PRAMANA — journal of physics

Vol. 65, No. 2
August 2005
pp. 165–176

Quantum states with continuous spectrum for a general time-dependent oscillator

JEONG-RYEOL CHOI
Department of New Material Science, Division of Natural Sciences, Sun Moon University, Asan 336-708, Korea
E-mail: choiardor@hanmail.net

MS received 14 November 2004; accepted 22 March 2005

Abstract. We investigated quantum states with continuous spectrum for a general time-dependent oscillator using invariant operator and unitary transformation methods together. The form of the transformed invariant operator by a unitary operator is the same as the Hamiltonian of the simple harmonic oscillator: $\hat{I}' = \hat{p}^2/2 + \omega^2\hat{q}^2/2$. The fact that ω^2 of the transformed invariant operator is constant enabled us to investigate the system separately for three cases, where $\omega^2 > 0$, $\omega^2 < 0$, and $\omega^2 = 0$. The eigenstates of the system are discrete for $\omega^2 > 0$. On the other hand, for $\omega^2 \leq 0$, the eigenstates are continuous. The time-dependent oscillators whose spectra of the wave function are continuous are not oscillatory. The wave function for $\omega^2 < 0$ is expressed in terms of the parabolic cylinder function. We applied our theory to the driven harmonic oscillator with strongly pulsating mass.

Keywords. Quantum states with continuous spectrum; time-dependent oscillator; invariant operator; unitary operator; propagator.

PACS Nos 03.65.Ca; 03.65.-w

1. Introduction

Harmonic oscillators which have time-variable mass and/or frequency may be good examples of the time-dependent quadratic Hamiltonian systems. Although various dynamical systems have been investigated using approximation and perturbation methods [1,2], we confine our concern to the research of exact quantum state of the general time-dependent harmonic oscillator. There are a large number of papers concerning the investigation of quantum states with discrete spectrum for time-dependent harmonic oscillator [3–8] and they can be applied to path-integral formulation of real-time finite-temperature field theory [9–11], dissipative quantum tunnelling effect in macroscopic system [12–15], scalar particle creation in cosmology [16], electrical behavior of RLC-circuit [17], and quantum motion of an ion in a Paul trap [3,18,19]. Even though the whole system is closed, it's subsystem may implicitly depend on time via interaction with the remnant of the system. The study of the properties for time-independent Hamiltonian systems whose eigenstates are continuous, such as energy dissipation, quantum and classical correspondence, decoherence and geometric/nongeometric effects are performed and found in [20–23]. The investigation of the eigenstates with continuous spectrum for the damped driven harmonic oscillator can be found in [24,25]. In this paper, we extended this idea to the general time-dependent oscillator using invariant operator and unitary transformation methods together. The quantum state whose spectrum is continuous has plentiful applications in physical branches such as FRW universe model [26,27], surface diffusion [28,29], and the analysis of electron beam in a magnetic field [30].

After the introduction of dynamical invariant operator by Lewis [31], the systematic investigation of quantum-mechanical time-dependent quadratic oscillator has been facilitated. The key idea of the invariant operator method to obtain the solution of the Schrödinger equation for the time-dependent system is that the wave function corresponding to the Hamiltonian is different from the eigenstate of dynamical invariant operator by only some time-dependent phase factors [32].

2. Preliminary concepts

The Hamiltonian describing quantum general time-dependent quadratic oscillator is given by [7]

$$\hat{H}(\hat{q},\hat{p},t) = A(t)\hat{p}^2 + B(t)(\hat{q}\hat{p}+\hat{p}\hat{q}) + C(t)\hat{p} + D(t)\hat{q}^2 + E(t)\hat{q} + F(t), \tag{1}$$

where $A(t)$–$F(t)$ are time-dependent coefficients which are real and differentiable with respect to t. Note that $A(t) \neq 0$. By applying Hamilton's equation into eq. (1), we can derive the classical equation of motion of the system as [33]

$$\ddot{\hat{q}} - \frac{\dot{A}}{A}\dot{\hat{q}} + \left(\frac{2\dot{A}B}{A} - 2\dot{B} - 4B^2 + 4AD\right)\hat{q} + \frac{\dot{A}C}{A} - \dot{C} - 2BC + 2AE = 0. \tag{2}$$

In fact, the Schrödinger equation for eq. (1) is too difficult to solve straightforward, since the separation of the coordinate and time is impossible.

