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Collineations of the curvature tensor in general relativity

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Abstract. Curvature collineations for the curvature tensor, constructed from a fundamental Bianchi Type-V metric, are studied. We are concerned with a symmetry property of space-time which is called curvature collineation, and we briefly discuss the physical and kinematical properties of the models.

Keywords. Collineation; Killing vectors; Ricci tensor; Riemannian curvature tensor.

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1. Introduction

The general theory of relativity, which is a field theory of gravitation, is described by the Einstein field equations. These equations whose fundamental constituent is the space-time metric g_{ij}, are highly non-linear partial differential equations and, therefore it is very difficult to obtain exact solutions. They become still more difficult to solve if the space-time metric depends on all coordinates [1–3]. This problem however, can be simplified to some extent if some geometric symmetry properties are assumed to be possessed by the metric tensor. These geometric symmetry properties are described by Killing vector fields and lead to conservation laws in the form of first integrals of a dynamical system [4,5]. There exists, by now, a reasonably large number of solutions of the Einstein field equations possessing different symmetry structure [6]. These solutions have been further classified according to their properties and groups of motions admitted by them [7].

Katzine *et al* [8] were the pioneers in carrying out a detailed study of curvature collineation in the context of the related particle and field conservation laws that may be admitted in the standard form of general relativity [9].

A Riemannian space V_n is said to admit a symmetry called a curvature collineation (CC) provided there exists a vector ξ^i such that [10–12]

$$\pounds_\xi (R^h_{ijk}) = 0,$$

where R^h_{ijk} is the Riemannian curvature tensor [13] and

$$R^h_{ijk} = \left\{ \begin{matrix} h \\ ik \end{matrix} \right\}_{,j} - \left\{ \begin{matrix} h \\ ij \end{matrix} \right\}_{,k} + \left\{ \begin{matrix} m \\ ik \end{matrix} \right\} \left\{ \begin{matrix} h \\ mj \end{matrix} \right\} - \left\{ \begin{matrix} m \\ ij \end{matrix} \right\} \left\{ \begin{matrix} h \\ mk \end{matrix} \right\} \quad (1.1)$$

and Ł_ξ denotes the Lie derivative with respect to vector ξ^i [14].

In this paper we are concerned with a symmetry property of space-time which is called curvature collineation and physical and kinematical properties of the models are discussed for the fundamental Bianchi Type-V metric in general relativity [15–18].

2. Curvature collineation

The fundamental form of Bianchi Type-V metric is

$$\pounds \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2\alpha}\mathrm{d}x^2 + \mathrm{e}^{2x+2\beta}\mathrm{d}y^2 + \mathrm{e}^{2x+2\gamma}\mathrm{d}z^2, \quad (2.1)$$

where α, β, γ are functions of t alone.

A brief outline of the procedure for finding the CC vector ξ^i admitted by (2.1) is presented. Starting with (1.1), the equations to be solved for the ξ^i can be expressed in the form

$$\pounds_\xi(R^h_{ijk}) = (R^h_{ijk,m})\xi^m - R^m_{ijk}\xi^h_{,m} + R^h_{mjk}\xi^m_{,i} + R^h_{imk}\xi^m_{,j} + R^h_{ijm}\xi^m_{,k} = 0. \quad (2.2)$$

From the algebraic symmetries on the indices we find that in a V_4-equation (2.2) formally represents 96 equations, evaluation of these equations by the use of the metric tensor defined by (2.1) leads to 84 sets of equations (redundant and trivial equations have been omitted).

