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Non-stationary probabilities for the asymmetric exclusion process on a ring

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Abstract. A solution of the master equation for a system of interacting particles for finite time and particle density is presented. By using a new form of the Bethe ansatz, the totally asymmetric exclusion process on a ring is solved for arbitrary initial conditions and time intervals.

Keywords. Asymmetric exclusion process; Bethe ansatz.

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1. Introduction

The theory of interacting particle systems with stochastic dynamics in one dimensions plays the role of a testing laboratory of non-equilibrium statistical mechanics because its relative simplicity allows exact consideration of kinetic phenomena.

Exact stationary and non-stationary solutions of evolution equations reveal a rich picture of relaxation processes which provides an extension of notions of equilibrium statistical mechanics such as phase transitions and spontaneous symmetry breaking to the non-equilibrium case. The asymmetric exclusion process (ASEP) is the simplest model of non-equilibrium theory of many interacting particles [1–3] admitting exact solutions [4–10]. Here we extend known solutions of the totally ASEP to the ring geometry and finite time intervals.

Let C be a configuration of P particles on a ring of L sites. The ASEP is defined by the master equation for the probability $P_t(C)$ of finding the system in the configuration C at time t

$$\partial_t P_t(C) = \sum_{\{C'\}} [M_0(C,C') + M_1(C,C')] P_t(C'), \tag{1}$$

where summation is over all possible configurations C', $M_1(C,C')$ is the probability of going from C' to C during a time interval $\mathrm{d}t$, and $M_0(C,C')$ is a diagonal matrix

$$M_0(C,C') = -\sum_{\{C'\neq C\}} M_1(C',C). \tag{2}$$

The matrix elements of $M_1(C,C')$ obey the exclusion rule that each particle during $\mathrm{d}t$ jumps with probability $\mathrm{d}t$ to its right if the target site is empty. Given the initial positions of particles $0 \le a_1 < a_2 < \cdots < a_P < L$ at the moment $t=0$, $P_t(C)$ is the conditional probability $P(x_1,...,x_P;t|a_1,...,a_P;0)$ of finding P particles on the sites $0 \le x_1 < \cdots < x_P < L$ at time t.

The solution of eq. (1) is [11]:

$$P_t(C) = \sum_{n_1=-\infty}^{\infty} ... \sum_{n_P=-\infty}^{\infty} (-1)^{\sum_{i<j}|n_i-n_j|} \det \mathbf{M}. \tag{3}$$

Elements of the $P\times P$ matrix $\mathbf{M}$ are

$$M_{ij} = F_{s_{ij}}(a_i, x_j + n_j L|t), \tag{4}$$

where

$$s_{ij} = (P-1)n_j - \sum_{k\neq j} n_k + j - i \tag{5}$$

and $F_m(a_i,x_j|t)$ are functions introduced by Schütz [9]:

$$F_m(a_i,x_j|t) = \sum_{k=0}^{\infty} \begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix} F_0(a_i-k,x_j|t) \tag{6}$$

if integer $m>0$, and

$$F_m(a_i,x_j|t) = \sum_{k=0}^{-m} (-1)^k \begin{pmatrix} -m \\ k \end{pmatrix} F_0(a_i-k,x_j|t) \tag{7}$$

if integer $m<0$. For $m=0$ and $x_j \ge a_i$,

$$F_0(a_i,x_j|t) = \frac{\mathrm{e}^{-t}t^K}{K!}, \tag{8}$$

where $K = x_j - a_i$. For $m=0$ and $x_j < a_i$ we put $F_0(a_i,x_j|t)=0$.