The introduction of the dynamical invariant operator $\hat{I}(t)$ may save us much time and labor in order to solve the quantum-mechanical solutions of the time-dependent Hamiltonian system because the wave function corresponding to the Hamiltonian is the same as the eigenstate of dynamical invariant operator except for only some time-dependent phase factors [32]. From $\mathrm{d}\hat{I}(t)/\mathrm{d}t = 0$, we obtain [7]

$$\begin{aligned}\hat{I}(t) = {}& \alpha(t)[\hat{p} - p_{\mathrm{p}}(t)]^2 + \beta(t)\{[\hat{q} - q_{\mathrm{p}}(t)][\hat{p} - p_{\mathrm{p}}(t)] \\ & +[\hat{p} - p_{\mathrm{p}}(t)][\hat{q} - q_{\mathrm{p}}(t)]\} + \gamma(t)[\hat{q} - q_{\mathrm{p}}(t)]^2.\end{aligned} \tag{3}$$

In the above equation, $q_{\mathrm{p}}(t)$ is the particular solution of classical equation (eq. (2)) of motion in q-space, and $p_{\mathrm{p}}(t)$ is the corresponding particular solution in p-space. Time-variable functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are

$$\alpha(t) = c_1\rho_1^2(t) + c_2\rho_1(t)\rho_2(t) + c_3\rho_2^2(t), \quad (4)$$

$$\beta(t) = \frac{1}{4A}\{4[c_1\rho_1^2(t) + c_2\rho_1(t)\rho_2(t) + c_3\rho_2^2(t)]B - [2c_1\rho_1(t)\dot\rho_1(t) + c_2\dot\rho_1(t)\rho_2(t) + c_2\dot\rho_2(t)\rho_1(t) + 2c_3\rho_2(t)\dot\rho_2(t)]\}, \quad (5)$$

$$\gamma(t) = \frac{1}{2A^2}\left\{\frac{1}{2}[c_1\dot\rho_1^2(t) + c_2\dot\rho_1(t)\dot\rho_2(t) + c_3\dot\rho_2^2(t)] - B[2c_1\rho_1(t)\dot\rho_1(t) + c_2\dot\rho_1(t)\rho_2(t) + c_2\dot\rho_2(t)\rho_1(t) + 2c_3\rho_2(t)\dot\rho_2(t)] + 2B^2[c_1\rho_1^2(t) + c_2\rho_1(t)\rho_2(t) + c_3\rho_2^2(t)]\right\}, \quad (6)$$

where c_1–c_3 are arbitrary constants and $\rho_{1,2}(t)$ are two independent homogeneous solutions of the following differential equation.

$$\ddot\rho_{1,2}(t) - \frac{\dot A}{A}\dot\rho_{1,2}(t) + \left(\frac{2\dot A B}{A} - 2\dot B - 4B^2 + 4AD\right)\rho_{1,2}(t) = 0. \quad (7)$$

For $B(t) = 0$, eqs (4)–(6) correspond to that of ref. [6].

We can convert the invariant operator into a simple form using unitary transformation method. To do this, let us introduce the following unitary operator:

$$\hat U = \hat U_3 \hat U_2 \hat U_1, \quad (8)$$

where

$$\hat U_1 = \exp\left(\frac{i}{\hbar}q_{\rm p}\hat p\right)\exp\left(-\frac{i}{\hbar}p_{\rm p}\hat q\right), \quad (9)$$

$$\hat U_2 = \exp\left(i\frac{\beta}{2\alpha\hbar}\hat q^2\right), \quad (10)$$

$$\hat U_3 = \exp\left[\frac{i}{4\hbar}(\hat q\hat p + \hat p\hat q)\ln(2\alpha)\right]. \quad (11)$$

Using eq. (8), we can transform invariant operator eq. (3):

$$\hat I' = \hat U \hat I \hat U^\dagger. \quad (12)$$

Then, after performing some algebra, $\hat I'$ reduces to the following simple form:

$$\hat I' = \frac{1}{2}\hat p^2 + \frac{1}{2}\omega^2\hat q^2, \quad (13)$$

where

$$\omega^2 = 4(\alpha\gamma - \beta^2)$$
$$= \frac{1}{4A^2}(\rho_1\dot{\rho}_2 - \dot{\rho}_1\rho_2)^2(4c_1c_3 - c_2^2) = \text{Constant}. \tag{14}$$

Note that eq. (14) is constant. This can be checked by direct differentiation of ω^2 with respect to time. The fact that ω^2 is constant enables us to investigate the system separately for three cases, where $\omega^2 > 0$, $\omega^2 < 0$, and $\omega^2 = 0$. The eigenstate of the system is discrete for $\omega^2 > 0$ since the transformed invariant operator eq. (13) corresponds to that of the oscillating system while that of the other two cases are continuous. The wave functions for $\omega^2 > 0$ are [7]

$$\langle q|\Psi_n(t)\rangle = \exp[i\epsilon_n(t)]\langle q|\Phi_n(t)\rangle, \tag{15}$$

where $\langle q|\Phi_n(t)\rangle$ are the eigenstates of the invariant operator eq. (3), that are given by

$$\langle q|\Phi_n(t)\rangle = \left(\frac{\omega}{2\alpha\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{\omega}{2\alpha\hbar}}(q - q_{\mathrm{p}})\right)\exp\left(\frac{i}{\hbar}p_{\mathrm{p}}q\right)$$
$$\times\exp\left[-\frac{1}{2\alpha\hbar}\left(\frac{\omega}{2} + i\beta\right)(q - q_{\mathrm{p}})^2\right], \tag{16}$$

