functions can be obtained. The interaction of waves and various limit cases are studied. Our results show that the interactions between the Jacobi elliptic waves, between x- or y- (doubly) periodic solitons, between two dromions and between dromion and solitoff for (2+1)-dimensional KdV equation are all nonelastic. It is well-known that the interactions of (1+1)-dimensional solitons are completely elastic. There is no energy and momentum exchange among solitons when they are interacting. The only effect of the soliton interaction is the phase shifts. However, for the (2+1)-dimensional KdV equation (1) there are quite different properties of the interaction. The long wave limit of periodic waves will typically generate solitary wave forms and such limit for the present family of doubly periodic waves is especially rich, since one can proceed with the long wave limit in one direction only. It is very interesting that eq. (8) with $A = 4$ is a dromion solution, and with $A = 2$ a solitoff solution, irrespective of the values of the other parameters k, ω and l_i. These two types of solutions are unified unexpectedly! It is the same in eq. (14). The two-dromion solution and the dromion–solitoff solution are also unified! More exciting discoveries of the complexity of the solution structures for nonlinear PDEs still remain ahead. It is seen from eq. (6) that eq. (1) possesses some special types of the localized coherent structures for the physical field u rather than the potential v. Now v has in its denominator f_x and therefore it is singular, i.e. for a fixed $t = 0$, there always exists $x = x_0$ at which the solutions blow up. Currently there is much interest in the blow up or hot spots of solutions for nonlinear PDEs [19,20]. It seems that these singular solutions will model this physical phenomena. Our algorithm is simple and direct. Whether the algorithm in this paper is applicable to other nonlinear PDEs or not is worth studying further.

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