

Coupled chaotic oscillators and their relation to a central pattern generator for artificial quadrupeds

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Abstract. Animal locomotion employs different periodic patterns known as animal gaits. In 1993, Collins and Stewart recognized that gaits possessed certain symmetries and characterized the gaits of quadrupeds and bipeds using permutation symmetry groups, which impose constraints on the locomotion center called the central pattern generator (CPG) in the animal brain. They modeled the CPG by coupling four nonlinear oscillators and found that it was possible to reproduce all symmetries of the gaits by changing the coupling strength. Here we propose to extend this idea using coupled chaotic oscillators synchronized using the Pyragas method in order to characterize the CPG symmetries. We also evaluate the time series behavior when the foot is in contact with the ground: this has potential robotic applications.

Keywords. Animal gaits; synchronization; chaos.

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1. Introduction

Animal gaits have always attracted interest not only from animal lovers, but also from scientists since ancient times, due to the apparently symmetric movements. They were probably first classified by Muybridge [1] in the 1880s, who, using many still cameras, took sequential pictures of a large variety of animals and determined the motion of their limbs. In this way, he was able to classify the motion of a large number of animals. In 1965, Hildebrand [2] showed that gaits follow certain patterns with correlation between length of cycles and individual foot strikes. Such organized behavior, which he established for a large number of horses, correspond very well to expectancy; he studied the distribution of gaits in parameter space, where the period of a cycle was plotted against the time lag between the foot strike defining the cycle and the next. Late in the twentieth century biologists and

Table 1. Symmetries associated with gaits.

Gait	Symmetry	Group
Stopped	$(I, \theta) (\alpha, \theta) (\beta, \theta) (\alpha\beta, \theta)$	$D_2 \times S^1$
Pronk	$(I, 0) (\alpha, 0) (\beta, 0) (\alpha\beta, 0)$	D_2
Pace	$(I, 0) (\alpha, \frac{1}{2}) (\beta, \frac{1}{2}) (\alpha\beta, 0)$	\tilde{D}_2^D
Bound	$(I, 0) (\alpha, 0) (\beta, \frac{1}{2}) (\alpha\beta, \frac{1}{2})$	\tilde{D}_2^F
Trot	$(I, 0) (\alpha, \frac{1}{2}) (\beta, 0) (\alpha\beta, \frac{1}{2})$	\tilde{D}_2^L
Rotatory gallop	$(I, 0) (\beta, \frac{1}{2})$	\tilde{Z}_2^L
Transverse gallop	$(I, 0) (\alpha\beta, \frac{1}{2})$	\tilde{Z}_2^F
Canter	$(I, 0)$	\mathbb{I}

$\alpha = (12)(34)$, $\beta = (13)(24)$, $\alpha\beta = (14)(23)$ and S^1 refers to all cyclic group of phase slip mod 1. D represents the dihedral subgroup and Z all the cyclic subgroups. The tilde indicates the existence of phase slip symmetry. The notation $\frac{1}{2}$ represents a half cycle out of phase.

mathematicians united their efforts in identifying the symmetry groups of common gaits: these are summarized in table 1.

Current analysis represents the gait as a cycle pattern in the movement of symmetrically placed limbs. A cycle is represented by the interval between successive foot strikes of the same foot, from where the sequence is repeated. The foot strike factor β is the fraction of a cycle when the foot is in contact with the ground [3]. The relative phase ϕ is defined as the fraction of a cycle between the contact of the surface by the reference foot and another one, when they touch the ground. For simplicity we assume β to be the same for all limbs, and it plays no role in this study, although it is quite obvious that it should in a more realistic model. Here we are looking for a model that not only has the appropriate symmetries, but that also reproduces the complete sequences including those which the animals learn.

