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Factorised steady states and condensation transitions in nonequilibrium systems

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Abstract. Systems driven out of equilibrium can often exhibit behaviour not seen in systems in thermal equilibrium – for example phase transitions in one-dimensional systems. In this talk I will review a simple model of a nonequilibrium system known as the 'zero-range process' and its recent developments. The nonequilibrium stationary state of this model factorises and this property allows a detailed analysis of several 'condensation' transitions wherein a finite fraction of the constituent particles condenses onto a single lattice site. I will then consider a more general class of mass transport models, encompassing continuous mass variables and discrete time updating, and present a necessary and sufficient condition for the steady state to factorise. The property of factorisation again allows an analysis of the condensation transitions which may occur.

Keywords. Nonequilibrium systems; zero-range process; factorised steady state; condensation transitions.

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1. Introduction

In this presentation I shall discuss some simple models of nonequilibrium systems defined by the stochastic dynamics of mass transfer between lattice sites. In these systems the steady state takes on a factorised form and the factorised steady state allows one to analyse a variety of 'condensation' transitions, wherein a finite faction of the mass condenses onto a single lattice site. The first part of the talk is concerned with a model known as the zero-range process [1,2] and is essentially an update of a previous work [3] to include recent developments. The last part of the talk discusses recent results obtained with S N Majumdar and R K P Zia which describe the general conditions for a mass transport model to have a factorised steady state.

Broadly speaking, one can consider two types of nonequilibrium systems: those relaxing towards thermal equilibrium and those held far from thermal equilibrium e.g. by the system being driven by some external field. In the latter case the steady state of the system will usually not be described by Gibbs–Boltzmann statistical weights rather it will be a nonequilibrium steady state. A natural way to construct a nonequilibrium steady state is to drive the system by forcing a current of some conserved quantity, for example energy or mass, through the system. Such systems are known as driven diffusive systems [4,5].

A class of phase transitions which may occur in such models is that involving spatial condensation, whereby a finite fraction of the constituent particles condenses onto the same site [6,7]. Of particular interest is the fact that transitions may occur in one-dimensional systems. Examples include the appearance of a large aggregate in models of aggregation and fragmentation [8] and the emergence of a single flock in dynamical models of flocking [9]. We will focus our discussion on the zero-range process which we first define and later will motivate both as model in its own right and as an effective description of many driven systems.

2. The zero-range process

The zero-range process (ZRP) was introduced some years ago by Spitzer [1] and recent interest has been reviewed in [2,3]. To begin with, we consider a one-dimensional lattice of N sites with sites labelled $i = 1 \ldots N$ and periodic boundary conditions (site $N+1=$ site 1). Each site can hold an integer number of indistinguishable particles. The configuration of the system is specified by the occupation numbers m_i of each site i. The total number of particles is denoted by M and is conserved under the dynamics. The dynamics of the system is given by the rates at which a particle leaves a site i (one can think of it as the topmost particle – see figure 1a) and moves to the left nearest neighbour site $i-1$. The hopping rates $u(m)$ are a function of the number of particles at the site of departure m. Some particular cases are: if $u(m) = m$ then the dynamics of each particle is independent of the others; if $u(m) =$ const. for $m > 0$ then the rate at which a particle leaves a site is unaffected by the number of particles at the site (as long as it is greater than zero).

As illustrated in figure 1 there exists an exact mapping from a ZRP to an asymmetric exclusion process, which is a driven system where there is at most one particle per site. The idea is to consider the particles of the ZRP as the holes (empty sites) of the exclusion process. Then the sites of the ZRP become the moving particles of the exclusion process. Note that in the exclusion process we have M particles hopping on a lattice of $M + N$ sites. A hopping rate in the ZRP $u(m)$ which is dependent on m corresponds to a hopping rate in the exclusion process which depends on the gap to the particle in front. So the particles can feel each other's presence and one can have a long-range interaction.

The important attribute of the ZRP is that it has a *factorised steady state*. By this it is meant that the steady state probability $P(\{m_i\})$ of finding the system in configuration $\{m_1, m_2 \ldots m_N\}$ is given by a product of factors $f(m_i)$

$$P(\{m_i\}) = \frac{1}{Z(M,N)} \prod_{i=1}^{N} f(m_i) \; . \qquad (1)$$

Here the normalisation $Z(M,N)$ is introduced so that the sum of the probabilities for all configurations, with the correct number of particles M, is one:

$$Z(M,N) = \sum_{m_1,m_2\ldots m_N} \delta\left(\sum_i m_i - M\right) \prod_{i=1}^{N} f(m_i). \tag{2}$$

The normalisation may usefully be considered as the analogue of a canonical partition function of a thermodynamic system [10].

