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Local dimension and finite time prediction in coupled map lattices

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Abstract. Forecasting, for obvious reasons, often become the most important goal to be achieved. For spatially extended systems (e.g. atmospheric system) where the local nonlinearities lead to the most unpredictable chaotic evolution, it is highly desirable to have a simple diagnostic tool to identify regions of predictable behaviour. In this paper, we discuss the use of the bred vector (BV) dimension, a recently introduced statistics, to identify the regimes where a finite time forecast is feasible. Using the tools from dynamical systems theory and Bayesian modelling, we show the finite time predictability in two-dimensional coupled map lattices in the regions of low BV dimension.

Keywords. Coupled map lattices; spatio-temporal chaos.

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1. Introduction

Spatially extended systems often arise in a variety of contexts in physical, biological and ecological sciences. In most of the cases these systems often exhibit highly complex type of evolutions such as high dimensional or spatio-temporal chaos. They are characterized by a large number of positive Lyapunov exponents which measures the exponential divergence of neighbouring trajectories. As a consequence, it is almost impossible to know precisely the long-term behaviour of these systems as they evolve. One can visualize the extended systems by considering a collection of a large number of sub-systems coupled either locally or globally which are very useful to understand the various features associated with spatio-temporal chaos. In this connection, coupled map lattices (CML) provide a prototype model to study chaos in spatially extended systems. In general, they occur in a variety of fields involving spatio-temporal complexity.

In this connection, the bred vector (BV) dimension or local dimension introduced recently in the atmospheric science literature [1] provides a better understanding of the dynamics, at least, in the local sense.

The aim of this paper is to study the bred vector dimension and its connection to the predictability in coupled map lattices. This paper is organized as follows: In §2, we present a rather brief overview of the bred vector (BV) dimension that is applied in the context of spatially extended systems. By considering coupled map lattices, we study the nature of BV dimension and its connection to predictability with the aid of a simple prediction tool and cluster weighted modelling in §3. Summary and conclusions are given in §4.

2. Bred vector dimension

The bred vectors are constructed in a similar way as Lyapunov vectors but in practical applications bred vectors are different in two aspects. Firstly, for bred vectors there is no global orthonormalization and secondly, they are finite-amplitude, finite-time vectors. In the following, we discuss how one can formulate BV dimension for spatio-extended systems.

Consider a 2D spatially distributed system whose state at a given time t_1 is defined over a collection of points (i,j). Here we take $M-1$ nearest neighbours for each point (i,j) in a square lattice with $M=25$, as illustrated in figure 1. Logistic maps are one-variable dynamical systems and in order to specify the corresponding state at a point including its neighbours we need an M-dimensional state vector called bred vector. Now generate k distinct perturbations of the state starting at $t_0 < t_1$ to obtain k local bred vectors. The $k \times k$ covariance matrix of the system is just $\mathbf{C} = \mathbf{B}^{\mathbf{T}}\mathbf{B}$, where $\mathbf{B}$ is the $M \times k$ matrix of local bred vectors each normalized to unity. In this paper we will fix $k=5$. We order the eigenvalues of the covariance matrix as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and define the singular values of $\mathbf{B}$ as $\sigma_i = \sqrt{\lambda_i}$. Here the connection with factor analysis is clear [2], since the eigenvalues of the covariance matrix give an idea of the local linear independence of the k local bred vectors. An effective dimension of the space of bred vectors can be obtained by fixing a threshold value corresponding to the highest l eigenvalues, as is done in principal components analysis. Thus an approximation of the data, supposing zero average for simplicity, is contained in the product $\mathcal{FL}$ were $\mathcal{F} = \mathbf{BL}^{\mathbf{T}}$ is called the factor and $\mathbf{L}$ the loading matrix of dimension $l \times k$. The rows of the loading matrix are the components of the eigenvectors corresponding to the dominant eigenvalues. Clearly there is an arbitrariness in the stipulation of l and this ambiguity is absent when using the concept of BV dimension.

