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The constancy of the contact angle in viscous liquid motions with pinned contact lines

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Abstract. Consider motion initiated in a viscous liquid in a smooth walled container. The liquid is initially at rest under uniform pressure from an inert gas of negligible inertia. We show that if the contact line is pinned and the interface is single valued, the contact angle has to remain constant throughout the motion. This is true even for motions of finite amplitude. Some implications of the result are discussed.

Keywords. Contact angle; viscous motions; pinned contact line.

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1. Introduction

The classical theory of wave motion in fluids [1] deals with waves in an inviscid fluid. Even in the simplest case of periodic waves in an infinite ocean one could, on account of symmetry, associate the motion with one between vertical plates spaced a wavelength apart, with the contact angle α between the wave surface and the plate being $\pi/2$. But even in the more complex case of inviscid motion in closed basins bounded by vertical walls, we find that in the classical theory the contact angle is always $\pi/2$. This is in fact purely a consequence of the fact that in the classical theory the undisturbed surface is always assumed to be flat with the wave motion being a perturbation about this surface; this is proved in Benjamin and Ursell [2], who remark that this is an important consequence of the boundary conditions.

The contact line is the line at which the liquid-free surface meets the walls of the container. In the inviscid wave motions mentioned above, the contact line is in general free to move. When the contact line moves, viscous effects in this neighbourhood which are not easy to estimate and which seriously affect damping rates add to the uncertainties in any attempt to compare the results of theoretical calculations with experimental results. A method of avoiding this difficulty was suggested by Benjamin and Scott [3]. They investigated a class of motions where the contact line was pinned; this can be done for example by filling the container until the contact line coincides with the sharp upper edge of the side wall. Benjamin and Scott [3] showed that for this class of pinned contact line motions, which avoid contact line hysteresis, there was excellent agreement between the theoretical predictions and the experimental results. Since then the class of pinned edge or fixed contact line motions have been used extensively in investigations of wave motion [4,5].

By the contact angle we mean the angle between the local normal to the bounding solid surface, pointing towards the liquid and the normal to the liquid–gas interface, pointing into the gas, at the contact line. One of the oldest and most applicable models of the neighbourhood of the contact line formed by a stationary liquid–gas interface and a solid boundary is one where the contact angle is assumed to be a constant $\alpha_{\rm s}$ [6]. $\alpha_{\rm s}$, the static or microscopic contact angle, is assumed to be a property of the liquid, gas and solid alone. For very smooth solid boundaries and many triplets prepared carefully, the model is a good representation of reality. On the other hand, if the walls are rough or if the fluids are contaminated, for example, the apparent or macroscopic contact angle $\alpha_{\rm a}$ can take a range of values around $\alpha_{\rm s}$, $\alpha_{\rm R} < \alpha_{\rm a} < \alpha_{\rm A}$, where $\alpha_{\rm A}$ and $\alpha_{\rm R}$ are the advancing and receding contact angles respectively. Naturally the situation is more complicated in the dynamic case where the contact line is in motion. As has been pointed out by Dussan [7] the measured dynamic contact angle, or the apparent dynamic contact angle, in these cases is probably not the actual contact angle because of the distortion of the meniscus in the neighbourhood of the moving contact line.

We wish here to investigate contact angle behaviour in the motion of a viscous liquid. In order to avoid the uncertainties associated with a moving contact line we will restrict ourselves to motions in which it is fixed. Also, since we do not want to deal with situations where the static meniscus itself is not unique, we will through out assume that there is no contact angle hysteresis. The method we use is very similar to the one used by Benjamin and Ursell [2]; however we will not restrict ourselves to the linearized case or to one where the mean interface is flat and there will be no restriction on the shape of the container. We will in fact show that the contact angle is actually constant for this class of motions. Some implications of this result will be pointed out in §3.

2. Analysis

We consider a viscous liquid in an arbitrary, smooth walled, rigid, three-dimensional container. The fluid motion is generated by the motion of the container which is, along with the fluid, initially at rest. To fix ideas, the motion is assumed to start and continue with a uniform pressure over the free surface; we will assume the gas to be passive, i.e. it only exerts a constant pressure on the liquid-free surface. In §2.1, we write down the equations governing the motion. In §2.2, we first treat the 2-D case where the analysis is simpler; the general 3-D case is dealt with in §2.3. The reader can get the gist of the argument from §2.2, with §2.3 differing from the former only in some technical details.