It is important to realise that due to the constraint of fixed particle number the single-site weight $f(m)$ is not the same as the single-site mass probability distribution $p(m)$ which would be calculated as

$$p(m) = \frac{f(m)Z(M-m,N-1)}{Z(M,N)} . \tag{3}$$

In other words, although the steady state factorises the constraint of fixed particle number still induces correlations between sites.

In the basic model described above, $f(m)$ is given by

$$\begin{aligned} f(m) &= \prod_{n=1}^{m} \frac{1}{u(n)} \quad \text{for} \quad m \geq 1, \\ &= 1 \qquad\qquad \text{for} \quad m = 0. \end{aligned} \tag{4}$$

To prove eqs (1) and (4) one simply considers the stationarity condition on the probability of a configuration (probability current out of the configuration due to hops is equal to probability current into the configuration due to hops):

$$\begin{aligned} &\sum_i \theta(m_i)u(m_i)P(m_1\ldots m_i\ldots m_N) \\ &\quad = \sum_i \theta(m_i)u(m_{i+1}{+}1)P(m_1\ldots m_i{-}1, m_{i+1}{+}1\ldots m_N) . \end{aligned} \tag{5}$$

The Heaviside function $\theta(m_i)$ highlights that it is the sites with $m > 1$ that allow exit from the configuration (left-hand side of (5)) but also allow entry to the configuration (right-hand side of (5)). Equating the terms i on both sides of (5) and cancelling common factors assuming (1), results in

$$u(m_i)f(m_i)f(m_{i+1}) = u(m_{i+1}+1)f(m_i-1)f(m_{i+1}+1). \tag{6}$$

This equation can be recast as

$$u(m_i)\frac{f(m_i)}{f(m_i-1)} = u(m_{i+1}+1)\frac{f(m_{i+1}+1)}{f(m_{i+1})} = \text{constant}. \tag{7}$$

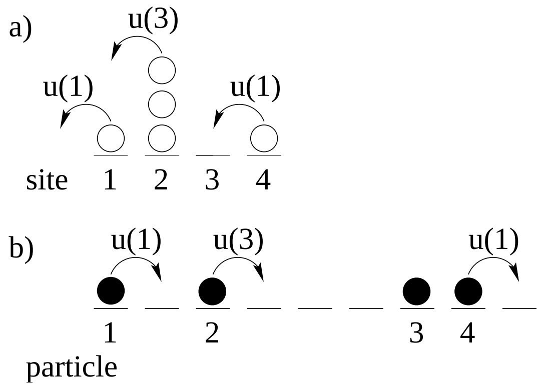

Figure 1. Equivalence of zero-range process and asymmetric exclusion process.

Setting the constant to unity implies

$$f(m_i) = \frac{f(m_i - 1)}{u(m_i)} \tag{8}$$

and iterating (8) leads to (4).

We can easily generalise to consider an heterogeneous system by which we mean that the hopping rates are site dependent: the hopping rate out of site i when it contains m_i particles is $u_i(m_i)$. It is easy to check that the steady state still factorises and the single-site weights are simply modified to

$$f_i(m) = \prod_{n=1}^{m} \frac{1}{u_i(n)} \quad \text{for} \quad m \geq 1\ . \tag{9}$$

The proof is identical to that for the homogeneous case, with the replacement of $u(m_i)$ by $u_i(m_i)$.

Finally, let us note that the ZRP can be considered on more complicated lattices, for example higher dimensional or random lattices. If the hop rate from site j to i has the form

$$u_{ij}(m_j) = u_j(m_j) W_{ij} \tag{10}$$

i.e., it factorises into a function $u_j(m_j)$ of the number of particles at the departure site j and a term W_{ij} which encodes the structure of the lattice, then the steady state is again factorised. The weights f_i are modified to

$$f_i(m) = \prod_{n=1}^{m} \frac{s_i}{u_i(n)} \quad \text{for} \quad m \geq 1, \tag{11}$$

where s_i satisfy

$$s_i = \sum_j W_{ij} s_j \tag{12}$$

i.e., s_i are the steady state weights of a single continuous time random walker moving from site j to i with rate W_{ij}.

We now return to the motivation for studying the ZRP. Firstly, there exist some exact mappings of particular nonequilibrium systems onto the ZRP. Examples include the repton model of polymer dynamics under periodic boundary conditions [11]; models of sandpile dynamics [12,13]; the drop-push model for the dynamics of a fluid moving through backbends in a porous medium [14].