and $\epsilon_n(t)$ are phases of the form

$$\epsilon_n(t) = -\omega T(t)\left(n + \frac{1}{2}\right)$$
$$-\frac{1}{\hbar}\int_0^t\left[\mathcal{L}_{\mathrm{p}}(q_{\mathrm{p}}(t'), \dot{q}_{\mathrm{p}}(t'), t') - \frac{C^2(t')}{4A(t')} + F(t')\right]\mathrm{d}t', \tag{17}$$

with

$$T(t) = \int_0^t \frac{A(t')}{\alpha}\mathrm{d}t', \tag{18}$$

$$\mathcal{L}_{\mathrm{p}}(q_{\mathrm{p}}(t), \dot{q}_{\mathrm{p}}(t), t) = \frac{1}{4A(t)}\dot{q}_{\mathrm{p}}^2(t) - \frac{B(t)}{A(t)}q_{\mathrm{p}}(t)\dot{q}_{\mathrm{p}}(t)$$
$$-\left(D(t) - \frac{B^2(t)}{A(t)}\right)q_{\mathrm{p}}^2(t). \tag{19}$$

The propagator which describes quantum mechanics in terms of the generalized path integral provides us the information about the evolution of the state for a particle. If an object placed originally q_1 at t_1 evolves to q_2 at t_2, we can express the propagator of the system whose spectrum of the eigenstate is discrete as [34]

$$K(q_2, t_2; q_1, t_1) = \sum_{n=0}^{\infty}\langle q_2|\Psi_n(t_2)\rangle\langle\Psi_n(t_1)|q_1\rangle. \tag{20}$$

When we perform the summation after inserting eq. (15) into the above equation with the aid of Mehler's formula [35],

$$\sum_{n=0}^{\infty}\frac{(z/2)^n}{n!}H_n(x)H_n(y)=\frac{1}{(1-z^2)^{1/2}}\exp\left[\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\right], \quad (21)$$

eq. (20) becomes

$$\begin{aligned}K(q_2,t_2;q_1,t_1)=&\left(\frac{\omega}{4i\sqrt{\alpha^*(t_1)\alpha(t_2)}\pi\hbar\sin[\omega(T(t_2)-T^*(t_1))]}\right)^{1/2}\\&\times\exp\left\{\frac{\omega}{4i\hbar\sin[\omega(T(t_2)-T^*(t_1))]}\right.\\&\times\left\{\frac{2}{\sqrt{\alpha^*(t_1)\alpha(t_2)}}(q_1-q_\mathrm{p}(t_1))(q_2-q_\mathrm{p}(t_2))\right.\\&\left.\left.-\left[\frac{(q_1-q_\mathrm{p}(t_1))^2}{\alpha^*(t_1)}+\frac{(q_2-q_\mathrm{p}(t_2))^2}{\alpha(t_2)}\right]\cos[\omega(T(t_2)-T^*(t_1))]\right\}\right\}\\&\times\exp\left\{-\frac{i}{2\hbar}\left[\frac{\beta(t_2)}{\alpha(t_2)}(q_2-q_\mathrm{p}(t_2))^2-\frac{\beta^*(t_1)}{\alpha^*(t_1)}(q_1-q_\mathrm{p}(t_1))^2\right]\right\}\\&\times\exp\left\{-\frac{i}{\hbar}\int_{t_1}^{t_2}\left[\mathcal{L}_\mathrm{p}(q_\mathrm{p}(t'),\dot{q}_\mathrm{p}(t'),t')-\frac{C^2(t')}{4A(t')}+F(t')\right]\mathrm{d}t'\right\}\\&\times\exp\left[\frac{i}{\hbar}(p_\mathrm{p}(t_2)q_2-p_\mathrm{p}(t_1)q_1)\right]. \quad (22)\end{aligned}$$

This is the probability amplitude for the evolution of an oscillator from q_1 to q_2 at time interval t_2-t_1:

$$\langle q_2|\Psi_n(t_2)\rangle=\int_{-\infty}^{\infty}K(q_2,t_2;q_1,t_1)\langle q_1|\Psi_n(t_1)\rangle\mathrm{d}q_1. \quad (23)$$

3. Quantum states with continuous spectrum

In this section, we investigate the quantum states with continuous spectrum for the time-dependent oscillator. In the case of $\omega^2\le 0$, eq. (13) says that the system is not oscillatory and its eigenvalue becomes continuous.