The eigenvalues λ_i represent the amount of variance in the set of k unit bred vectors. In order to estimate unambiguously the value of the threshold, one defines the following statistic [1]:

$$\psi_{i,j}(\sigma_1, \sigma_2, \ldots, \sigma_k) = \frac{\left(\sum_{l=1}^{k} \sigma_l\right)^2}{\sum_{l=1}^{k} \sigma_l^2}. \tag{1}$$

As each of the k bred vectors is normalized to unity, ψ assumes values in the interval $(0,k)$. Examples of the values of this statistic for several distributions of bred vectors can be found in [1]. A property of the statistic just defined is its robustness under noise or numerical errors. It can be used to determine the

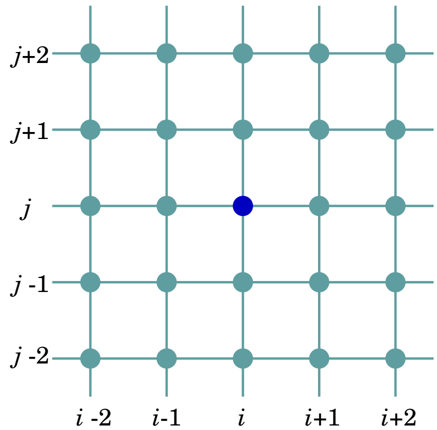

Figure 1. Schematic diagram showing the choice of the nearest neighbours at site (i, j) for local dimension. The bred vectors are the dynamical variables associated with these sites.

dominant eigenvalues l: just take it to be the smallest integer bigger than ψ. In this sense an approximation to the bred vectors is obtained as a product of the corresponding factor and the loading matrix.

3. Local dimension in coupled map lattices

We consider the case of two-dimensional coupled map lattices consisting of logistic maps [3].

$$x_{n+1}^{i,j} = (1-\epsilon)f(x_n^{i,j}) + \frac{\epsilon}{4}[f(x_n^{i-1,j}) + f(x_n^{i+1,j}) + f(x_n^{i,j-1}) + f(x_n^{i,j+1})], \tag{2}$$

with

$$f(x) = \mu x(1-x), \quad x \in (0,1), \quad \mu \in (0,4), \tag{3}$$

where $i,\ j = 1, 2, \ldots, N$ and ϵ represents the coupling strength. Here, we use $N = 50$, $\mu = 4$, $\epsilon = 0.4$ and periodic boundary conditions. The reference spatial variables $x^{i,j}$, $i, j = 1, 2 \ldots, N$, are obtained by evolving system (2) from random initial conditions. The number of iterations is chosen to be 5015 so that transients are removed. We generate additional spatial variables by adding small perturbations to the reference variables at time 5000 to measure the local instability of the coupled maps. Thus at time 5015 we have spatial distributions corresponding to the reference variable and four distinct perturbations. The local dimension at the spatial points are computed at this time value using the statistic defined in eq. (1) as discussed in the previous section. Figure 2 illustrates the results of the logistic maps (2) calculations where dark regions correspond to low dimensions and bright

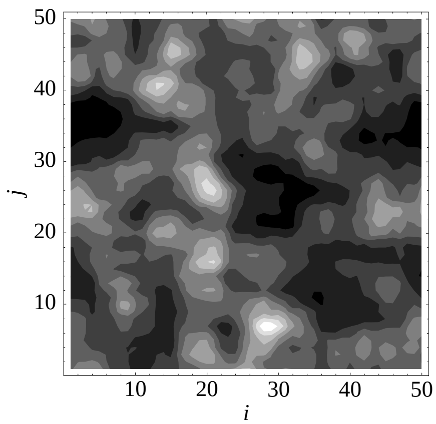

Figure 2. Gray scale plot showing the regions of low (dark) and high (bright) local dimensions for the coupled logistic maps (2).

regions represent high dimensions. We found that the local dimension has a minimum value, $\psi = 1.0323$, at $i = 2$, $j = 37$ and a maximum value, $\psi = 2.3636$, at $i = 28$, $j = 7$. We note that the maximum and minimum values at the spatial points (i, j) are practically constant under evolution of system (2), a few steps forward or backward.