2.1 *Governing equations*

We write the equations in a reference frame attached to the moving container. The container wall is given by $f(x, y, z) = 0$. Rectangular Cartesian coordinates are employed with gravity in the negative z-direction. Our analysis is restricted to the case when the interface is representable by a single valued smooth function, i.e., by $z = \eta(x, y, t)$. The interface motion can be of finite amplitude however. The equations governing the liquid motion are the continuity and the Navier–Stokes equations:

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mathbf{F} + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u}, \tag{2.2}$$

where $\mathbf{u}$ is the liquid velocity, $\mathbf{F}$ is the net body force, both real and fictitious and Re a suitably defined Reynolds number. $\mathbf{F}$ can be an arbitrary function of time. These have to be solved subject to the boundary conditions on (a) the container wall and (b) the interface. The condition on the wall is the no-slip condition $u = v = w = 0$ where u, v and w are the x, y and z components of velocity. The conditions on the interface $z = \eta(x, y, t)$ are (for example Johnson [8])

$$\eta_t + u\eta_x + v\eta_y = w, \tag{2.3a}$$

$$\eta_x(v_z + w_y) - \eta_y(u_z + w_x) + 2\eta_x\eta_y(u_x - v_y) - (\eta_x^2 - \eta_y^2)(u_y + v_x) = 0, \tag{2.3b}$$

$$\begin{aligned} 2\eta_x^2(u_x - w_z) &+ 2\eta_y^2(v_y - w_z) + 2\eta_x\eta_y(u_y + v_x) \\ &+ (\eta_x^2 + \eta_y^2 - 1)\{\eta_x(u_z + w_x) + \eta_y(v_z + w_y)\} = 0, \end{aligned} \tag{2.3c}$$

$$\begin{aligned} p - \frac{2}{\mathrm{Re}} & \frac{\eta_x^2 u_x + \eta_y^2 v_y - \eta_x(u_z + w_x) - \eta_y(v_z + w_y) + \eta_x\eta_y(u_y + v_x) + w_z}{1 + \eta_x^2 + \eta_y^2} \\ &= p_{\mathrm{a}} - \frac{1}{\mathrm{Bo}}\kappa. \end{aligned} \tag{2.3d}$$

Equation (2.3a) is a statement of the interface integrity; material particles originally on the interface remain on it. Equations (2.3b–d) are conditions on the surface stresses at the interface. The surface stresses are resolved to produce any two independent tangential stresses and the normal stress; eqs (2.3b,c) state that the tangential stresses at the interface are zero (no wind) and eq. (2.3d) is a description of the normal stress which is prescribed predominantly by the ambient pressure above the surface but also includes viscous and surface tension contributions. Bo is a suitably defined Bond number, p_{a} is the constant ambient pressure over the free surface and κ is the local interface curvature. The container wall is smooth, i.e., the normal to the wall exists at each point. The interface $z = \eta(x, y, t)$ is assumed to intersect the smooth container wall $f(x, y, z) = 0$ in a smooth curve called the contact line. The possibly time dependent angle α made by the contact line with the wall is given by

$$\frac{f_z - f_x\eta_x - f_y\eta_y}{\|\nabla f\|[1+\eta_x^2+\eta_y^2]^{1/2}} = \cos\alpha(t), \tag{2.4}$$

where the LHS is evaluated at any point $(x_c, y_c, \eta(x_c, y_c, t))$ on the contact line. We will show, over the next two sections, that $\alpha(t)$ is actually constant and equal to its initial value α_s.

2.2 *Two-dimensional motions*

The container is of arbitrary cross-section in the x–z plane and the motion is 2-D. The contact line in this case consists of just two points (A and B say). Only the appropriate terms are retained in eqs (2.3a,c,d); eq. (2.3b) is entirely superfluous. The no-slip condition and (2.3a,c,d) are valid at A and B, represented in general by $(x_c, \eta(x_c, t))$. The no-slip condition and (2.3a) together imply $\eta_t(x_c, t) = 0$ or that $\eta(x_c, t) = \eta(x_c, 0) \equiv \eta_c(x_c)$.