Case I: When $\alpha = \beta = \gamma$, we get the following sets of equations:

$$\pounds_\xi(R^4_{223}) = 0 \Rightarrow \xi^4_{,3} = 0 \quad (2.3)$$

$$\pounds_\xi(R^4_{323}) = 0 \Rightarrow \xi^4_{,2} = 0 \quad (2.4)$$

$$\pounds_\xi(R^4_{331}) = 0 \Rightarrow \xi^4_{,1} = 0 \quad (2.5)$$

$$\pounds_\xi(R^3_{431}) = 0 \Rightarrow \xi^1_{,4} = 0 \quad (2.6)$$

$$\pounds_\xi(R^1_{412}) = 0 \Rightarrow \xi^4_{,2} = 0 \quad (2.7)$$

$$\pounds_\xi(R^1_{314}) = 0 \Rightarrow \xi^3_{,4} = 0 \quad (2.8)$$

$$\pounds_\xi(R^1_{123}) = 0 \Rightarrow \xi^1_{,3} + \mathrm{e}^{2x}\xi^3_{,1} = 0 \quad (2.9)$$

$$\pounds_\xi(R^2_{223}) = 0 \Rightarrow \xi^2_{,3} + \xi^2_{,2} = 0 \quad (2.10)$$

$$\pounds_\xi(R^3_{123}) = 0 \Rightarrow \xi^1_{,2} + \mathrm{e}^{2x}\xi^2_{,1} = 0 \quad (2.11)$$

$$\pounds_\xi(R^2_{323}) = 0 \Rightarrow (\alpha_4^2\mathrm{e}^{2\alpha} - 1)_{,4}\xi^4 + 2(\alpha_4^2\mathrm{e}^{2\alpha} - 1)(\xi^2_{,2} + \xi^1) = 0 \quad (2.12)$$

$$\pounds_\xi(R^3_{131}) = 0 \Rightarrow (\alpha_4^2\mathrm{e}^{2\alpha} - 1)_{,4}\xi^4 + 2(\alpha_4^2\mathrm{e}^{2\alpha} - 1)\xi^1_{,1} = 0 \quad (2.13)$$

$$\pounds_\xi(R^1_{414}) = 0 \Rightarrow (\alpha_{44} + \alpha_4^2)_{,4}\xi^4 + 2(\alpha_{44} + \alpha_4^2)\xi^4_{,4} = 0 \tag{2.14}$$

$$\pounds_\xi(R^4_{114}) = 0 \Rightarrow \{e^{2\alpha}(\alpha_{44} + \alpha_4^2)\}_{,4}\xi^4 + 2e^{2\alpha}(\alpha_{44} + \alpha_4^2)\xi^1_{,1} = 0 \tag{2.15}$$

$$\pounds_\xi(R^4_{224}) = 0 \Rightarrow \{e^{2\alpha}(\alpha_{44} + \alpha_4^2)\}_{,4}\xi^4 + 2e^{2\alpha}(\alpha_{44} + \alpha_4^2)(\xi^1 + \xi^2_{,2}) = 0. \tag{2.16}$$

By inspection we find that the following relations exist between equations of the set:

$$(2.9) \text{ and } (2.10) \Rightarrow e^{2x}\xi^2_{,31} = \xi^1_{,32}. \tag{2.17}$$

Also, by (2.17) and (2.11) $\Rightarrow \xi^1_{,23} = 0$

$$\Rightarrow \xi^1 = F(x,y,t) + G(x,z,t) \tag{2.18}$$

$$(2.15) \text{ and } (2.16) \Rightarrow \xi^1_{,1} - \xi^1 = \xi^2_{,2} \tag{2.19}$$

$$(2.14) \Rightarrow \xi^4 = \frac{k}{[\alpha_{44} + \alpha_4^2]^{1/2}}, \quad k \text{ is a constant.} \tag{2.20}$$

$$(2.11) \Rightarrow \xi^2 = A(x,y,t) + B(y,z,t) \tag{2.21}$$

and

$$\xi^3 = C(x,z,t) + D(y,z,t), \quad \text{where } B_z + D_y = 0. \tag{2.22}$$

Thus, for the case in which $k = 0$ we get

$$\begin{aligned} \xi^1 &= py + q + G(z) \\ \xi^2 &= (1/2)pe^{-2x} - (1/2)py^2 - qy - yG(z) + r(z) \\ \xi^3 &= (1/2)e^{-x}(dG/dz) + (1/2)y^2(dG/dz) - y(dr/dz) + s(z) \\ \xi^4 &= 0, \end{aligned} \tag{2.23}$$

where p, q are integral constants.