The discrete space-time version of the totally ASEP is defined as follows. Consider the infinite triangular lattice Λ obtained from the square lattice by adding a diagonal between the upper left and the lower right corners of each elementary square. Let (x,T) be integer space-time coordinates of a particle on Λ, where the vertical time axis is directed down and the horizontal space axis is directed right. A trajectory of a particle is a sequence of connected vertical and diagonal bonds of Λ. Each diagonal bond corresponds to a jump of the particle to its nearest neighbor on the right for unit time and has a probability z. Each vertical bond corresponds to a stay of the particle at the site corresponding to its spatial coordinate during the unit time interval and has the probability $y = 1-z$. The statistical weight of all one-particle trajectories starting at point $(a, 0)$ and ending at point (x, T) is

$$B_0(a, x|T) \equiv B(x-a, T) = \binom{T}{N} z^N y^{T-N}, \tag{9}$$

where $N = x - a$ is the distance travelled for time T.

Consider the set of trajectories of n particles on the lattice Λ which start at points $(a_1, 0), \dots, (a_P, 0)$; $0 \le a_1 < a_2 < \cdots < a_P < L$ and end at points $(x_1, t), \dots, (x_P, t)$; $0 \le x_1 < x_2 < \cdots < x_P < L$. The exclusion rules read:

(a) trajectories of particles do not intersect;
(b) if two vertical bonds of any elementary square of Λ are occupied by adjacent trajectories, the probability of the left bond is changed from $1 - z$ to 1.

Rule (a) is the usual condition for occupation of every site by at most one particle. Rule (b) implies that the particle stays at the given site with probability 1 if the target site is occupied by a particle in the next moment of discrete time.

The solution of the discrete ASEP [11] coincides with eqs (3)–(7). The only difference is that the Poisson distribution equation (8) should be replaced by its Bernoulli analogue equation (9). Setting $T = Mt$, $z = 1/M$, and passing to the limit $M \to \infty$, we obtain as usual

$$\lim_{M\to\infty} B_0(a, x|Mt)|_{z=1/M} = F_0(a, x|t), \tag{10}$$

where t is the re-scaled continuous time.

In the next section, we give geometrical arguments explaining the above results. The analytical derivation of eqs (3)–(9) is given in §3.

2. Geometrical interpretation

The derivation of eqs (3)–(9) is based on a simple property of integrable models admitting a two-dimensional graphic representation: interchanging of end points of two trajectories leads to their crossing. The idea of the Bethe ansatz is to represent trajectories of interacting particles by a set of free trajectories given by eq. (8) or eq. (9). Then, using the one-to-one correspondence between intersections and permutations, one can reduce enumerating all interacting trajectories to a proper choice of signs of permutations.

We start with the case of two particles $P = 2$. According to the Bethe ansatz, we try to represent the motion of particles by free trajectories from $(a_i, 0)$ to (x_i, t), $i = 1, 2$. Consider an elementary square of Λ with space coordinates x of the left side and $x + 1$ of the right side. Assume that particles come for the first time to neighboring sites at a moment t' when one trajectory reaches the site (x, t') from $(a_1, 0)$ and another reaches the site $(x+1, t')$ from $(a_2, 0)$. To ensure correct weights of the next steps of interacting particles after moment t', we have to exclude two possibilities from all continuations of trajectories:

(i) For the first particle,the step from (x,t') to $(x+1,t'+1)$ with weight z and then from $(x+1,t'+1)$ to (x_1,t); for the second particle, the step from $(x+1,t')$ to $(x+1,t'+1)$ with weight y and then from $(x+1,t'+1)$ to (x_2,t).

(ii) For the first particle, the step from (x,t') to $(x,t'+1)$ with weight $y-1=-z$ and then from $(x,t'+1)$ to (x_1,t); for the second particle, the step from $(x+1,t')$ to $(x+1,t'+1)$ with weight y and then from $(x+1,t'+1)$ to (x_2,t).