The no-slip condition also implies

$$\nabla u \times \nabla f = \nabla w \times \nabla f = 0$$

from which we get the relations

$$u_x\, f_z = u_z\, f_x, \quad w_x f_z = w_z\, f_x. \tag{2.5}$$

Equation (2.5) is valid everywhere on the wall and in particular on the contact line. We now have two cases to consider – when neither f_x nor f_z vanishes on the contact line and when either of them vanishes. The former condition is obtained when the contact line does not pass through an extremum of f and is typical even though the proof is simpler for the atypical case when either f_x or f_z vanishes, as we will see in the next section.

2.2.1 *Case* I. Let us assume $f_x \neq 0$ and $f_z \neq 0$. Using the relations (2.5) and the continuity equation in (2.3c), we get for the points $(x_c, \eta_c(x_c))$ on the contact line the equation

$$u_x \left\{4\eta_x + (\eta_x^2 - 1)\left(\frac{f_z}{f_x} - \frac{f_x}{f_z}\right)\right\}\eta_x = 0. \tag{2.6}$$

The possible solutions are that for given $t = t_0 \geq 0$, (i) $u_x(x_c, \eta_c(x_c), t_0) = 0$ or (ii) $4\eta_x + (\eta_x^2 - 1)\,(f_z/f_x - f_x/f_z) = 0$ or (iii) $\eta_x = 0$.

(i) If $u_x(x_c, \eta_c(x_c), t_0) = 0$, then by continuity and (2.5), $u_z(x_c, \eta_c(x_c), t_0) = w_x(x_c, \eta_c(x_c), t_0) = w_z(x_c, \eta_c(x_c), t_0) = 0$.

Now, differentiate (2.3a) with respect to x to obtain an equation valid on the interface and hence on the contact line:

$$\begin{aligned}\eta_{xt}(x,t) + u(x,\eta(x,t),t)\eta_{xx}(x,t) + [u_z(x,\eta(x,t),t)\eta_x + u_x(x,\eta,t)]\eta_x(x,t)\\ = w_x(x,\eta,t) + w_z(x,\eta,t)\eta_x.\end{aligned} \tag{2.7}$$

Using the no-slip condition and the fact that all spatial velocity derivatives are zero on the contact line, we get

$$\frac{\partial}{\partial t}\frac{\partial \eta}{\partial x}\bigg|_{x=x_c,t=t_0} = 0. \tag{2.8}$$

(ii) The other possibility $4\eta_x + (\eta_x^2 - 1)(f_z/f_x - f_x/f_z) = 0 \Rightarrow$ the existence of atmost two real solutions $\eta_x^{\pm}$ provided $f_x(x_c, \eta_c(x_c)) \neq \pm f_z(x_c, \eta_c(x_c))$ and one solution $\eta_x = 0$ if $f_x(x_c, \eta_c(x_c)) = \pm f_z(x_c, \eta_c(x_c))$. In either case, suppose $u_x(x_c, \eta_c(x_c), t_0) \neq 0$. Equation (2.7) $\Rightarrow \eta_{xt}(x_c, t_0) \neq 0$ in general. This means that for arbitrarily small $\epsilon > 0$, we have $\eta_x(x_c, t_0 + \epsilon) \neq \eta_x^{\pm}$ or 0 depending on the case being considered which then implies, by (2.6), that $u_x(x_c, \eta_c(x_c), t_0 + \epsilon) = 0$. Since this holds true for arbitrarily small $\epsilon > 0$ and since u_x is continuous at $t = t_0$, we then have $u_x(x_c, \eta_c(x_c), t_0) = 0$. It was shown earlier that this leads to (2.8).

(iii) $\eta_x = 0$ was already considered as part of (ii). Since all three cases lead to $u_x(x_c, \eta_c(x_c), t_0) = 0$ and since (2.6) is true for all time, it follows that $u_x(x_c, \eta_c(x_c), t) = 0$ for all time. From the arguments leading to (2.8), it follows that

$$\frac{\partial}{\partial t}\frac{\partial \eta(x,t)}{\partial x}\bigg|_{x=x_c} = 0 \tag{2.9}$$

for all time. However, from (2.4), it is clear that the LHS is independent of time and thus $\alpha(t) = \alpha_{\mathrm{s}}$.