By use of (2.23), the definition $\pounds_\xi(g_{ij}) = h_{ij} = \xi_{i;j} + \xi_{j;i} = 0$ is satisfied. Therefore $\xi^i (i = 1,2,3,4)$ are Killing vectors.

Case II: $\alpha \neq \beta = \gamma$. In this case (2.1) leads to the following set of equations (redundant and trivial equations have been omitted).

$$\pounds_\xi(R^2_{323}) = 0 \Rightarrow \{e^{2\beta}(e^{-2\alpha} - \beta_4^2)\}_{,4}\xi^4 + 2e^{2\beta}(e^{-2\alpha} - \beta_4^2)(\xi^1 + \xi^3_{,3}) = 0 \tag{2.24}$$

$$\pounds_\xi(R^3_{131}) = 0 \Rightarrow (\alpha_4\beta_4 e^{2\alpha} - 1)_{,4}\xi^4 + 2(\alpha_4\beta_4 e^{2\alpha} - 1)\xi^1_{,1}) = 0 \tag{2.25}$$

$$\pounds_\xi(R^3_{431}) = 0 \Rightarrow (\alpha_4 - \beta_4)_{,4}\xi^4 + (\alpha_4 - \beta_4)(\xi^4_{,4} + \xi^1_{,1}) = 0 \tag{2.26}$$

$$\pounds_\xi(R^4_{331}) = 0 \Rightarrow \{e^{2\beta}(\beta_4 - \alpha_4)\}_{,4}\xi^4 + e^{2\beta}(\beta_4 - \alpha_4)[(2\xi^1 + \xi^1_{,1} + 2\xi^3_{,3} - \xi^4_{,4}) = 0 \tag{2.27}$$

$$\pounds_\xi(R^1_{212}) = 0 \Rightarrow [\mathrm{e}^{2\beta}(\alpha_4\beta_4 - \mathrm{e}^{-2\alpha})]_{,4}\xi^4 + 2\mathrm{e}^{2\beta}(\alpha_4\beta_4 - \mathrm{e}^{-2\alpha})(\xi^1 + \xi^2_{,2}) = 0 \quad (2.28)$$

$$\pounds_\xi(R^1_{414}) = 0 \Rightarrow (\alpha_{44} + \alpha_4^2)_{,4}\xi^4 + 2(\alpha_{44} + \alpha_4^2)\xi^4_{,4} = 0 \quad (2.29)$$

$$\pounds_\xi(R^4_{114}) = 0 \Rightarrow \{\mathrm{e}^{2\alpha}(\alpha_{44} + \alpha_4^2)\}_{,4}\xi^4 + 2\mathrm{e}^{2\alpha}(\alpha_{44} + \alpha_4^2)\xi^1_{,1} = 0 \quad (2.30)$$

$$\pounds_\xi(R^2_{424}) = 0 \Rightarrow (\beta_{44} + \beta_4^2)_{,4}\xi^4 + 2(\beta_{44} + \beta_4^2)\xi^4_{,4} = 0 \quad (2.31)$$

$$\pounds_\xi(R^1_{334}) = 0 \Rightarrow \{\mathrm{e}^{2\beta-2\alpha}(\beta_4 - \alpha_4)\}_{,4}\xi^4 + \mathrm{e}^{2\beta-2\alpha}(\beta_4 - \alpha_4)(2\xi^1 - \xi^1_{,1} + 2\xi^3_{,3} + \xi^4_{,4}) = 0 \quad (2.32)$$

$$\pounds_\xi(R^4_{334}) = 0 \Rightarrow [\mathrm{e}^{2\beta}(\beta_{44} + \beta_4^2)]_{,4}\xi^4 + \mathrm{e}^{2\beta}(\beta_{44} + \beta_{44}^2)(\xi^1 + \xi^3_{,3}) = 0 \quad (2.33)$$

and

$$\xi^i_{,j} = 0, \quad i \neq j \quad (2.34)$$

$$\xi^2_{,2} = \xi^3_{,3}$$

$$(2.34) \Rightarrow \xi^1 \equiv \xi^1(x), \quad \xi^2 \equiv \xi^2(y), \quad \xi^3 \equiv \xi^3(z), \quad \xi^4 \equiv \xi^4(t).$$

Now, (2.25) $\Rightarrow \xi^1_{,1} = m_1 \Rightarrow \xi^1 = m_1 x + c_1$. Similarly,

$$\xi^2 = m_2 y + c_2$$
$$\xi^3 = m_2 z + c_3,$$

where m_1, m_2, c_1, c_2, c_3 are integral constants.