More generally, however, one may think of the sites of the ZRP as representing domains of some driven system – this is the most natural picture within the exclusion process interpretation of the ZRP (figure 1). The domains may have some internal structure, for example further degrees of freedom, but all this is integrated out, and one is left with an effective dynamics of exchange of length between domains. An example of such a description is the bus route model which will be briefly described below. More recently, by using the ZRP as the effective description, a general criterion for phase separation in one-dimensional driven systems has been developed [15–17]. Within this description phase separation is manifested by the emergence of one large domain and this corresponds to the phenomenon of condensation in the ZRP which we discuss now.

3. Condensation transitions

To analyse the condensation transitions which may occur, it is the simplest to use the grand canonical ensemble and define the grand canonical partition function as

$$\mathcal{Z}_{\rm gc}(z) = \sum_{\{m_i=0\}}^{\infty} z^{\sum_i m_i} \prod_{i=1}^{N} f_i(m_i) \; = \prod_{i=1}^{N} F_i(z) \; , \tag{13}$$

where

$$F_i(z) = \sum_{m=0}^{\infty} z^m f_i(m). \tag{14}$$

The fugacity z is fixed by the equation for the average density $\rho = M/N$

$$\rho = \frac{z}{N}\frac{\partial \ln \mathcal{Z}_{\rm gc}(z)}{\partial z} = \frac{z}{N}\sum_{i=1}^{N}\frac{F_i'(z)}{F_i(z)} \; . \tag{15}$$

In the thermodynamic limit,

$$N \to \infty \quad \text{with} \quad M = \rho N \; , \tag{16}$$

where the density ρ is held fixed and the condensation mechanism reduces to the question of whether one can satisfy (15). Although there are some subtle differences between the heterogeneous and homogeneous cases to be described below, the basic mechanism is as follows. First note that the right-hand side of (15) is a monotonically increasing function of z. Thus as ρ increases the required value of z

increases. However, there is a maximum value that z can take which is the radius of convergence of (14). If at this maximum value of z the right-hand side of (15) takes a finite value ρ_c, then for $\rho > \rho_c$ (15) cannot be satisfied and we have condensation. We expect the excess number of particles $N(\rho - \rho_c)$ to condense onto a single site.

3.1 *Heterogeneous case*

To give an idea of how a condensation transition may occur we consider the case $u_i(m) = u_i$ for $m > 0$, i.e., the hopping rate does not depend on the number of particles at a site. In this case f_i is given by

$$f_i(m) = \left(\frac{1}{u_i}\right)^m . \tag{17}$$

The mapping to an ideal Bose gas is evident: the M particles of the ZRP are viewed as bosons which may reside in N states with energies E_i determined by the site hopping rates: $\exp(-\beta E_i) = 1/u_i$. Thus the ground state corresponds to the site with the lowest hopping rate. We can sum the geometric series (14) to obtain F_i and F_i'. Then taking the large N limit allows the sum over i to be written as an integral

$$\rho = \int_{u_{\min}}^{\infty} \mathrm{d}u \mathcal{P}(u)\ \frac{z}{u-z}, \tag{18}$$

where $\mathcal{P}(u)$ is the probability distribution of site hopping rates with $u_{\min}$ the lowest possible site hopping rate. Interpreting $\mathcal{P}(u)$ as a density of states, eq. (18) corresponds to the condition that in the grand canonical ensemble of an ideal Bose gas the number of bosons per state is ρ. The theory of Bose condensation tells us that when certain conditions on the density of low energy states pertain we can have a condensation transition. Then (15) can no longer be satisfied and we have a condensation of particles into the ground state, which is here the site with the slowest hopping rate.

A very simple example is to have just one 'slow site', i.e. $u_1 = p$ while the other $N-1$ sites have hopping rates $u_i = 1$ when $i > 1$. Using the mapping to an exclusion process, this corresponds to one slow particle. One can show [18,19] that for a high density of particles in the ZRP (low density of particles in the corresponding asymmetric exclusion process) we have a condensate since site 1 contains a finite fraction of the particles. In the low-density phase the particles are evenly spread between all sites.