We shall begin the investigation of the quantum state with $\omega^2<0$. For convenience, let us introduce a notation $\tilde{\omega}^2$ that is given by

$$\tilde{\omega}^2=-\omega^2>0. \quad (24)$$

Then, eq. (13) can be rewritten as

$$\hat{I}'=\frac{1}{2}\hat{p}^2-\frac{1}{2}\tilde{\omega}^2\hat{q}^2. \quad (25)$$

This is the same as the Hamiltonian of the harmonic parabola potential system. The eigenvalue equation for eq. (25) can be represented as

$$\hat{I}'|\phi_\lambda(t)\rangle = \lambda|\phi_\lambda(t)\rangle. \tag{26}$$

By substitution of eq. (25) into eq. (26), we can obtain the following differential equation in q-space:

$$\frac{\partial^2\langle q|\phi_\lambda(t)\rangle}{\partial Q^2} + \left(\Lambda + \frac{1}{4}Q^2\right)\langle q|\phi_\lambda(t)\rangle = 0, \tag{27}$$

where

$$Q = \sqrt{\frac{2\tilde{\omega}}{\hbar}}\; q, \tag{28}$$

$$\Lambda = \frac{\lambda}{\hbar\tilde{\omega}}. \tag{29}$$

By solving eq. (27), we can derive the eigenstate of transformed invariant operator as

$$\begin{aligned}\langle q|\phi_\lambda(t)\rangle = &\left(\frac{\tilde{\omega}}{8\pi^2\hbar}\right)^{1/4} \\ &\times\left[\mathrm{e}^{i\pi/4}D_{-i\Lambda-1/2}\left(\frac{1+i}{\sqrt{2}}Q\right)\right. \\ &\left.+\mathrm{e}^{-i\pi/4}D_{-i\Lambda-1/2}\left(-\frac{1+i}{\sqrt{2}}Q\right)\right],\end{aligned} \tag{30}$$

where $D_\nu(x)$ is the parabolic cylinder function which is defined as [35,36]

$$D_\nu(x) = 2^{(\nu-1)/2}\exp(-x^2/4)x\bar{\Psi}(1/2-\nu/2, 3/2; x^2/2), \tag{31}$$

with $\bar{\Psi}$ given by [36]

$$\bar{\Psi}(a, c; y) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-yt}t^{a-1}(1+t)^{c-a-1}\mathrm{d}t. \tag{32}$$

The eigenstate of the untransformed invariant operator $\langle q|\Phi_\lambda(t)\rangle$ can be derived from [37]

$$\langle q|\Phi_\lambda(t)\rangle = \hat{U}^\dagger\langle q|\phi_\lambda(t)\rangle. \tag{33}$$

Using eq. (8), the above equation can be evaluated as

$$\begin{aligned}\langle q|\Phi_\lambda(t)\rangle = &\frac{1}{(2\alpha)^{1/4}}\left(\frac{\tilde{\omega}}{8\pi^2\hbar}\right)^{1/4}\exp\left[-\frac{i\beta}{2\alpha\hbar}(q-q_{\mathrm{p}}(t))^2\right]\mathrm{e}^{ip_{\mathrm{p}}(t)q/\hbar} \\ &\times\left[\mathrm{e}^{i\pi/4}D_{-i\Lambda-1/2}\left(\frac{1+i}{2\sqrt{\alpha}}Q'\right)\right. \\ &\left.+\,\mathrm{e}^{-i\pi/4}D_{-i\Lambda-1/2}\left(-\frac{1+i}{2\sqrt{\alpha}}Q'\right)\right],\end{aligned} \tag{34}$$

where

$$Q' = \sqrt{\frac{2\tilde{\omega}}{\hbar}}\ (q - q_{\rm p}(t)). \tag{35}$$

It is known that the wave function of the system is different from the eigenstate of the invariant operator by some time-dependent phase factor:

$$\langle q|\Psi_\lambda(t)\rangle = \exp[i\epsilon(t)]\langle q|\Phi_\lambda(t)\rangle. \tag{36}$$

By substituting eq. (36) into Schrödinger equation, we can obtain that

$$\hbar\dot{\epsilon}(t) = \langle\Phi_\lambda(t)|\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)|\Phi_\lambda(t)\rangle. \tag{37}$$

The phase can be derived from eq. (37) with eqs (1) and (34) as

$$\epsilon(t) = -\frac{\lambda}{\hbar}t - \frac{1}{\hbar}\int_0^t \left[\mathcal{L}_{\rm p}(q_{\rm p}(t'), \dot{q}_{\rm p}(t'), t') - \frac{C^2(t')}{4A(t')} + F(t')\right]{\rm d}t'. \tag{38}$$