2.2.2 *Case* II. Assume without loss of generality $f_z(x_c, \eta_c(x_c)) = 0$ for some point $(x_c, \eta_c(x_c))$ on the contact line. This implies $u_z(x_c, \eta_c(x_c), t) = 0$ and $w_z(x_c, \eta_c(x_c), t) = 0$ by (2.5). By continuity we have $u_x(x_c, \eta_c(x_c), t) = 0$. The above relations are valid for all time. Now, from (2.3c), if at some $t = t_0$, $\eta_x(x_c, t) \neq 0, \pm 1$, then we must have $w_x(x_c, \eta_c(x_c), t_0) = 0$ and we obtain (2.8). $\eta_x(x_c, t_0) = 0, \pm 1$ leads to (2.8) again, by arguments similar to the ones outlined in the previous section. Once again, we arrive at the conclusion that the contact angle is independent of time.

2.3 *Three-dimensional motions*

The no-slip condition and eqs (2.3a–d) are valid at any point $(x_c, y_c, \eta(x_c, y_c, t))$ on the contact line, it being the intersection of $f(x, y, z) = 0$ and $z = \eta(x, y, t)$. The no-slip condition and eq. (2.3a) together imply $\eta_t(x_c, y_c, t) = 0$ or that $\eta(x_c, y_c, t) = \eta(x_c, y_c, 0) \equiv \eta_c(x_c, y_c)$.

The no-slip condition also implies

$$\nabla u \times \nabla f = \nabla v \times \nabla f = \nabla w \times \nabla f = 0$$

from which we get six relations of the type

$$\frac{u_x}{f_x} = \frac{u_y}{f_y} = \frac{u_z}{f_z}. \tag{2.10}$$

We now have three cases to consider: (a) $f_x, f_y, f_z \neq 0$ anywhere on the contact line, (b) only one of f_x, f_y, f_z vanish at a given point on the contact line and (c) two of f_x, f_y, f_z vanish at a given point on the contact line. Again, the first case is the typical one just like in the 2-D case. Case (b) will in general define curves on the container surface on which one of the derivatives will vanish; we are interested in points of intersection of such curves with the contact line. When two of the derivatives vanish, they do so at isolated points; if these points happen to lie on the contact line, case (c) has to be considered.

2.3.1 *Case* I. Using the relations (2.10) and the continuity equation in (2.3b,c), we get for any point $(x_c, y_c, \eta_c(x_c, y_c))$ on the contact line the two equations

$$\begin{aligned} &u_x \left\{ 2\eta_x\eta_y - \eta_x \frac{f_y}{f_z} + \eta_y \left(\frac{f_x}{f_z} - \frac{f_z}{f_x} \right) + (\eta_y^2 - \eta_x^2) \frac{f_y}{f_x} \right\} \\ &+ v_y \left\{ \eta_x \left(\frac{f_z}{f_y} - \frac{f_y}{f_z} \right) + \eta_y \frac{f_x}{f_z} - 2\eta_x\eta_y + (\eta_y^2 - \eta_x^2) \frac{f_x}{f_y} \right\} = 0, \end{aligned} \tag{2.11a}$$

$$\begin{aligned} &u_x \left\{ 4\eta_x^2 + 2\eta_y^2 + 2\eta_x\eta_y \frac{f_y}{f_x} + \left(\eta_x^2 + \eta_y^2 - 1 \right) \left[\eta_x \left(\frac{f_z}{f_x} - \frac{f_x}{f_z} \right) - \eta_y \frac{f_y}{f_z} \right] \right\} \\ &+ v_y \left\{ 2\eta_x^2 + 4\eta_y^2 + 2\eta_x\eta_y \frac{f_x}{f_y} + \left(\eta_x^2 + \eta_y^2 - 1 \right) \right. \\ &\times \left. \left[\eta_y \left(\frac{f_z}{f_y} - \frac{f_y}{f_z} \right) - \eta_x \frac{f_x}{f_z} \right] \right\} = 0. \end{aligned} \tag{2.11b}$$