Case (i) is the forbidden step of the first particle toward the standing second particle. Case (ii) is a correction of the weight of the vertical step of the first particle which must be 1 instead of y according to the ASEP rule (b). The statistical weight of paths of the first particle in case (i) is a product of three factors $B(x-x_1^0,t')zB(x_1-x-1,t-t'-1)$. Using symbolic notations $W(a,x|z|x+1,b)$ for the statistical weight of trajectories passing points $a,x,x+1,b$ at moments $0,t',t'+1,t$ and making a step with weight z between t' and $t'+1$, we can write the contribution from case (i) in the form

$$W_1 = W(a_1,x|z|x+1,x_1)W(a_2,x+1|y|x+1,x_2). \tag{11}$$

The contribution from the case (ii) is

$$W_2 = -W(a_1,x|z|x,x_1)W(a_2,x+1|y|x+1,x_2). \tag{12}$$

Consider now the trajectories where the end points are interchanged, denoting the corresponding cases by $\tilde{(\mathrm{i})}$ and $\tilde{(\mathrm{ii})}$. The contribution from these cases is $W(a_1,x|z|x+1,x_2)W(a_2,x+1|y|x+1,x_1)-W(a_1,x|z|x,x_2)W(a_2,x+1|y|x+1,x_1)$. We are going to take these cases with opposite signs to cancel W_1+W_2. The cases (i) and $\tilde{(\mathrm{i})}$ are equivalent, however (ii) and $\tilde{(\mathrm{ii})}$ are different. To cancel all unwanted cases, we add to the cases $\tilde{(\mathrm{i})}$ and $\tilde{(\mathrm{ii})}$ a set of auxiliary trajectories. Namely, add to trajectories of the second particle those starting in point a_2-1 and taken with minus sign. Also, we add to trajectories of the first particle a set of trajectories starting at the points shifted along the ring in negative directions: a_1-1, a_1-2, a_1-3,... . If a shift exceeds mL, where $m>0$ is integer, the trajectory wraps the cylinder Λm times. Then, the contribution from case $\tilde{(\mathrm{i})}$ will be $\tilde{W}_1 = W_1^+W_1^-$ where

$$W_1^+ = \sum_{k=0}^{\infty} W(a_1-k,x-k|z|x+1-k,x_2) \tag{13}$$

and

$$W_1^- = W(a_2,x+1|y|x+1,x_1) - W(a_2-1,x|y|x,x_1). \tag{14}$$

Correspondingly, for the case $\tilde{(\mathrm{ii})}$ we have $\tilde{W}_2 = W_2^+W_2^-$ where

$$W_2^+ = -\sum_{k=0}^{\infty} W(a_1-k,x-k|z|x-k,x_2) \tag{15}$$

and $W_2^- = W_1^-$. Taking into account that statistical weights of trajectories from $(a_i-k,0)$, $i=1,2$, to $(x-k,t')$ are equal for all k due to translation invariance, one can check the identity $W_1 + W_2 - \tilde{W}_1 - \tilde{W}_2 = 0$ comparing all positive and negative terms.

Consider now the evaluation of probability $P(x_1, x_2; t|a_1, a_2; 0)$ in the case $L \gg t$ and $L \gg a_2 > a_1 \gg 0$ which is equivalent to the ASEP on an infinite lattice solved in [9]. In this case,

$$P = B(x_1 - a_1, t)B(x_2 - a_2, t) - (B(x_1 - a_2, t) - B(x_1 - a_2 + 1, t)) \sum_{k=0}^{\infty} B(x_2 - a_1 + k, t). \tag{16}$$

Indeed, the first term in eq. (16) generates all possible free trajectories from initial to end points. When one particle approaches another, the second term produces trajectories cancelling unwanted terms. On the other hand, the order of starting and ending points in the second term is interchanged. Therefore, each trajectory from the second term starting at $a_1 - k$ or $a_2 - k$ approaches at least once the point $(x - k, t')$ or $(x + 1 - k, t')$ where it participates in the cancellation procedure.