Then, substitution of eqs (34) and (38) into eq. (36), we can express the full wave function of the system as

$$\begin{aligned}\langle q|\Psi_\lambda(t)\rangle = {} & \frac{1}{(2\alpha)^{1/4}}\left(\frac{\tilde{\omega}}{8\pi^2\hbar}\right)^{1/4}\exp\left[-\frac{i\beta}{2\alpha\hbar}(q-q_{\rm p}(t))^2\right]{\rm e}^{ip_{\rm p}(t)q/\hbar} \\ & \times\left[{\rm e}^{i\pi/4}D_{-i\Lambda-1/2}\left(\frac{1+i}{2\sqrt{\alpha}}\ Q'\right)\right. \\ & \left. + {\rm e}^{-i\pi/4}D_{-i\Lambda-1/2}\left(-\frac{1+i}{2\sqrt{\alpha}}Q'\right)\right] \\ & \times\exp\left\{-\frac{i\lambda}{\hbar}t - \frac{i}{\hbar}\int_0^t\left[\mathcal{L}_{\rm p}(q_{\rm p}(t'), \dot{q}_{\rm p}(t'), t')\right.\right. \\ & \left.\left. -\frac{C^2(t')}{4A(t')} + F(t')\right]{\rm d}t'\right\},\end{aligned} \tag{39}$$

where the eigenvalue λ is continuous.

The propagator of the system whose spectrum of the eigenstate is continuous can be derived from [24]

$$K(q_2, t_2; q_1, t_1) = \frac{1}{\hbar\tilde{\omega}}\int_{-\infty}^{\infty}\langle q_2|\Psi_\lambda(t_2)\rangle\langle\Psi_\lambda(t_1)|q_1\rangle{\rm d}\lambda. \tag{40}$$

After substituting eq. (39) into the above equation, the integration can be performed using the following integral formula [35],

$$\begin{aligned}&\int_{c-i\infty}^{c+i\infty}[D_\nu(x)D_{-\nu-1}(iy) + D_\nu(-x)D_{-\nu-1}(-iy)]\frac{t^{-\nu-1}{\rm d}\nu}{\sin(-\nu\pi)} \\ &\quad = \frac{2\sqrt{2\pi}i}{\sqrt{1+t^2}}\exp\left[\frac{1-t^2}{4(1+t^2)}(x^2+y^2) + i\frac{xyt}{1+t^2}\right] \\ &\qquad\qquad -1 < c < 0, \qquad |\arg\ t| < \pi/2.\end{aligned} \tag{41}$$

Then, eq. (40) becomes

$$K(q_2,t_2;q_1,t_1) = \left(\frac{\tilde{\omega}}{4i\sqrt{\alpha^*(t_1)\alpha(t_2)}\pi\hbar\sinh[\tilde{\omega}(t_2-t_1)]}\right)^{1/2}$$
$$\times \exp\left[\frac{i}{\hbar}(p_{\rm p}(t_2)q_2 - p_{\rm p}(t_1)q_1)\right]\exp\left\{\frac{\tilde{\omega}}{4i\hbar\sinh[\tilde{\omega}(t_2-t_1)]}\right.$$
$$\times\left\{\frac{2}{\sqrt{\alpha^*(t_1)\alpha(t_2)}}(q_1-q_{\rm p}(t_1))(q_2-q_{\rm p}(t_2))\right.$$
$$\left.\left.-\left[\frac{(q_1-q_{\rm p}(t_1))^2}{\alpha^*(t_1)}+\frac{(q_2-q_{\rm p}(t_2))^2}{\alpha(t_2)}\right]\cosh[\tilde{\omega}(t_2-t_1)]\right\}\right\}$$
$$\times\exp\left\{-\frac{i}{2\hbar}\left[\frac{\beta(t_2)}{\alpha(t_2)}(q_2-q_{\rm p}(t_2))^2-\frac{\beta^*(t_1)}{\alpha^*(t_1)}(q_1-q_{\rm p}(t_1))^2\right]\right\}$$
$$\times\exp\left\{-\frac{i}{\hbar}\int_{t_1}^{t_2}\left[\mathcal{L}_{\rm p}(q_{\rm p}(t'),\dot{q}_{\rm p}(t'),t')-\frac{C^2(t')}{4A(t')}+F(t')\right]{\rm d}t'\right\}. \quad (42)$$

Now, let us investigate the system for $\omega^2 = 0$, which is another case whose spectrum of quantum state is continuous. In this case eq. (13) can be more simplified to

$$\hat{I}' = \frac{1}{2}\hat{p}^2. \quad (43)$$

Note that eq. (43) is the same as the Hamiltonian of the free particle. We can represent the eigenvalue equation of eq. (43) as

$$\hat{I}'|\phi_{\lambda'}(t)\rangle = \lambda'|\phi_{\lambda'}(t)\rangle. \quad (44)$$