3.1 *Simple nonlinear prediction*

In order to establish a relation between dimension and predictability, we analyse the time series at the lattice points $(2, 37)$ and $(28, 7)$ discussed above. For a chaotic time series $\{x_j\}_{j=1}^{L}$, one can define a simple prediction as an average over *futures* of the neighbours, $\mathcal{U}_n$ [4],

$$\tilde{x}_{n+1} = \frac{1}{|\mathcal{U}_n|} \sum_{x_j \in \mathcal{U}_n} x_{j+1}. \tag{4}$$

A point x_j belongs to $\mathcal{U}_n$ if $|x_n - x_j| \leq \delta$. Here $|\mathcal{U}_n|$ denotes the number of elements of the neighbourhood $\mathcal{U}_n$. Then the prediction error is given by

$$\eta = \left| x_{\mathrm{orig}}^{i,j} - x_{\mathrm{pred}}^{i,j} \right|. \tag{5}$$

We apply the simple prediction algorithm during the interval from $n = 5015$ to $n = 5040$. Figure 3 shows the prediction errors for $x^{2,37}$ and $x^{28,7}$ with $\delta = 0.05$. It is evident that the prediction error is small at the lattice point $(2, 37)$ where the local dimension is low, and large at the point $(28, 7)$ where the local dimension is maximum. In addition, similar results were obtained for points with other local dimension values. We found that the prediction error is consistently low in regions of low local dimension and high in the regions of relatively high local dimension.

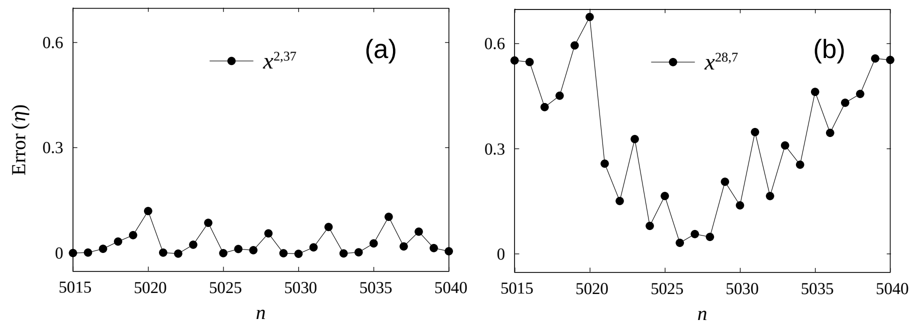

Figure 3. Simple finite prediction error for the coupled logistic maps. The error remains minimum for $x^{2,37}$ (low BVD point) (**a**) and high for $x^{28,7}$ (high BVD point) (**b**).

3.2 *Cluster-weighted modelling*

In order to characterize the time series data at the lattice points where the local dimensions are low or high, we use a more powerful technique known as cluster-weighted modelling. This method is based on the probability density estimation approach developed by Gershenfeld, using probabilistic dependence of local models [5]. The cluster-weighted modelling technique essentially estimates the functional dependence of time series in terms of delay coordinates. The main task of this approach is to find the conditional forecast by estimating the joint probability density.

Let $\{y_n, \vec{x}_n\}_{n=1}^N$ be the N observations in which $\vec{x}_n$ are the known inputs and y_n are the corresponding outputs. By knowing the joint probability density $p(y, \vec{x})$, we can derive the conditional forecast, the expectation value of y, given $\vec{x}$, $\langle y|\vec{x}\rangle$. We can also deduce other quantities such as the variance of the above estimation. Actually, the joint density $p(y, \vec{x})$ is the expanded terms of clusters which describe the local models. Each cluster contains three terms, namely, the weight $p(c_m)$, the domain of influence in the input space $p(\vec{x}|c_m)$, and finally the dependence in the output space $p(y|\vec{x}, c_m)$. Thus the joint density can be written as [5]

$$p(y, \vec{x}) = \sum_{m=1}^{M} p(y|\vec{x}, c_m)\, p(\vec{x}|c_m)\, p(c_m), \tag{6}$$