The movement of mammalian phenotypes evolved into two kinds of gaits: bipedal, where the limbs can be out of phase (walking or running) or in phase (jumping or hopping), or quadrupedal, which have a more complicated behavior of the relative phase. The natural gaits [4] are:

- *Walk*: The limbs move with a quarter of a cycle out of phase, designing an eight.
- *Trot*: The diagonal legs move in phase and this pair is half a cycle out of phase with the other one.
- *Pace*: The fore and hind left or right limbs are paired and move half cycle out of phase in between the pairs.
- *Canter*: A diagonal pair moves in phase, the other move half cycle out of phase between themselves and out of phase with respect to the first pair.
- *Bound*: The fore legs move in phase, as do the hind legs, and they move half cycle out of phase.
- *Transverse gallop*: The front and hind left (right) legs move one quarter of a cycle out of phase, the front (hind) limbs are slightly out of phase between themselves.

Coupled chaotic oscillators

- *Rotatory gallop*: It is the specular reflection of the transverse gallop along the front-hind axis.
- *Pronk*: The four limbs move together and in phase.

All these gaits correspond to the symmetrical groups listed on table 1.

It is clear that any modeling of gaits has to have solutions with the symmetries of these groups, but they also have to have the sequences of several of the gaits described above obtainable by changing the parameter, which we reproduce in figure 1 for clarity [2,5]. For instance for a horse, they need to pace, trot, gallop and canter gaits as solutions, as well as the complete stop. Some of these gaits are not natural in the horse and must be learned.

Here we make the assumption that an animal like a horse can perform a gait only when that pattern exists in its brain. Therefore, if we are going to model the locomotion of a horse, the model has to have all the sequences as their solutions.

Biological models assume that the animal nervous system contains a variety of central pattern generators (CPG), each oriented to a specific action [6]. For instance, the locomotion CPG controls the rhythm of mammal gait, in the case of quadrupedal mammals this is modeled by a system of coupled cells where each cell is composed of a set of neurons directly responsible to harmonize the movement of the leg [7]. A simplified mathematical model of locomotion CPG consists in replacing each cell by a nonlinear oscillator. This model has been studied using different methods: bifurcation theory, numerical simulations and phase response [2,5,8–11].

The idea to study rhythmic patterns in animal gaits using symmetries in models was introduced by Hildebrand [10] and Schöner *et al* [12]. The concept of symmetries in coupled cells as a model of the CPG for locomotion in quadrupedal mammals was first used by Collins and coworkers [5,13,14]. The model consists of a ring of four coupled nonlinear oscillators. Each oscillator may represent limb of the animal or a neuron that controls it. The stability and breakdown of the symmetries play an effective role in the validity of the model. Golubitsky *et al* argued that symmetries present in the above model for walk, trot and pace are not adequate for quadrupeds, since the trot and pace correspond to conjugate solutions which have the same stability and which depend on initial conditions [15]. Many quadrupeds

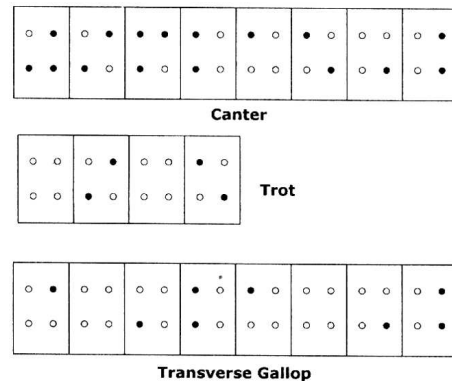


Figure 1. Temporal sequences for different gaits.

move with pace but do not trot (camels) or vice versa (horse), unless trained. In this work we propose a different coupling mechanism in order to avoid the problem of multiple conjugated solutions.