Each free trajectory from a to b making the vertical step at the collision site x can be decomposed into two parts $W(a, x|1|x, b) + W(a, x| - z|x, b)$. The second part is unwanted and is cancelled, the first one corresponds to trajectories which continue with true weights up to the next collision. As the second term in eq. (16) contains intersecting trajectories only, all of them will be cancelled eventually and only true allowed trajectories from the first term survive.

The ASEP on the ring has several peculiarities. To fix them, let us note that two intersecting trajectories are non-equivalent: one of them belongs to the overtaking particle and we may call it 'active'. On the contrary, the second particle can be called 'passive'. In the case of infinite lattice, the active and passive trajectories are ordered: for each pair $i, i+1$, the trajectory of ith particle with respect to $i+1$ particle is always active. On the ring, each of the two trajectories can be active or passive independently on initial conditions. Moreover, one trajectory can intersect another m times if the number of rotations differ by m for two particles.

Assume, that the trajectory of a given particle has m active intersections. It means that it participates m times in the cancellation procedure and its starting point is shifted m times to arbitrary distances in the negative direction of the ring. As a result, the auxiliary set associated with the free trajectory between points a_i and x_j becomes

$$B(x_j - a_i, t) \to \sum_{k=0}^{\infty} \begin{pmatrix} k + m - 1 \\ m - 1 \end{pmatrix} B(x_j - a_i + k, t) \tag{17}$$

because the shift by k positions for m attempts can be done in $(k+m-1)!/(m-1)!k!$ ways. The above can be expressed in an operator form

$$B(x_j - a_i, t) \to \frac{1}{(1 - \hat{a}_i)^m} B(x_j - a_i, t), \tag{18}$$

where the operator $\hat{a}_i$ shifts a_i by one step in the negative direction. Similarly, for trajectories having m passive intersections we get

$$B(x_j - a_i, t) \to \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} B(x_j - a_i + k, t) \tag{19}$$

because the right-hand side is the result of action of the operator $(1 - \hat{a}_i)^m$. Note that eq. (17) coincides with eq. (6) and eq. (19) with eq. (7) where the index m can be called activity.

To find the conditional probabilities for two particles on the ring, we map the trajectories wrapping the cylinder Λ on an infinite plane introducing coordinates $x + nL$, n integer, for equivalent points. We call trajectories of two particles compatible if there is at least one possibility to draw them without intersections. Given starting points a_1 and a_2 at $t = 0$, the pairs of compatible trajectories correspond to end points x_1, x_2; $x_2, x_1 + L$;...; $x_1 + nL, x_2 + nL$; $x_2 + nL, x_1 + (n+1)L$;... at time t. The index of activity of compatible trajectories is 0. A trajectory ending at $x_1 + nL$ may interact with the trajectories ending at $x_2 + (n-1)L$ or $x_2 + nL$.

Following the Bethe-ansatz prescription, we should add to the set of compatible trajectories two sets of intersecting trajectories taken with minus sign: the set which is obtained by permuting end points $x_2 + (n-1)L$ and $x_1 + nL$ and the second one obtained by permuting $x_1 + nL$ and $x_2 + nL$. Each new trajectory, in its turn, interacts with neighbouring trajectories and we should permute their end points again. Continuing this procedure, we obtain all possible pairs of trajectories of the first particle with end point $x_1 + n_1 L$, the second one with end point $x_2 + n_2 L$, or vice versa, for arbitrary integers n_1 and n_2. The permutations of end points can be expressed by the determinant as can be seen in eq. (3). The sign of a pair is defined by the number of permutations needed to obtain a given pair from a compatible one. The index of activity of each trajectory is defined by the number of overtakes to the moment t. Evaluation of the number of permutations gives the pre-factor in eq. (3). The number of overtakes is given by eq. (5).