3.2 *Homogeneous case*

We now consider the homogeneous ZRP where the hopping rates $u(m)$ are site independent. Then (14) is independent of i and reads

$$F(z) = \sum_{m=0}^{\infty} \prod_{n=1}^{m} \left[\frac{z}{u(n)}\right] . \tag{19}$$

The fugacity z is restricted to $z \leq \beta$ where we define β to be the radius of convergence of $F(z)$. From (19) we see that β is the limiting value of $u(m)$, i.e., the limiting value of the hopping rate out of a site for a large number of particles at a site. The condition (15) becomes

$$\rho = \frac{zF'(z)}{F(z)}. \tag{20}$$

Let us fix the maximum allowed value of z as $z = \beta$. Then, for condensation to occur we require

$$\lim_{z\to\beta} \frac{F'(z)}{F(z)} < \infty\ . \tag{21}$$

Take for example $u(m) = \beta(1+b/m)$ so that the chipping probability decreases to an asymptotic value β as the mass at the site increases. The criterion for condensation is that

$$F'(\beta) = \sum_{m=0}^{\infty} m \exp\left[-\sum_{n=1}^{m} \ln(1 + b/n)\right] \tag{22}$$

converges. Now asymptotically (22) is

$$\int^{\infty} \mathrm{d}m\, m\, \exp\left[-\int^{m} \mathrm{d}n \frac{b}{n}\right] \sim \int^{\infty} \mathrm{d}m\, m^{1-b}$$

which converges if $b > 2$. Thus the condition for condensation is simply that $u(m)$ decays to β more slowly than $\beta(1 + 2/m)$. Note that in the homogeneous system the particle excess $N(\rho - \rho_{\mathrm{c}})$ condenses onto a spontaneously selected site.

In the previous section we mentioned the use of the ZRP as an effective description for models involving domain dynamics. As an example of this let us consider the 'bus route model' [20]. The model is defined on a $1d$ lattice. Each site (bus-stop) is either empty, contains a bus (a conserved particle) or contains a passenger (non-conserved quantity). The dynamical processes are that passengers arrive at an empty site with rate λ; a bus moves forward to the next stop with rate 1 if that stop is empty; if the next stop contains passengers the bus moves forward with rate β and removes the passengers. The bus route model can be related to the ZRP by a mean-field approximation in which we integrate out the non-conserved quantity (passengers). The idea is that a bus-stop, next to bus 1 say, will last have been visited by a bus (bus 2) a mean time ago of m/v where m is the distance between bus 2 and bus 1 and v is the steady-state speed. Therefore the mean-field probability that the site next to bus 1 is not occupied by a passenger is $\exp(-\lambda m/v)$. From this probability an effective hopping rate for a bus into a gap of size m is obtained by averaging the two possible hop rates $1, \beta$:

$$u(m) = \beta + (1 - \beta) \exp(-\lambda m/v)\ . \tag{23}$$

We can now see that this mean-field approximation to the bus-route model is equivalent to a homogeneous ZRP discussed earlier.

Since $u(m)$ decays exponentially the condition for a strict phase transition in the thermodynamic limit is not met, unless we take the passenger arrival rate $\lambda \to 0$. However on any *finite* system for λ sufficiently small, an apparent condensation will be seen. In the bus route problem this corresponds to the universally irritating situation of all the buses on the route arriving at once.

More generally the result that $u(m)$ should decay to β more slowly than $\beta(1 + 2/m)$ for condensation to occur has been used as a criterion for phase separation in one dimension [15–17]. In this context $u(m)$ corresponds to a conserved current flowing out of a domain of size m.

3.3 *Recent developments on ZRP*

Before closing this section let us mention some recent developments in the study of the ZRP. Firstly, one can generalise the ZRP to contain two or more species of particles whose hop rates are functions of the number of particles of each species at the departure site. Under certain conditions on these hop rates, a factorised steady state is obtained [21,22]. The multispecies system allows new condensation mechanisms whereby one species can induce condensation in the other [23].

Secondly, the dynamics of condensation has been of considerable interest. Starting from some random initial condition a coarsening process ensues from which a single condensate ultimately emerges. A variety of approaches have been used to study this process both for heterogeneous condensation [19,24] and homogeneous condensation [25–27].

Finally, although it has been shown that in the thermodynamic limit the canonical and grand canonical ensembles are equivalent [26], on large but finite systems there may be strong finite size effects making calculations ensemble dependent. For example, in a system with a single heterogeneity it has been shown that the canonical and grand canonical ensembles give different predictions for the behaviour of the current of particles on a finite system [28]. In particle there is an interesting 'overshoot' phenomenon whereby the fluid (uncondensed) phase continues to a higher density and maintains a higher current than would be expected in the thermodynamic limit. It would be of interest to have a detailed and general analysis of the finite-size effects associated with the condensation mechanisms.

4. General mass transport model

So far we have considered the ZRP and taken advantage of the fact that it has a factorised steady state. Actually there are some other models known to have factorised steady states, for example, the asymmetric random average process (ARAP) [29–32]. In that case, sites contain a quantity of mass that is a continuous rather than discrete variable. A natural question is: under what conditions does a model obtain a factorised steady state? To answer this we will consider a general mass transport model where the mass may be continuous and also time is discrete (the cases of discrete mass and continuous time may be considered as special limits) [33].