The CPG is modeled by the following system of ordinary differential equations:

$$\frac{dX_j}{dt} = f(X_j) + h_j(X_{j-1}, X_{j+1}), \quad (1)$$

where $j = 1 \dots n \bmod 4$ is the index of the cell, $X \in \mathbb{R}^\kappa$ is the state vector and $f: \mathbb{R}^\kappa \rightarrow \mathbb{R}^\kappa$ is a nonlinear velocity vector field. $h_j: \mathbb{R}^\kappa \times \mathbb{R}^\kappa \rightarrow \mathbb{R}^\kappa$ are coupling vector fields. We define the symmetry of the ring as the permutations of cells that preserve the coupling. A permutation σ of $\{1 \dots 4\}$ numbers on the phase space $X = (X_1, X_2, X_3, X_4)$,

$$\sigma X = (X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, X_{\sigma^{-1}(3)}, X_{\sigma^{-1}(4)}) \quad (2)$$

then σ is the symmetry of the ring in

$$F(\sigma X) = \sigma F(X), \quad (3)$$

where $F(\cdot) = f(\cdot) + h_j(\cdot, \cdot)$ and $h_j(\cdot, \cdot)$ with $j = 1, 2, 3, 4$ are the coupling vectors. Since all cells (oscillators) are identical, the coupling must fulfill

$$h_j(\sigma X_{j-1}, \sigma X_{j+1}) = h_{\sigma(j)}(X_{j-1}, X_{j+1}) \quad (4)$$

all indices being considered modulo 4.

If we define $[i, j]$ as the action of exchange h_i with h_j , then we fixed the field vector coupling as well as its symmetries of the ring are: $\{[1,4]; [2,3]; [1,2]; [4,2]\}$. These symmetries are called *Type-2* by Collins and Stewart [5]. Another kind of symmetry is called symmetry of phase change [15]. Assuming that $X(t)$ is a periodic solution with minimal period (cycle) T , and γ represents the symmetry (ij) to permute X_i for X_j , then $\gamma X(t)$ will be a periodic solution if the trajectories $\{X(t)\}_t$ and $\{\gamma X(t)\}_t$ coincide. Therefore, the only solution is the existence of phase delay θ such that $\gamma X_j(t) = X_j(t + \theta)$. The pair (γ, θ) is a spatio-temporal symmetry where θ is a phase delay. Finally we define as primary gait, those gaits modeled by identical output signals of each cell but out of phase.

We associate the index of a cell to each limb as follows: $j = 1$ hind left, $j = 2$ fore left, $j = 3$ fore right and $j = 4$ hind right. The possible symmetries of the primary gait for four-legged animals characterized by Type-2 arrays are shown in table 1. Here we study the possibility of using a ring of coupled chaotic oscillators to produce the locomotion of quadrupeds based on the symmetries of primary gaits using Pyragas control theory [16].

2. A model for the CPG

We represent each cell by a Rössler oscillator coupled using Pyragas method [16]. We use Rössler chaotic oscillators since these are the only ones that have shown to synchronize simulating the primary gait. This behavior does not happen for Van der Pol [14] or Showalter [17] oscillators even if the single oscillator reproduces

the necessary output. The primary gait is important for any digital application on mechanical limbs. Besides, Hildebrand [2] already mentions that a given gait, i.e., a given time series, repeats periodically within a certain error, up to 15 per cent. Recently, Hu *et al* [18] showed that exact symmetries persist on time averaging for systems of oscillators in the chaotic state, thus guaranteeing the symmetries needed, at the same time accounting for the errors reported by Hildebrand.

$$\begin{cases} \frac{dx}{dt} = -(y+x) \\ \frac{dy}{dt} = x+0.2y \\ \frac{dz}{dt} = 0.2+z(x-c) \end{cases} . \quad (5)$$

Random initial conditions are taken. A direct synchronization mechanism where the master variable is ‘y’ and the others are the slave variables and a delay time series are used to obtain the delay feedback value. Then the coupling functions are

$$\begin{aligned} h_i(X_i, X_{i+1}, X_{i-1}) &= k_i(y_{i-1}(t-\tau) - y_i(t)) \\ &+ g_i(y_{i+1}(t-\tau) - y_i(t)). \end{aligned} \quad (6)$$