Letting $a_{11}(x_c, y_c, \eta_c(x_c, y_c), t), a_{21}(x_c, y_c, \eta_c(x_c, y_c), t)$ be the coefficients of u_x and $a_{12}(x_c, y_c, \eta_c(x_c, y_c), t)$, $a_{22}(x_c, y_c, \eta_c(x_c, y_c), t)$ be the coefficients of v_y, the possible solutions are that for given $t = t_0$, $u_x(x_c, y_c, \eta_c(x_c, y_c), t_0) = v_y(x_c, y_c, \eta_c(x_c, y_c), t_0) = 0$ or $\det(A) = 0$ where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The trivial solution $u_x(x_c, y_c, \eta_c(x_c, y_c), t_0) = v_y(x_c, y_c, \eta_c(x_c, y_c), t_0) = 0$ implies, on using (2.10) and continuity, that all the other derivatives on the contact line are zero as well. Now, differentiate (2.3a) with respect to x and y to obtain a pair of equations valid on the interface and hence on the contact line:

$$\begin{aligned} &\eta_{xt}(x, y, t) + u(x, y, \eta(x, y, t), t)\eta_{xx}(x, y, t) + [u_z(x, y, \eta(x, y, t), t)\eta_x \\ &+ u_x(x, y, \eta, t)]\eta_x(x, y, t) + v(x, y, \eta, t)\eta_{xy}(x, y, t) \\ &+ [v_z(x, y, \eta, t)\eta_x + v_x(x, y, \eta, t)]\eta_y(x, y, t) \\ &= w_x(x, y, \eta, t) + w_z(x, y, \eta, t)\eta_x, \end{aligned} \tag{2.12a}$$

$$\begin{aligned} &\eta_{yt}(x, y, t) + u(x, y, \eta(x, y, t), t)\eta_{xy}(x, y, t) + [u_z(x, y, \eta(x, y, t), t)\eta_y \\ &+ u_y(x, y, \eta, t)]\eta_x(x, y, t) + v(x, y, \eta, t)\eta_{yy}(x, y, t) \\ &+ [v_z(x, y, \eta, t)\eta_y + v_y(x, y, \eta, t)]\eta_y(x, y, t) \\ &= w_y(x, y, \eta, t) + w_z(x, y, \eta, t)\eta_y. \end{aligned} \tag{2.12b}$$

Using the no-slip condition and the fact that all spatial velocity derivatives are zero on the contact line, we get that

$$\frac{\partial}{\partial t}\frac{\partial\eta}{\partial x}\bigg|_{x=x_c,y=y_c,t=t_0} = \frac{\partial}{\partial t}\frac{\partial\eta}{\partial y}\bigg|_{x=x_c,y=y_c,t=t_0} = 0. \tag{2.13}$$

If on the other hand $\det(A) = 0$ at $t = t_0$, it cannot be concluded that $u_x(x_c, y_c, \eta_c(x_c, y_c), t_0) = v_y(x_c, y_c, \eta_c(x_c, y_c), t_0) = 0$. Supposing that these are not zero, we have, from (2.12), $\eta_{xt}(x_c, y_c, t_0) \neq 0, \eta_{yt}(x_c, y_c, t_0) \neq 0$ in general. Now, there are two possibilities:

(i) $\exists n$ such that $\partial^n \det[A(t)] \,/\, \partial t^n \neq 0|_{t=t_0}$ for some n or

(ii) $\det[A(t)] = 0$ for all t.

The first possibility means, that for arbitrarily small $\epsilon > 0$, $\det[A(t_0 + \epsilon)] \neq 0$ which implies that $u_x(x_c, y_c, \eta_c(x_c, y_c), t_0 + \epsilon) = v_y(x_c, y_c, \eta_c(x_c, y_c), t_0 + \epsilon) = 0$. Since this holds true for arbitrarily small $\epsilon > 0$ and since u_x and v_y are continuous at $t = t_0$, we then have $u_x(x_c, y_c, \eta_c(x_c, y_c), t_0) = v_y(x_c, y_c, \eta_c(x_c, y_c), t_0) = 0$. It was shown earlier that this leads to (2.13).

If $\det[A(t)] = 0$ for all t, we proceed as follows. For clarity, we introduce β and γ to represent $\eta_x(x = x_c, y = y_c, t)$ and $\eta_y(x = x_c, y = y_c, t)$, i.e., with the derivatives evaluated on the contact line. The definition of $\det[A]$ yields a relation between β and γ which we write as

$$g_1(\beta, \gamma) = \det[A]. \tag{2.14}$$

Now the contact line can be parametrized by an arc-length parameter s as $\eta = \eta_{\mathrm{s}}(s)$. Note that η_{s} is independent of time as the contact line is pinned. We have

$$\frac{\mathrm{d}\eta_{\mathrm{s}}}{\mathrm{d}s} = \beta\frac{\mathrm{d}x}{\mathrm{d}s} + \gamma\frac{\mathrm{d}y}{\mathrm{d}s}. \tag{2.15}$$

