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Charged fluid distribution in higher dimensional spheroidal space-time

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Abstract. A general solution of Einstein field equations corresponding to a charged fluid distribution on the background of higher dimensional spheroidal space-time is obtained. The solution generates several known solutions for superdense star having spheroidal space-time geometry.

Keywords. Charged fluid distribution; higher dimensional space-time.

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1. Introduction

Higher dimensional view of the world geometry suggests that the universe started in $(4+D)$-dimensional phase with extra D dimensions either collapsing and stabilizing or remain at a size close to the Plank length while three others continued to expand. In a pioneering work, Tangerlini [1] has investigated higher dimensional Schwarzschild-like metric which describes the gravitational field of a spherically symmetric star. The study of physics in higher dimensional space-time has been stimulated by investigations of superstring and supergravity theories. For more than a decade there have been a number of articles on higher dimensional space-times both in localized and cosmological domains. In connection with localized sources, higher dimensional generalizations of the spherically symmetric Schwarzschild, Reissner Nordström and Ker space-time have been studied by Myers and Perry [2]. Subsequently, Dianyan [3] has presented higher dimensional version of Schwarzschild–de Sitter, Reissner Nordström–de Sitter, Ker–de Sitter and Ker–Newman space-times with cosmological constant term Λ. The higher dimensional generalizations of Vaidya metric representing a model of radiating star without and with electromagnetic field are presented in [4] and [5] respectively. Subsequently, many authors [6–9] have taken interest in studying the implications of Einstein field equations in higher dimensions. Singh *et al* [10] have discussed higher dimensional analogue of Tolman's solutions. Recently, Ponce de Leon and Cruz [11] have considered higher dimensional Schwarzschild space-time and studied the influence of the extra dimensions on the equilibrium configuration of stars. Vaidya and Tikekar [12] have discussed spheroidal space-time and obtained an exact model of a superdense star. It has been shown that the relativistic space-times which have the associated 3-spaces obtained as hypersurfaces t = constant, 3-spheroids, are suitable to describe the gravitational field in the interior of superdense spherical stars in which the collapse under gravitational attraction is countered by repulsive fluid pressure. Considering the Vaidya–Tikekar [12] spheroidal geometry, Maharaj and Leach [13] have obtained a class of solutions expressible in terms of polynomial and algebraic functions. Mukherjee *et al* [14] have found the general solution for a relativistic star in hydrostatic equilibrium having the spheroidal geometry of the 3-spaces. Their solution possesses all the desirable physical features of a relativistic star. The charged analogue of the above solution is discussed by Tikekar and Singh [15]. Patel and Singh [16] have presented higher dimensional solution of a relativistic star which generates the four-dimensional solution of Mukherjee *et al* [14]. Several authors [17–20] have studied models for a class of compact stars, employing Vaidya–Tikekar [12] geometry of space-time. These studies suggest the spheroidal geometry of the physical space, for distribution of matter in equilibrium, may be of astrophysical interest. In the context of the above we have thought it is worthwhile to obtain the higher dimensional generalization of the solution discussed in [15].

2. Field equations

The investigation begin with the static spherically symmetric metric of $(n+3)$-dimensional space-time

$$\mathrm{d}s^2 = e^{\nu}\mathrm{d}t^2 - e^{\alpha}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2_{n+1}, \tag{1}$$

where

$$\mathrm{d}\Omega^2_{n+1} = \mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\theta_2^2 + \cdots + (\sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_n)\mathrm{d}\theta^2_{n+1}. \tag{2}$$

The metric (1) is geometrically not a product of $4D$ space-time and extra $(n-1)$ internal space. It rather has a topology of R^{n+1} and has been discussed in detail by Wetterich [21].

The Einstein–Maxwell equation may be written as

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi T_{ij}, \tag{3}$$

where the physical content of the space-time is described by [22,23]

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij} + \frac{1}{H}\left[-F_i^k F_{jk} + \frac{1}{4}g_{ij}F_{kl}F^{kl}\right],$$

where $u_i u^i = 1$,

$$H = \frac{(n+2)\pi^{(n+2)/2}}{\Gamma(n+4)/2}, \quad \text{for} \quad n = 1, H = 4\pi. \tag{4}$$