By substitution of eq. (43) into eq. (44) and after some algebra, we can obtain the eigenstate of transformed invariant operator in q-space:

$$\langle q|\phi_{\lambda'}(t)\rangle = \left(\frac{\lambda_0}{8\hbar^2\pi^2}\right)^{1/4}$$
$$\times\left[{\rm e}^{i\pi/4}\exp\left(\frac{i}{\hbar}\sqrt{2\lambda'}\ q\right)+{\rm e}^{-i\pi/4}\exp\left(-\frac{i}{\hbar}\sqrt{2\lambda'}\ q\right)\right], \quad (45)$$

where λ_0 is some constant with dimension of energy. The eigenstate of untransformed invariant operator can be calculated using eqs (8) and (45) to be

$$\langle q|\Phi_{\lambda'}(t)\rangle = \hat{U}^\dagger\langle q|\phi_{\lambda'}(t)\rangle$$
$$= \frac{1}{(2\alpha)^{1/4}}\left(\frac{\lambda_0}{8\hbar^2\pi^2}\right)^{1/4}\exp\left(-\frac{i\beta}{2\alpha\hbar}(q-q_{\rm p})^2\right)\exp\left(\frac{i}{\hbar}p_{\rm p}q\right)$$
$$\times\left[{\rm e}^{i\pi/4}\exp\left(\frac{i}{\hbar}\sqrt{\frac{\lambda'}{\alpha}}(q-q_{\rm p})\right)\right.$$
$$\left.+{\rm e}^{-i\pi/4}\exp\left(-\frac{i}{\hbar}\sqrt{\frac{\lambda'}{\alpha}}(q-q_{\rm p})\right)\right]. \quad (46)$$

The relation between eigenstate of the invariant operator and wave function is also the same as the previous one, eq. (36) and we can easily derive the corresponding phase as

$$\epsilon(t) = -\frac{\lambda'}{\hbar}t - \frac{1}{\hbar}\int_0^t \left[\mathcal{L}_{\rm p}(q_{\rm p}(t'), \dot{q}_{\rm p}(t'), t') - \frac{C^2(t')}{4A(t')} + F(t')\right] {\rm d}t'. \tag{47}$$

Then, substitution of eqs (46) and (47) into eq. (36), we can express the full wave function as

$$\begin{aligned}\langle q|\Psi_{\lambda'}(t)\rangle = {} & \frac{1}{(2\alpha)^{1/4}}\left(\frac{\lambda_0}{8\hbar^2\pi^2}\right)^{1/4} \exp\left(-\frac{i\beta}{2\alpha\hbar}(q-q_{\rm p})^2\right)\exp\left(\frac{i}{\hbar}p_{\rm p}q\right) \\ & \times \left[{\rm e}^{i\pi/4}\exp\left(\frac{i}{\hbar}\sqrt{\frac{\lambda'}{\alpha}}\,(q-q_{\rm p})\right)\right. \\ & \left. + {\rm e}^{-i\pi/4}\exp\left(-\frac{i}{\hbar}\sqrt{\frac{\lambda'}{\alpha}}\,(q-q_{\rm p})\right)\right] \\ & \times \exp\left\{-i\frac{\lambda'}{\hbar}t - \frac{i}{\hbar}\int_0^t \left[\mathcal{L}_{\rm p}(q_{\rm p}(t'), \dot{q}_{\rm p}(t'), t')\right.\right. \\ & \left.\left. - \frac{C^2(t')}{4A(t')} + F(t')\right]{\rm d}t'\right\}. \end{aligned} \tag{48}$$

In eq. (48), the eigenvalue λ' is also continuous as λ in Eq. (39). The propagator of the system can be derived from [24]

$$K(q_2, t_2; q_1, t_1) = \frac{1}{\sqrt{\lambda_0}}\int_{-\infty}^{\infty} \langle q_2|\Psi_{\lambda'}(t_2)\rangle\langle\Psi_{\lambda'}(t_1)|q_1\rangle {\rm d}\sqrt{\lambda'}. \tag{49}$$

By substituting eq. (48) into the above equation we can obtain that

$$\begin{aligned} K(q_2, t_2; q_1, t_1) = {} & \left(\frac{1}{4i\sqrt{\alpha^*(t_1)\alpha(t_2)}\pi\hbar(t_2-t_1)}\right)^{1/2} \\ & \times \exp\left[\frac{i}{\hbar}(p_{\rm p}(t_2)q_2 - p_{\rm p}(t_1)q_1)\right] \\ & \times \exp\left[\frac{i}{4\hbar(t_2-t_1)}\left(\frac{q_1 - q_{\rm p}(t_1)}{\sqrt{\alpha^*(t_1)}} - \frac{q_2 - q_{\rm p}(t_2)}{\sqrt{\alpha(t_2)}}\right)^2\right] \\ & \times \exp\left\{-\frac{i}{2\hbar}\left[\frac{\beta(t_2)}{\alpha(t_2)}(q_2 - q_{\rm p}(t_2))^2\right.\right. \\ & \left.\left. - \frac{\beta^*(t_1)}{\alpha^*(t_1)}(q_1 - q_{\rm p}(t_1))^2\right]\right\} \\ & \times \exp\left\{-\frac{i}{\hbar}\int_{t_1}^{t_2} [\mathcal{L}_{\rm p}(q_{\rm p}(t'), \dot{q}_{\rm p}(t'), t')\right. \\ & \left. - \frac{C^2(t')}{4A(t')} + F(t')\right]{\rm d}t'\right\}. \end{aligned} \tag{50}$$