The symmetry conditions associated to a Type-2 array limits the range of k_i and g_i values. In this case $g_1 = k_2$, $g_2 = k_3$, $g_3 = k_4$ and $g_4 = k_1$ (see figure 2). The delay time $\tau \in \mathbb{N}$, and the nonlinear constant c play an important role in the wave pattern obtained. We consider as the output of each cell (oscillator) the value of the variable $x_i(t)$ and we define a threshold function:

$$Q(x) = \begin{cases} 0 & \text{if } x > 2.0 \\ 1 & \text{if } x \leq 2.0 \end{cases} . \quad (7)$$

This defines a mapping from phase space into a binary matrix space of 2×2 . We associate the value ‘1’ to the state ‘limb on ground’ and the value ‘0’ to the state ‘limb in movement’, not on the ground. Finally the matrix representation is

$$\mathbf{C} = \begin{pmatrix} \text{Fore left} & \text{Fore right} \\ \text{Hind left} & \text{Hind right} \end{pmatrix} .$$

Therefore, the gait is nothing but the sequence of matrices of successive states representing the symmetry of CPG. For instance, the pronk is given by the sequence of matrices:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} .$$

This allows a clearer visualization of the symmetries of the primary gait, even if we lose the time interval between different patterns. This is highly important for application in robotics; it conditions the actuator responses, but also controls undesirable instabilities produced by inertial mechanical movements. For this reason we need to analyse the time each pattern stays in a periodic sequence.

We consider two types of combinations for the coupling constants, which are the most representatives in between the values tried. We call it SA model when $k_1 = k_3 = 0.1$ and $k_2 = k_4 = 0.001$, and SB when $k_1 = k_3 = 0.1$ and $k_2 = k_4 = -0.001$. Those values were selected under the assumption that strong coordination exists between the limbs associated to the same cerebral hemisphere, while they are weakly correlated when they belong to different hemispheres.

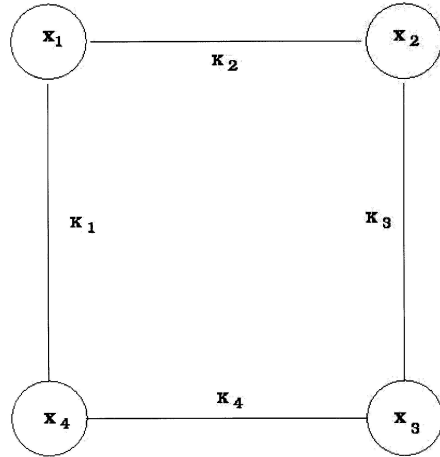


Figure 2. Cell coupling diagram.

3. Numerical results

In this section we show some numerical simulations that we have performed for the above models with and without delay. We notice that, independent of the selection of the coupling strength, SA or SB, there is a strong behavior as a function of the time delay, which is robust under changes in the variable c in eq. (5). Increasing the delay we observe the transitions chaos \rightarrow periodic orbits \rightarrow stable fixed point \rightarrow primary Hopf bifurcation \rightarrow secondary Hopf bifurcation \rightarrow chaos. In figure 3 we plot the largest Lyapunov exponent as a function of the delay, where the chaotic regions can be clearly seen. As mentioned before, the chaotic behavior does not affect the symmetries, but it interferes with the time of stability of each pattern, which is troublesome for robotic design.

3.1 SA coupling

For $6 \leq \tau \leq 13$, the periodic gait obtained is

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

which corresponds to the ‘prank’ and the symmetry is D_2 . On the other hand, for $\tau = 14$ the limit cycle is not stable any longer and it appears as an asymptotic stationary state, which produces a single sequence which corresponds to the symmetry $D_2 \times S^1$, therefore a ‘stop’, since all limbs are on the ground. For $\tau = 34$, the fixed point loses its stability and it becomes unstable, where the orbits converge to a single limit cycle. The pattern found for the delay $34 \leq \tau \leq 38$ is