We adopt the comoving coordinates so that $u_i = e^{\nu/2}\delta_i^t$. Here F_{ij} denotes the skew symmetric Maxwell tensor satisfying the Maxwell equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \tag{5}$$

$$\frac{\partial}{\partial x^j}\left(\sqrt{-g}F^{ij}\right) = H\sqrt{-g}J^i, \tag{6}$$

where

$$J^i = \sigma u^i$$

is the $(n+3)$-current and σ is the charge density. A spherical distribution of charged perfect fluid at rest will be accompanied by electrostatic field along the radial direction, so that the only non-vanishing component of F_{ij} is F_{14} which from Maxwell equation yields

$$-F_{14}F^{14} = E^2(r). \tag{7}$$

The 'electric field intensity' E can be obtained by

$$E = \frac{q(r)}{r^{n+1}}, \tag{8}$$

where

$$q(r) = H\int_0^r \sigma r^{n+1} e^{\alpha/2}\mathrm{d}r \tag{9}$$

represents an electric charge within the $(n+3)$-dimensional spherical region of radius r.

The set of Einstein–Maxwell equations (3) for the metric (1) reduces to the following system of equations:

$$e^{-\alpha}\left(\frac{\alpha'}{r} - \frac{n}{r^2}\right) + \frac{n}{r^2} = \frac{16\pi\rho}{n+1} + \frac{2E^2}{n+1}, \tag{10}$$

$$e^{-\alpha}\left(\frac{\nu'}{r} + \frac{n}{r^2}\right) - \frac{n}{r^2} = \frac{16\pi p}{n+1} - \frac{2E^2}{n+1}, \tag{11}$$

$$e^{-\alpha}\left[\frac{\nu''}{2} - \frac{\alpha'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\nu' + n\alpha'}{2r} - \frac{n}{r^2}\right] + \frac{n}{r^2} = \frac{4E^2}{n+1}. \tag{12}$$

Here the prime denotes derivative with respect to the radial coordinate r and eq. (12) is the condition for pressure isotropy.

3. Solution of the field equations

A solution for superdense star has been presented by Vaidya and Tikekar [12] by proposing an ansatz for the geometry of the 3-space embedded in a 4-Euclidean space. The ansatz prescribes a spheroidal geometry for the 3-surface governed by two curvature parameters R and K. For the spheroidal space-times the metric potential e^α is given by

$$e^\alpha = \frac{(1 - Kr^2/R^2)}{(1 - r^2/R^2)}. \tag{13}$$

It is evident that a spheroidal 3-space is essentially spherically symmetric. Its space-time metric is regular and positive definite at all points $r < R$ for $K < 1$. When $K = 1$, the spheroidal 3-space degenerates into a flat 3-space and when $K = 0$ it becomes spherical.

The set of three eqs (10)–(12) containing four variables (ν, ρ, p, E) can be solved only when one more relation among the variables is furnished. For this purpose we introduce the electric field intensity distribution by,

$$E^2 = \frac{\mu^2 r^2}{R^4(1 - Kr^2/R^2)^2}, \tag{14}$$

where μ is a constant. It can be easily seen that, for $K < 0$, the electric field intensity E is always regular. However, if $0 < K < 1$, E is regular for $0 \leq r < (R/\sqrt{K})$.

Using eqs (13) and (14) and introducing $e^\nu = \psi^2(r)$, eqs (10) and (12) become

$$\frac{(1-K)}{R^2}\frac{[2 + n(1 - Kr^2/R^2)]}{[1 - Kr^2/R^2]^2} = \frac{16\pi\rho}{n+1} + \frac{2\mu^2 r^2}{(n+1)R^4(1 - Kr^2/R^2)^2}, \tag{15}$$

$$\left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{Kr^2}{R^2}\right)^{-1}\left[\frac{2\psi'}{r\psi} + \frac{n}{r^2}\right] - \frac{n}{r^2} = \frac{16\pi p}{n+1} - \frac{2\mu^2 r^2}{(n+1)R^4(1 - Kr^2/R^2)^2}, \tag{16}$$

$$\left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{Kr^2}{R^2}\right)\left[\frac{\psi''}{\psi} - \frac{\psi'}{r\psi}\right] - \frac{(1-K)}{R^2}\left[\frac{r\psi'}{\psi} + \frac{nKr^2}{R^2}\right] = \frac{4\mu^2 r^2}{(n+1)R^4}. \tag{17}$$