If the number of particles $P \geq 3$, the interaction at elementary squares considered above may occur several times in one horizontal strip of Λ. If squares filled by interacting trajectories are separated from one another by a gap of empty sites, the above arguments can be applied to each pair of interacting trajectories separately. The crucial case for the Bethe ansatz is a situation when the elementary squares are nearest neighbors. The specific property of the totally ASEP is that, in each pair of interacting trajectories, the right trajectory remains free and interacts with the next trajectory independently on its left neighbors. Therefore, we can analyse the interaction between particles considering successively elementary squares in each row from left to right starting from an arbitrary empty square and then from the top to bottom of the lattice until all unwanted trajectories on Λ will be removed. As above, all trajectories ending at points $x_i + n_i L, i = 1, ..., P$ are to be taken into consideration. Then, eq. (3) is a straightforward generalisation of the case $P = 2$, and eq. (5) gives the index of activity for an arbitrary number of intersecting trajectories. A new element in the many-particle case is that each trajectory may get m active intersections and n passive ones. The operator form of eq. (18) and (19) shows that the resulting activity in this case is $m - n$.

3. Analytical derivation

We start the analytical derivation with the case of two particles $P = 2$. It is convenient to map the trajectories wrapping the ring on an infinite 1D lattice, introducing the coordinates $X_i = x_i + n_i L$, n_i is an integer, along with the ring coordinates $0 \leq x_i < L$. The conditional probability can be written in the form

$$P(x_1, x_2; t|a_1, a_2; 0) \equiv P(x_1, x_2; t) = \sum_{X_1, X_2} \psi(X_1, X_2; t), \tag{20}$$

where summation is over all X_1, X_2 which satisfy the condition (a) $X_1 = x_1 + n_1^* L$; $X_2 = x_2 + n_2^* L$ or (b) $X_1 = x_2 + n_2^* L$; $X_2 = x_1 + n_1^* L$, n_1^*, n_2^* are integers. The function $\psi(X_1, X_2; t)$ is the probability for the first particle to reach the point X_1 from the starting point x_1^0 and for the second particle to reach X_2 from x_2^0 provided the trajectories of particles obey the ASEP rules. The exclusion rule says that the probability $\psi(X_1, X_2; t)$ is non-zero if and only if the trajectories of two particles are compatible. Respectively, $\psi > 0$ if $n_1^* = n_2^*$ for the condition (a) and $n_1^* = n_2^* + 1$ for (b).

Like $P(x_1, x_2; t)$, the function $\psi(X_1, X_2; t)$ satisfies the master equation (1) which can be converted into the eigenvalue problem

$$\lambda\psi = \psi(X_1 - 1, X_2) + \psi(X_1, X_2 - 1) - 2\psi(X_1, X_2) \tag{21}$$

by the substitution $\psi(X_1, X_2; t) = \mathrm{e}^{\lambda t}\psi(X_1, X_2)$ quite similar to the usual one, $P(x_1, x_2; t) = \mathrm{e}^{\lambda t} P(x_1, x_2)$. Equation (21) has to be supplemented by boundary conditions

$$\psi(X, X) = \psi(X, X + 1) \tag{22}$$

if the first particle overtakes the second one, and

$$\psi(X, X + L) = \psi(X + 1, X + L) \tag{23}$$

if the second particle, making an additional loop, overtakes the first one.

Instead of the standard Bethe ansatz

$$P(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1} \tag{24}$$

we introduce a detailed Bethe ansatz for each pair of compatible coordinates X_1, X_2,

$$\psi(X_1, X_2) = \sum_{n_1, n_2} A_{12}^{(n_1, n_2)} z_1^{x_1 + n_1 L} z_2^{x_2 + n_2 L} + A_{21}^{(n_2, n_1)} z_1^{x_2 + n_2 L} z_2^{x_1 + n_1 L}, \tag{25}$$

where summation is over all n_1, n_2 obeying $n_1 + n_2 = n_1^* + n_2^*$ and parameters A depend on ordering of trajectories. The new form of the Bethe ansatz is a natural extension of eq. (24). Indeed, the permutation of coordinates x_1, x_2 in eq. (24) serves for a correction of free trajectories of two particles due to their interaction. The coordinates X_1, X_2 distinguish trajectories with different number of rotations n_1, n_2

around the ring and, therefore, the interactions between them need using all possible permutations starting from the compatible ones. The equation $n_1 + n_2 = n_1^* + n_2^*$ is a conservation rule of the total number of rotations under permutations.