γ can be eliminated between (2.14) and (2.15) to obtain $\det[A]$ entirely in terms of β which is written, say as,

$$\det[A] = g_2(\beta; f_x, f_y, f_z). \tag{2.16}$$

Since the LHS of (2.16) is independent of time, the RHS is too which implies β is independent of time. In a similar manner, it is shown that γ is independent of time as well. Thus, (2.13) holds even in this case. Since this exhausts all cases and since t_0 is arbitrary, it is thus established that

$$\frac{\partial}{\partial t}\frac{\partial\eta}{\partial x}\bigg|_{x=x_c,y=y_c} = \frac{\partial}{\partial t}\frac{\partial\eta}{\partial y}\bigg|_{x=x_c,y=y_c} = 0 \tag{2.17}$$

for all time. From (2.4), it is readily seen that the LHS is independent of time and so is the contact angle, α.

2.3.2 *Case* II. We can assume, without loss of generality, $f_z(x_c, y_c, \eta_c(x_c, y_c)) = 0$ at one or more points $(x_c^*, y_c^*, \eta_c(x_c^*, y_c^*))$ on the contact line. Using (2.10) and continuity, we can again write eqs (2.3b,c) in terms of only two of the velocity derivatives (this time we choose u_x, w_x):

$$u_x\left[4\eta_x\eta_y + \left(\eta_y^2 - \eta_x^2\right)\left(\frac{f_y}{f_x} - \frac{f_x}{f_y}\right) - \eta_x\frac{f_z}{f_y} - \eta_y\frac{f_z}{f_x}\right]$$
$$+w_x\left[\eta_x\frac{f_y}{f_x}\left(1-\frac{f_z^2}{f_y^2}\right) + 2\eta_x\eta_y\frac{f_z}{f_x} - \eta_y + \left(\eta_x^2-\eta_y^2\right)\frac{f_z}{f_y}\right] = 0, \tag{2.18a}$$

$$u_x\left[2\left(\eta_x^2-\eta_y^2\right) + 2\eta_x\eta_y\left(\frac{f_y}{f_x}-\frac{f_x}{f_y}\right) + \left(\eta_x^2+\eta_y^2-1\right)\left(\eta_x\frac{f_z}{f_x} - \eta_y\frac{f_z}{f_y}\right)\right]$$
$$+w_x\left[\left(\eta_x^2+\eta_y^2-1\right)\left(\eta_x + \eta_y\frac{f_y}{f_x}\left(1-\frac{f_z^2}{f_y^2}\right)\right)\right.$$
$$\left.-2\eta_x^2\frac{f_z}{f_x} - 4\eta_y^2\frac{f_z}{f_x} - 2\eta_x\eta_y\frac{f_z}{f_y}\right] = 0. \tag{2.18b}$$

This is similar to the system (2.11) and the analysis following (2.11) applies exactly leading to the same conclusion of constant angle.

2.3.3 *Case* III. Assume that there exist points $(\hat{x}_c, \hat{y}_c, \eta_c(\hat{x}_c, \hat{y}_c))$ on the contact line such that $f_x(\hat{x}_c, \hat{y}_c, \eta_c(\hat{x}_c, \hat{y}_c)) = f_y(\hat{x}_c, \hat{y}_c, \eta_c(\hat{x}_c, \hat{y}_c)) = 0$. It is easy to see that $u_x, u_y, v_x, v_y, w_x, w_y$ and w_z are all zero at the points $(\hat{x}_c, \hat{y}_c, \eta_c(\hat{x}_c, \hat{y}_c))$. Equations (2.3b,c) are now written in terms of the possibly non-zero u_z and v_z as

$$-\eta_y u_z + \eta_x v_z = 0, \tag{2.19a}$$

$$\eta_x u_z + \eta_y v_z = 0. \tag{2.19b}$$

If $\Delta \equiv \eta_x^2 + \eta_y^2 \neq 0$, then $u_z = v_z = 0$ and $\eta_{xt} = \eta_{yt} = 0$. If $\Delta = 0$ which can happen only if $\eta_x = \eta_y = 0$, then eqs (2.12a,b) show that $\eta_{xt} = \eta_{yt} = 0$ at the points $(\hat{x}_c, \hat{y}_c, \eta_c(\hat{x}_c, \hat{y}_c))$ which is what we were required to prove.