To solve the above set of eqs (15)–(17), we introduce a new variable z which is defined as

$$z^2 = \frac{K}{K-1}\left(1 - \frac{r^2}{R^2}\right). \tag{18}$$

With the help of eqs (14) and (18), eq. (17) reduces to

$$(1-z^2)\ddot{\psi}(z) + z\dot{\psi}(z) + (1+\lambda)\psi(z) = 0, \tag{19}$$

where overhead dot stands for derivative with respect to z, and

$$\lambda = n - nK - 1 + \frac{4\mu^2}{(n+1)K}. \tag{20}$$

Equation (19) yields the general solution

$$\psi = A_1 T_{N+1}^{-3/2}(z) + A_2 (1-z^2)^{3/2} T_{N-2}^{3/2}(z), \tag{21}$$

where $N = \left[1 + n - nK + \frac{4\mu^2}{(n+1)K}\right]^{1/2}$ and T_α^β is the Gegenbauer function [24]. Further, it should be noted that

$$\psi'(z) = A_1 T_N^{-1/2}(z) + A_2 (N+1)(N+3)(1-z^2)^{1/2} T_{N-1}^{1/2}(z). \tag{22}$$

$NT_N^{-1/2}(z)$ and $(1-z^2)^{1/2} T_{N-1}^{1/2}(z)$ are Tschebyscheff polynomials which take the following form:

$$NT_N^{-1/2}(z) = (2/\pi)^{1/2} \cos(N \cos^{-1} z), \tag{23}$$

$$(1-z^2)^{1/2} T_{N-1}^{1/2}(z) = (2/\pi)^{1/2} \sin(N \cos^{-1} z). \tag{24}$$

By using eqs (23) and (24), eq. (22) on integration yields a general solution

$$\psi(z) = A\left[\frac{\cos[(N+1)\cos^{-1} z + B]}{N+1} - \frac{\cos[(N-1)\cos^{-1} z + B]}{N-1}\right], \tag{25}$$

where A, B are constants. The metric (1) with ψ determined by (25) describe the higher dimensional space-time of a spherical charged fluid distribution with electric field intensity given by (14). Further, using eqs (14)–(18), one can easily obtain the expressions for density and pressure as

$$\rho = \frac{n+1}{16\pi R^2(1-z^2)}\left[n - \frac{2\beta^2}{K(1-z^2)} - \frac{2\mu^2\beta^2[\beta^2 - z^2]}{K^2(n+1)(1-z^2)}\right], \tag{26}$$

$$p = \frac{n+1}{16\pi R^2(1-z^2)} \times \left[\frac{2\mu^2\beta^4 - n(n+1)K^2 + [n(n+1)K^2 - 2\mu^2\beta^2]z^2}{(n+1)K^2(1-z^2)} + \frac{2\beta^2 z}{K}\frac{\dot{\psi}}{\psi}\right], \tag{27}$$

where $\beta = [K/(K-1)]^{1/2}$. The requirement of $\rho > 0$ for physical plausibility of the solution is satisfying for $\mu^2 < (1-K)(n+1), 0 < K < 1$.

If the charged fluid distribution extends up to a finite radius $r = b$, the space-time in the exterior region $r > b$ is described by the higher dimensional Reissner–Nordström metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r^n} + \frac{q^2}{r^{2n}}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r^n} + \frac{q^2}{r^{2n}}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega_{n+1}^2. \tag{28}$$

The metric (1) with e^α and ψ given by eqs (13) and (25) respectively should smoothly match with the metric (28) on the boundary $r = b$, where the pressure is vanishing. These boundary conditions determine the constants A, B and the total mass m of the charged fluid sphere. Further, we know that the mass contained inside a radius r is given by

$$m(r) = H\int_0^r r'^{(n+1)}\rho(r')\mathrm{d}r'. \tag{29}$$

The value of mass parameter can be obtained by using eq. (15) and integrating the above equation for $r = b$. Owing to the complexity of the expressions it is not possible to examine other physical viability conditions. This result relates the mass parameter with the electric field intensity, geometrical parameter K and extra dimensions leading to a clear dependence of physical properties of the star on specific geometrical property (curvature of 3-spaces) and dimensionality of the space-time.

4. Discussion

In this paper we have replaced the usual procedure of prescribing an equation of state $p = p(\rho)$ by the ansatz 3-spheroidal geometry to the physical 3-spaces of the higher dimensional space-time. In this paper it is not possible to find explicit relation between energy density and pressure concerning to the equation of state. The solution presented here reduces to an earlier one obtained by Patel and Singh [16] when the electromagnetic field is switched off. Our solution produces various other four-dimensional well-known results. For example, when $n = 1$, our solution reduces to the result obtained in [15] and further if electric field is switched off, it gives the solution discussed by Mukherjee *et al* [14]. One can easily obtain the Reissner–Nordström metric given by Patel and Pandya [25] by substituting $n = 1$, $K = -2$, and $\lambda = -(\mu^2 + 2)$ in our solution.

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