4. Application to the driven oscillator with strongly pulsating mass

The discussions in the previous sections may be applied to various kinds of time-dependent oscillators. As an example, we apply them to the driven oscillator with strongly pulsating mass [38,39]. In this case the Hamiltonian is given by

$$\hat{H}(t) = \frac{\hat{p}^2}{2M(t)} + \frac{1}{2}M(t)\omega_0^2\hat{q}^2 - M(t)f(t)\hat{q}, \tag{51}$$

where

$$M(t) = M_0 \cos^2 \omega_M t, \tag{52}$$

$$f(t) = f_0 \cos(\omega_f t + \theta), \tag{53}$$

where M_0 is the mass at $t = 0$, ω_M and ω_f are arbitrary constant frequencies, f_0 is the amplitude of the driving force, and θ is the initial phase of the driving force. If we apply Hamilton's equation of motion into eq. (51), we can obtain that

$$\ddot{\hat{q}} + \frac{\dot{M}}{M}\dot{\hat{q}} + \omega_0^2\hat{q} = f(t), \tag{54}$$

$$\ddot{\hat{p}} - \frac{\dot{M}}{M}\dot{\hat{p}} + \omega_0^2\hat{p} = M\dot{f}(t). \tag{55}$$

By applying eqs (52) and (53), the above two equations become

$$\ddot{\hat{q}} - 2\omega_M \tan(\omega_M t)\dot{\hat{q}} + \omega_0^2\hat{q} = f_0 \cos(\omega_f t + \theta), \tag{56}$$

$$\ddot{\hat{p}} + 2\omega_M \tan(\omega_M t)\dot{\hat{p}} + \omega_0^2\hat{p} = -f_0 M_0 \omega_f \cos^2(\omega_M t)\sin(\omega_f t + \theta), \tag{57}$$

and, eq. (7) becomes

$$\ddot{\rho}_{1,2}(t) + \frac{\dot{M}}{M}\dot{\rho}_{1,2}(t) + \omega_0^2\rho_{1,2}(t) = 0. \tag{58}$$

The two c-number solutions of the above equation are

$$\rho_1(t) = \rho_1(0)\sec(\omega_M t)\mathrm{e}^{i\Omega t}, \tag{59}$$

$$\rho_2(t) = \rho_2(0)\sec(\omega_M t)\mathrm{e}^{-i\Omega t}, \tag{60}$$

where

$$\Omega = \sqrt{\omega_0^2 + \omega_M^2}. \tag{61}$$

If we consider eqs (54) and (55), the two particular solutions q_p and p_p follow that

$$\ddot{q}_\mathrm{p} + \frac{\dot{M}}{M}\dot{q}_\mathrm{p} + \omega_0^2 q_\mathrm{p} = f(t), \tag{62}$$

$$\ddot{p}_\mathrm{p} - \frac{\dot{M}}{M}\dot{p}_\mathrm{p} + \omega_0^2 p_\mathrm{p} = M\dot{f}(t). \tag{63}$$

The particular solutions satisfying the above two equations are given by [38]

$$q_\mathrm{p} = \frac{1}{2} f_0 \sec(\omega_M t)\left[\frac{\cos[(\omega_f + \omega_M)t + \theta] - \cos(\Omega t)\cos\theta}{\Omega^2 - (\omega_f + \omega_M)^2} + \frac{\cos[(\omega_f - \omega_M)t + \theta] - \cos(\Omega t)\cos\theta}{\Omega^2 - (\omega_f - \omega_M)^2}\right], \tag{64}$$

$$p_\mathrm{p} = [M_0 M(t)]^{1/2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(q_\mathrm{p}\cos\omega_M t) + \omega_M q_\mathrm{p}\sin\omega_M t\right]. \tag{65}$$

In terms of eqs (59), (60), (64), and (65) the quantum solution of the system can be completely described.

5. Summary

We used both invariant operator and unitary transformation methods in order to investigate the quantum eigenstates with continuous spectrum for the general time-dependent oscillator. If we choose unitary operator as eq. (8) with eqs (9)–(11), the invariant operator can be transformed to eq. (13) which is the same as the Hamiltonian of the simple harmonic oscillator. The fact that ω^2 in eq. (14) is constant enabled us to investigate the system separately for three cases, where $\omega^2 > 0$, $\omega^2 < 0$, and $\omega^2 = 0$. The eigenvalue of the system for $\omega^2 \leq 0$ is continuous while that for $\omega^2 > 0$ is discrete. For the latter case, the system is oscillatory and quantized. On the other hand, for the former case, the system is not oscillatory. We obtained exact solutions of the Schrödinger equation for the system having continuous eigenvalue. The wave function for $\omega^2 < 0$ is expressed in terms of the parabolic cylinder function while that for $\omega^2 > 0$ is expressed in terms of the well-known Hermite polynomial. We derived propagator of the system whose quantum states are both discrete and continuous by using the wave function, which is the probability amplitude for the evolution of an oscillator.