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

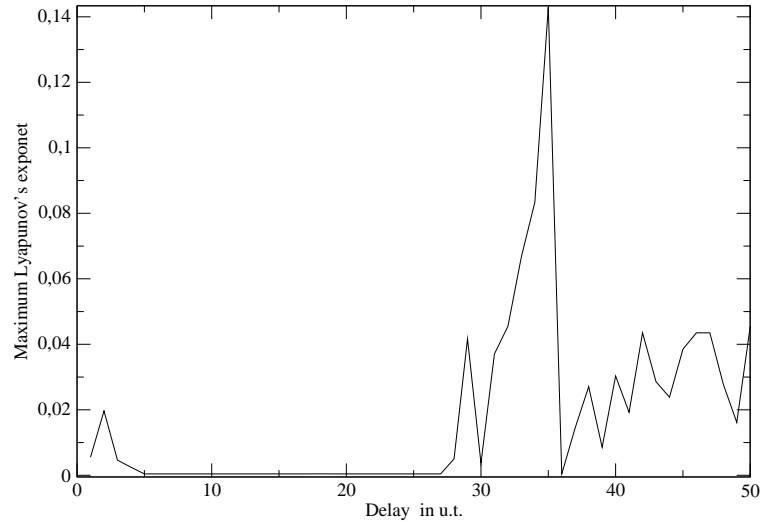


Figure 3. Largest Lyapunov exponent λ obtained from the time-series of the oscillators in the ring, using the algorithm of Aurell *et al* available in the TISEAN [19] package.

This has a symmetry \tilde{D}_2^L , which corresponds to the ‘trot’. In this case each step involves the movement of all the limbs, gait which is only observed in heavy quadrupeds, above 1 ton, such as giraffes and buffaloes. We could not find any other patterns in this interval of time delay.

3.2 SB Coupling

For $6 \leq \tau \leq 13$ the coupled chaotic system oscillates in a stable limit cycle, which in this case produces the patterns:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

which has a symmetry \tilde{D}_2^F and corresponds to the ‘bound’. We should mention that the true bound, as the one observed in the Siberian squirrel, corresponds to the state of all limbs on the air. On the other hand, in this case, all four limbs are on the ground in order to generate another step, which does not exist in nature. We resolve this drawback by applying the ‘not’ operator each matrix. As in the SA case, for $14 \leq \tau \leq 32$ the coupled system has a stable fixed point, and the limit cycle becomes unstable, which correspond to stop. For $\tau = 33$ the fixed point become unstable, and a stable cycle appears producing the patterns:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

which corresponds to gait pace with symmetry \tilde{D}_2^D . As described above, the horse never sets all four limbs on the ground. However, applying the ‘not’ operator each

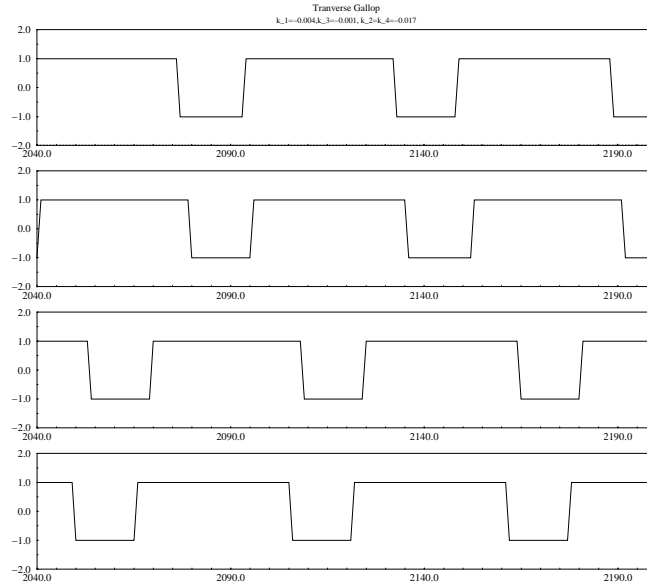


Figure 4. Gallop time graph.

matrix, we have resolved this problem. For this coupling, the gait is not structurally stable for all time delay values. From $\tau = 34$ until $4\tau = 37$ the gait changes to another periodic pattern:

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \right.$$

$$\left. \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

with symmetry $\{(I, 0)(\alpha, \frac{2}{3})(\beta, \frac{2}{3})\}$. This symmetry does not correspond to any primary gait observed in nature. For $\tau = 38$ the system generates again a gait-like ‘pace’.