From eq. (21), we have $\lambda = -2 + z_1^{-1} + z_2^{-1}$. The conditions (22), (23) are satisfied if

$$\frac{A_{12}^{(n_1,n_2)}}{A_{21}^{(n_1,n_2)}} = -\frac{(1-z_1)}{(1-z_2)} \tag{26}$$

and

$$\frac{A_{12}^{(n_1,n_2)}}{A_{21}^{(n_1-1,n_2+1)}} = -\frac{(1-z_2)}{(1-z_1)}. \tag{27}$$

Therefore, the parameters $A_{ij}^{(n_1,n_2)}$ can be found recursively:

$$A_{ij}^{(n_1,n_2)} = (-)^{\pi}(-1)^{|n_1-n_2|}\frac{f(z_1,z_2)}{(1-z_1)^{s_{1i}}(1-z_2)^{s_{2j}}}, \tag{28}$$

where s_{ij} is given by eq. (5), $f(z_1, z_2)$ is an unknown function and the sign of permutation $(-)^{\pi}$ is positive for $i, j = 1, 2$ and negative for $i, j = 2, 1$.

Using $z_j = \mathrm{e}^{ip_j}$, we can write the general solution

$$\psi(X_1, X_2; t) = \int_0^{2\pi} \mathrm{d}p_1 \int_0^{2\pi} \mathrm{d}p_2 \mathrm{e}^{\lambda t} \psi(X_1, X_2) \tag{29}$$

and determine the function $f(z_1, z_2)$ from the initial conditions. As was noticed in [9] for the case of the infinite lattice, the choice

$$f(z_1, z_2) = \mathrm{e}^{-ip_1 a_1 - ip_2 a_2} \tag{30}$$

and the definition of poles in eq. (28) by $p_j \to p_j + i0$ provides the correct initial conditions

$$P(x_1, x_2; 0|x_1^0, x_2^0; 0) = \delta_{x_1,x_1^0}\delta_{x_2,x_2^0}. \tag{31}$$

Substituting eqs (28) and (30) into eq. (25) and integrating in eq. (29), we get

$$\psi(X_1, X_2; t) = \sum_{n_1+n_2=n_1^*+n_2^*} (-1)^{|n_1-n_2|} \det M, \tag{32}$$

where M is the 2×2 matrix defined by eq. (4). The summation over all n_1^*, n_2^* leads to eq. (3) for $P = 2$.

For P particles, the conditional probability becomes

$$P(x_1, ..., x_P; t) = \sum_{\{X\}} \psi(X_1, ..., X_P; t), \tag{33}$$

where all X_i satisfy $X_i = x_{\pi(i)} + n^*_{\pi(i)}L$ and π is the permutation of numbers $1, 2, ..., P$. The function $\psi(X_1, ..., X_P; t)$ is the probability for ith particle to reach the point X_i from the starting point a_i, $i = 1, 2, ..., P$ provided the trajectories of particles obey the ASEP rules. Again, we use the substitution $\psi(X_1, ..., X_P; t) = \mathrm{e}^{\lambda t}\psi(X_1, ..., X_P)$. The eigenvalue problem is a direct generalization of eq. (21) and has to be supplemented by P boundary conditions corresponding to P possible overtakings:

$$\psi(..., X_i, X_i, ...) = \psi(..., X_i, X_i + 1, ...) \tag{34}$$

and

$$\psi(X_1, X_2, ..., X_1 + L) = \psi(X_1 + 1, X_2, ..., X_1 + L). \tag{35}$$

The Bethe ansatz, eq. (25), takes the form

$$\psi(X_1, ..., X_P) = \sum_{\pi}\sum_{\{n\}} A^{(n_{\pi(1)},...,n_{\pi(P)})}_{\pi(1)...\pi(P)} \times \prod_{j=1}^{P} z_j^{x_{\pi(j)}+n_{\pi(j)}L}, \tag{36}$$

where the first summation is over all permutations π and the second one obeys $n_1 + \cdots + n_P = n^*_1 + \cdots + n^*_P$. The eigenvalue for P particles is

$$\lambda = -P + \sum_{i=1}^{P} \frac{1}{z_i}. \tag{37}$$

It follows from eqs (34) and (35) that

$$\frac{A^{(...,n_i,n_j,...)}_{...ij...}}{A^{(...,n_i,n_j,...)}_{...ji...}} = -\frac{(1 - z_i)}{(1 - z_j)} \tag{38}$$

and

$$\frac{A^{(n_{\pi(1)},...,n_{\pi(P)})}_{\pi(1)...\pi(P)}}{A^{(n_{\pi(1)}-1,...,n_{\pi(P)}+1)}_{\pi(P)...\pi(1)}} = -\frac{(1 - z_{\pi(P)})}{(1 - z_{\pi(1)})}. \tag{39}$$

The ratio of parameters A in eqs (38) and (39) is similar to eqs (26) and (27) and can be considered as a scattering factor corresponding to the intersection of two particles leading to the permutation of their end points. The trajectories of particles i and $\pi(P)$ in eqs (38) and (39) are active, and the trajectories j and $\pi(1)$ are passive. We can conclude that each intersection of the active trajectory i gives the factor $(1 - z_i)^{-1}$ in the parameter $A^{(...,n_i,n_j,...)}_{...ji...}$ whereas each intersection of the passive trajectory j gives the factor $(1 - z_j)$. The resulting exponent s_{ij} of $(1 - z_i)$ for the particle starting at x^0_i and ending at X_j is defined by the difference between total number of active and passive intersections (eq. (5)). Summarizing, we can write

$$A_{\pi(1)\ldots\pi(P)}^{(n_{\pi(1)},\ldots,n_{\pi(P)})} = \frac{(-)^{\pi}(-1)^{\sum_{i<j}|n_i-n_j|} f(z_1,...,z_P)}{(1-z_1)^{s_{1\pi(1)}}...(1-z_P)^{s_{P\pi(P)}}}, \tag{40}$$

where the sign on the right-hand side is defined by the number of permutations and $f(z_1,...,z_P)$ is to be found from the initial conditions.

The boundary conditions (38) and (39) take into account the pair interactions between particles when two particles are on neighbouring sites of the ring. The configurations containing 'dense packed' intervals of three or more sites produce no new constraints because the boundary conditions for any pair are independent of coordinates of the remaining particles [9].

As above, the choice

$$f(z_1,...,z_P) = \prod_{i=1}^{P} z_i^{-a_i} \tag{41}$$

ensures the initial conditions

$$P(x_1,...,x_P;0|a_1,...,a_P;0) = \delta_{x_1,a_1}...\delta_{x_P,a_P}. \tag{42}$$

Integrating $\mathrm{e}^{\lambda t}\psi(X_1,...,X_P)$ over $p_j = -i\ln z_j$ with parameters A given by eq. (40), we obtain the function $\psi(X_1,...,X_P;t)$ in the form

$$\psi(X_1,...,X_P;t) = \sum_{\{n\}}(-1)^{\sum_{i<j}|n_i-n_j|}\det\mathbf{M}, \tag{43}$$

where summation $\{n\}$ is restricted by the same conditions as eq. (36). Continuing the summation in eq. (33) we obtain the unrestricted sum over all $n_1,...,n_P$ that is eq. (3).

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