Sub-case I: $m_1 \neq 0$, $\xi^4 \neq 0$.

$$(2.24) \text{ to } (2.34) \Rightarrow \mathrm{e}^\alpha = t + c, \text{ where } c \text{ is a constant.}$$

Therefore, in this case the line element reduces to the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (t+c)\mathrm{d}x^2 + \mathrm{e}^{2x}(t+c)^{2k}\mathrm{d}y^2 + \mathrm{e}^{2x}(t+c)^2\mathrm{d}z^2, \quad k \text{ is a constant} \quad (2.35)$$

which is a flat space.

Sub-case II: If $m_1 = 0$, $\xi^4 = 0$.

In this case the Killing vectors are

$$\xi^i_{(1)} = (0, 1, -y, -z)$$
$$\xi^i_{(2)} = (0, 0, c_2, 0)$$
$$\xi^i_{(3)} = (0, 0, 0, c_3).$$

Sub-case III: If $m_1 = 0, \xi^4 \neq 0$, then by using of (2.24) to (2.34) we get

$$\alpha = \log(at + b),$$
$$\beta = d\log(at + b) + c,$$

where a, b, c, d are integral constants.

Therefore, the line element reduces to the form

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + a^2T^2\mathrm{d}X^2 + \mathrm{e}^{2X}T^{2b}(\mathrm{d}Y^2 + \mathrm{d}Z^2), \tag{2.36}$$

where

$$T = t = b/a,$$
$$X = ax,$$
$$Y = ca^d y,$$
$$Z = ca^d y.$$

With the metric (2.36) the non-vanishing components of the Ricci tensor are

$$R_1^1 = \frac{2}{a^2T^2}(1 - a^2b) \tag{2.37}$$

$$R_2^1 = R_3^3 = \frac{2}{a^2T^2} \tag{2.38}$$

$$R_4^4 = \frac{2b}{T^2}(1 - b) \tag{2.39}$$

$$R = \frac{6}{a^2T^2} - \frac{2b^2}{T^2}. \tag{2.40}$$

The energy momentum tensor T_{ij} in the Einstein field equations is of the form

$$T_{ij} = (\varepsilon + p)v_i v_j + pg_{ij}$$
$$T_1^1 = T_2^2 = T_3^3 = p, T_4^4 = \varepsilon.$$

The Einstein field equations

$$-8\pi GT_{ij} = R_{ij} - 1/2Rg_{ij} \tag{2.41}$$

with $\Lambda = 0$ for the model (2.36) give rise to

$$8\pi Gp = 1/T^2(1/a^2 - b^2) \tag{2.42}$$

$$8\pi G\varepsilon = 1/T^2(2b^2 + b - 3/a^2). \tag{2.43}$$

3. Discussion

In this paper we worked out an important characterization of curvature collineations for the curvature tensor which is constructed from a fundamental Bianchi Type-V metric.

In case $\alpha = \beta = \gamma, \xi^i$ are the Killing vectors. But in case $\alpha \neq \beta = \gamma$ we get the result (2.35) which is a flat space. For the model (2.36) the strong energy conditions $\varepsilon - p > 0$ and $\varepsilon + p > 0$ are satisfied provided $b > a(> 1)$. Each of the energy density ε and pressure p tends to ∞ and zero, respectively, when $T \to 0$ and ∞. In addition, it is noted that ε and p vanish at $a = b = 1$. The expansion scalar $\theta = (2b + 1)/T$, tends to infinity and zero respectively when $T \to 0$ and ∞.

Thus, the model starts with a big bang at $T = 0$ and its rate of expansion vanishes asymptotically. The temperature decreases gradually to zero when the expansion stops.

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