Thus we consider a one-dimensional lattice of N sites with periodic boundary conditions: associated with each site is a mass m_i which is most generally a continuous variable. The total mass is given by $M = \sum_{i=1}^{N} m_i$. The dynamics is as follows: in each time-step, at each site i, mass μ_i drawn from a distribution $\phi_i(\mu_i|m_i)$ 'chips off' the mass m_i, and moves to site $i-1$. Thus the master equation reads

$$P_{T+1}(\{m\}) = \prod_{i=1}^{N} \int_0^\infty \mathrm{d}m_i' \int_0^{m_i'} \mathrm{d}\mu_i \; \phi_i(\mu_i|m_i') \times \prod_{j=1}^{N} \delta(m_j - m_j' + \mu_j - \mu_{j+1})\, P_T(\{m'\})\,. \tag{24}$$

We show in [33] that a necessary and sufficient condition for the steady state to factorise is

$$\phi_i(\mu|m) = \frac{v\,(\mu) w_i(m-\mu)}{[v * w_i]\,(m)}\,, \tag{25}$$

in which case the single-site steady-state weights are given by

$$f_i\,(m) = [v * w_i]\,(m) \equiv \int_0^m \mathrm{d}\mu\, v\,(\mu)\, w_i\,(m-\mu)\,. \tag{26}$$

Here v and w_i are arbitrary functions but v must be the same for each site. Note that $\phi_i(\mu|m)$ factorises into a function of the mass which moves, $v(\mu)$, and a function $w_i(\sigma)$ of the mass which stays, $\sigma = m - \mu$.

Let us stress that this simple condition (25) determines a very general class of mass transport models with factorised steady states. This class encompasses both continuous and discrete mass, as well as parallel and random sequential dynamics. This approach provides a unified perspective of all previously known models and includes the ZRP and the ARAP as special cases [33].

We now analyse the condensation phenomena arising from $f_i(m)$ of the form (26). For simplicity we consider the homogeneous case $w_i(\sigma) = w(\sigma)$ which implies $f_i(m) = f(m)$. The grand canonical partition function (13) is given by

$$\mathcal{Z}_{\mathrm{gc}}(z) = \left[\int_0^\infty \mathrm{d}m\, z^m\, [v * w]\,(m)\right]^N = [V(z)\, W(z)]^N\,, \tag{27}$$

where

$$V(z) = \int_0^\infty \mathrm{d}\mu\, z^\mu v(\mu) \quad ; \quad W(z) = \int_0^\infty \mathrm{d}\sigma\, z^\sigma w(\sigma)\,. \tag{28}$$

As in (20) the mass density ρ determines the fugacity z through

$$\rho = \frac{zV'(z)}{V(z)} + \frac{zW'(z)}{W(z)}\,. \tag{29}$$

As before, the condensation mechanism is as follows: since (29) is an increasing function of z, and if there exists a finite $z_{\max} = \min(z_v, z_w)$ where z_v and z_w

are respectively the radii of convergences of $V(z)$ and $W(z)$, then as long as (29) diverges as $z \to z_{\max}$, one can always find a finite solution of $z < z_{\max}$ of eq. (29). In this case there is no condensation. On the other hand, a condensation transition can occur if $zW'(z)/W(z)$ and $zV'(z)/V(z)$ have finite limits as $z \to z_{\max}$. Then one can no longer satisfy (29) above a critical density set by ρ_c, the value of the right-hand side of (29) at $z = z_{\max}$. For $\rho > \rho_c$ we expect the excess density to be contained in a single site which forms the condensate.

Note that the condensation mechanism may be driven by either or both of $v(\mu)$, $w(\sigma)$ according to which $V(z)$, $W(z)$ has a finite radius of convergence. The condensation in the ZRP corresponds to condensation driven by $w(\sigma)$ since in the ZRP the amount of mass that can chip off is restricted to one unit. Condensation driven by $v(\mu)$ would require a slowly decaying $v(\mu)$ which would correspond to chipping off large fractions of mass. One would expect that under these conditions the condensate is mobile since events where a large fraction of mass is chipped off will move the condensate around the lattice.

Thus in addition to the two types of condensation seen in the ZRP, i.e. condensation of particles onto the site with the slowest hopping rate in a heterogeneous system and condensation due to slowly decaying hop rates on a homogeneous system, we identify a third mechanism for condensation due to slowly decaying size distribution of the chunks of mass transferred.

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