3.3 Without delay

Here, we show some numerical simulations, which are carried out simply by changing the coupling constants. As mentioned before, we were trying to look for the sequences for all possible gaits of a given animal, in this case the horse. For each selection of the coupling constants, after obtaining the output we apply a threshold which is chosen in such a way that it reproduces one of the states of the sequence, for instance, all legs in the air, and with this one we obtain the complete sequence. In figures 4–6 we show the time series for transverse gallop, canter and trot respectively, for which we showed the corresponding sequences in figure 1.

Coupled chaotic oscillators

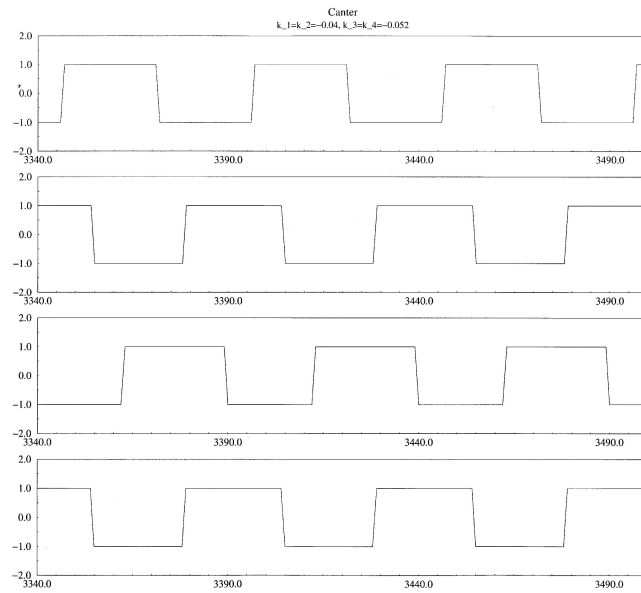


Figure 5. Canter time graph.

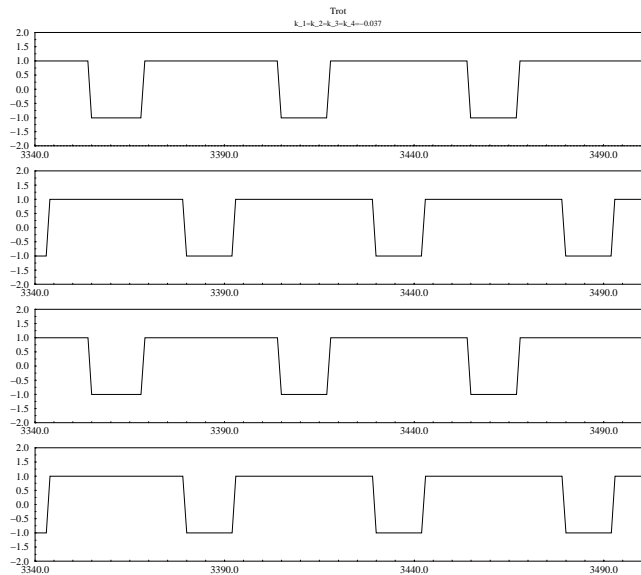


Figure 6. Trot time graph.

These preliminary results show that there is a good possibility that an improved model can have a realistic application in robotics, even though a lot of work lies ahead, before it can be applied.

4. Conclusions

We have studied a system of chaotic oscillators in a ring, with and without delay diffusive interaction, as a model for animal locomotion. We had shown that a number of natural gaits, as well as some which are known to be learned (in particular by the horse) are possible solutions of the system upon variation of the coupling constants or the delay time.

This supports the idea that a particular gait is possible only if it is imprinted in the animal's brain. The present model can be considered as a starting point for a realistic application in robotics, although many problems need to be solved before it can be applied.

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