Our results can be applied to various time-dependent quadratic Hamiltonian systems beyond damped driven harmonic oscillator. As an example, we applied our theory to the driven oscillator with strongly pulsating mass. For damped driven harmonic oscillator, our results reduces to that of ref. [25].

References

[1] G E Giacaglia, *Perturbation methods in nonlinear systems* (Springer, Berlin, 1972), Applied Mathematical Science Series, Vol. 18
[2] W T Van Horssen, *SIAM J. Appl. Math.* **59**, 1444 (1999)
[3] L S Brown, *Phys. Rev. Lett.* **66**, 527 (1991)
[4] Fu-li Li, S J Wang, A Weiguny and D L Lin, *J. Phys.* **A27**, 985 (1994)
[5] C I Um, J R Choi and K H Yeon, *J. Korean Phys. Soc.* **38**, 447 (2001)

[6] J Y Ji, J K Kim and S P Kim, *Phys. Rev.* **A51**, 4268 (1995)
[7] J R Choi, *Pramana – J. Phys.* **61**, 7 (2003)
[8] C I Um, J R Choi, K H Yeon and T F George, *J. Korean Phys. Soc.* **40**, 969 (2002)
[9] R L Kobes and K L Kowalski, *Phys. Rev.* **D34**, 513 (1986)
[10] N P Landsman and Ch G van Weert, *Phys. Rep.* **145**, 141 (1987)
[11] E Calzetta and B L Hu, *Phys. Rev.* **D37**, 2878 (1988)
[12] A O Caldeira and A J Leggett, *Ann. Phys.* **149**, 374 (1983)
[13] S Baskoutas, A Jannussis and R Mignani, *J. Phys.* **A27**, 2189 (1994)
[14] A O Caldeira and A J Leggett, *Phys. Rev. Lett.* **46**, 211 (1981)
[15] A Widom and T D Clark, *Phys. Rev. Lett.* **48**, 63 (1982)
[16] B K Berger, *Phys. Rev.* **D12**, 368 (1975)
[17] H Lamba, S McKee and R Simpson, *J. Phys.* **A31**, 7065 (1998)
[18] H Dehmelt, *Rev. Mod. Phys.* **62**, 525 (1990)
[19] W Paul, *Rev. Mod. Phys.* **62**, 531 (1990)
[20] J B Maddox and E R Bittner, *Phys. Rev.* **E65**, 026143 (2002)
[21] Z Qi, *Phys. Rev.* **A53**, 3805 (1996)
[22] S Yngve and S Linnaeus, *J. Phys.* **A19**, 3017 (1986)
[23] S Guillouzic, I L'Heureux and A Longtin, *Phys. Rev.* **E61**, 4906 (2000)
[24] K H Yeon, S S Kim, Y M Moon, S K Hong, C I Um and T F George, *J. Phys.* **A34**, 7719 (2001)
[25] J R Choi, *Mod. Phys. Lett.* **B17**, 1365 (2003)
[26] M A Castagnino, H Giacomini and L Lara, *Phys. Rev.* **D63**, 044003 (2001)
[27] H Suzuki and M Tanaka, *Phys. Rev.* **D49**, 6692 (1994)
[28] R Haydock, *Phys. Rev.* **B63**, 125413 (2001)
[29] T Geisel, R Ketzmerick and G Petschel, *Phys. Rev. Lett.* **66**, 1651 (1991)
[30] O Watanabe, K Ogura, T Cho and Md R Amin, *Phys. Rev.* **E63**, 056503 (2001)
[31] H R Lewis Jr, *Phys. Rev. Lett.* **27**, 510 (1967)
[32] H R Lewis Jr and W B Riesenfeld, *J. Math. Phys.* **10**, 1458 (1969)
[33] K-H Yeon, D-H Kim, C-I Um, T F George and L N Pandey, *Phys. Rev.* **A55**, 4023 (1997)
[34] D C Khandekar, S V Lawande and K V Bhagwat, *Path-integral methods and their applications* (World Scientific, Singapore, 1993) p. 10
[35] A Erdély, *Higher transcendental functions* (McGraw-Hill, New York, 1953) Vol. II, pp. 116–196
[36] A Erdély, *Higher transcendental functions* (McGraw-Hill, New York, 1953) Vol. I, pp. 255–267
[37] I A Pedrosa, *Phys. Rev.* **A55**, 3219 (1997)
[38] M S Abdalla and R K Colegrave, *Phys. Rev.* **A32**, 1958 (1985)
[39] R K Colegrave and M S Abdalla, *J. Phys.* **A15**, 1549 (1982)