3. Discussion

We have shown in the previous section that if the contact line is pinned in a viscous motion, the contact angle will remain constant. This result holds even for motions of finite amplitude and for essentially arbitrary containers. More surprising is that not only is the result independent of the Bond number, but that it holds even if capillarity is absent or assumed to be negligible. Of course, if surface tension is absent and the motion starts from rest, the interface in this case will be flat and the contact angle of $\pi/2$ will be maintained.

A natural question that arises is: what is the equivalent result in inviscid flow? In this case, slip occurs at the walls and so the contact line cannot be pinned and has to be allowed to move. Benjamin and Ursell [2] proved that for small irrotational, three-dimensional motions about a flat interface in a vertical container, the contact angle would be preserved. We had originally thought that the result would be generally true. However, on more careful analysis, Shankar and Kidambi [9], show that it is only in the case of $\alpha_s = \pi/2$ that the contact angle is preserved in inviscid, irrotational motions. Even this result is restricted to arbitrary two-dimensional motions and linearized 3D motions in containers with flat sides. It may appear puzzling that the absence of vorticity is crucial for the result to hold in inviscid motion but that it still holds in viscous motion where the flow is in general vortical. However, as was shown, all the velocity gradients and hence the vorticity too vanish at the contact line.

It was to avoid the complications arising from a moving contact line that we chose to deal only with viscous motions involving pinned contact lines. In §2 we showed that the contact angle remained unchanged from its initial, static value α_s.

When one first comes across the notion of a pinned contact line in viscous flow one is a little surprised that that there can be any other type of motion; after all in viscous flow the no-slip condition holds and one would guess that this would guarantee that the contact line stays fixed. Of course all such misconceptions vanish when we actually look at the contact line in a glass of water which is given a shake – the contact line certainly moves in general. There is of course no violation of the no-slip condition. The moving contact line is made up of new particles of fluid brought to the wall from the interface, for example, by the lateral motion of the wall or interface. However, there are a number of circumstances where the contact line is fixed. If the central portion of fluid in a container is stirred gently the contact line can be observed to be stationary. In certain low-amplitude slosh regimes too, the contact line can remain fixed [10]. The third example is the one mentioned in §1 where the contact line is pinned at the sharp upper edge of the side wall of a container. Here, however, the static contact angle is not unique and lies in a range of values from the normal α_s to some maximum that can even exceed $\pi/2$. Our analysis would not quite apply here as we have assumed the existence of a normal to the wall, which is not the case at an edge.

It was to solve the general three-dimensional case that in §2 we assumed some hypotheses that are not really necessary for certain simpler situations. For example, the analysis can be extended to show that the contact angle continues to remain constant for some cases where no reference frame exists in which all boundary surfaces are at rest, provided the boundaries move in such a way that they are described at all times by their initial definitions like $f(x, y, z) = 0$ which do not involve time. An example is the evolution of an air–liquid interface between two parallel vertical plates one of which is at rest and the other is dragged out of the liquid in its own plane. The analysis shows that the contact angles at both the plates have to be maintained at the initial value even though no rest frame exists. This result may appear surprising because if the moving plate is now stopped, we know that the static interface will ultimately have to take on its original shape even though the contact line on the moving plate is now stationary above the liquid. There is no contradiction here. As the liquid is dragged up, the contact angle will remain constant at the upward moving contact line. If the plate is now stopped and liquid starts draining down, the contact angle at the contact line, provided it is the original line, will remain constant even as the draining film keeps thinning. The original static meniscus will be approached only as $t \to \infty$ with an ever thinning film on the wall. Of course, our result will fail once the film is so thin that continuum ideas fail.

A similar, perhaps simpler, situation is one where fluid drains out of a tube. Consider a tube or container partially filled with viscous liquid which is initially stationary. The fluid is slowly drained out of the container by opening a valve near the bottom. Analysis similar to the one in §2 shows that the contact angle has to remain constant as the liquid drains out. Once again experience with thin liquids like water make the result appear surprising as experience suggests the rapid formation of a film on the wall as the free surface appears to descend almost unchanged in shape. The present analysis suggests that at least initially, until the film thickness is too small to be described as a continuum, the microscopic contact angle will be constant. Things become clearer if we consider the draining of a very viscous liquid. Now the film thickness in slow draining will evidently be large for even casually observable times, and the idea of a constant contact angle is not too surprising.

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