where M corresponds to the number of clusters. Once the joint density is known, the other quantities can be derived from $p(y, \vec{x})$. For example, the conditional forecast is given by

$$\langle y|\vec{x}\rangle = \int y\, p(y|\vec{x})\, \mathrm{d}y = \frac{\sum_{m=1}^{M} f(\vec{x}, \beta_m)\, p(\vec{x}|c_m)\, p(c_m)}{\sum_{m=1}^{M} p(\vec{x}|c_m)\, p(c_m)}. \tag{7}$$

Here $f(\vec{x}, \beta_m)$ describes the local relationship between $\vec{x}$ and y. The parameters β_m are found by maximizing the cluster-weighted log-likelihood. The simplest approximation for the local model is with linear coefficients of the form

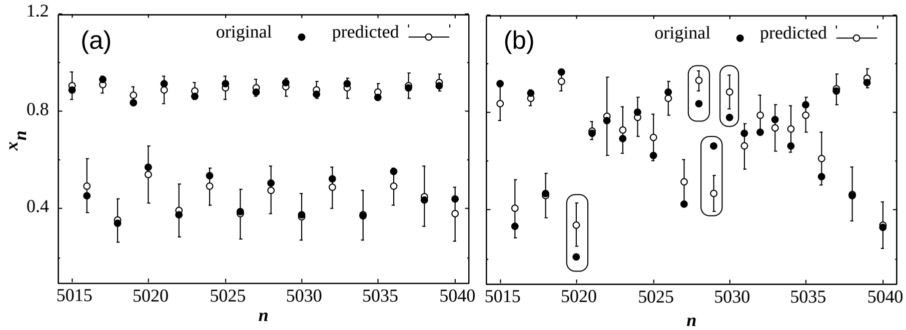

Figure 4. Prediction of time series for the coupled logistic maps by using cluster-weighted modelling. The original (filled circles) and predicted values (circles with error bars) of the time series at (**a**) low BVD location $(2, 37)$ and (**b**) high BVD location $(28, 7)$.

$$f(\vec{x}, \beta_m) = \sum_{i=1}^{I} \beta_{m,i}\, f_i(\vec{x}). \tag{8}$$

The method just described above provides more insights into the predictability issue of the present problem. For the present problem, we consider the average prediction $\langle y|\vec{x}\rangle$ using the conditional distribution obtained from the joint distribution $p(y, \vec{x})$ given in (6). Using the time series data at the lattice points at each site for definiteness, y the value to be predicted is taken as $x(n + 2\tau)$ (e.g. with embedding dimension $m = 3$ and delay τ), given the vector of delayed components $\vec{x} = \{x(n), x(n + \tau), n\}$; here we take $\tau = 1$. Figure 4 presents a series of predicted values, using always the two most recent original values, and the corresponding variance. In most cases analysed, we found the following behaviour. Lattice sites with high BV dimension result in predicted values with larger variances than those with low BV dimension, or either the prediction tend to fall outside the confidence interval defined by the variance of the future value. These facts are shown in figures 4a and 4b for the low and high BV dimensions, 1.0323 and 2.3636, respectively. In this case the simple prediction results are more compelling than the CWM since the difference between the highest and lowest BV dimension is not big enough. In this case the conclusion that the uncertainty in prediction is related to dimension is even more forceful.

4. Summary and conclusions

In the present paper, we have considered the issue of forcasting based on the BV (local) dimension for spatially extended systems. In particular, we have shown that the BV dimension has direct relation with the finite time predictability. By considering two-dimensional coupled map lattices and with simple nonlinear prediction tools, we pointed out that the spatial points of small BV dimension are more predictable than the regions of high BV dimension. If the dimension changed substantially during the short time evolution then the relationship between dimension and prediction could not be maintained. In this work predictions are made over intervals of 20 or 25 units of time units and under such circumstances the value of BV dimension is practically constant.

For deterministic evolution some systems are more predictable than others and this can be measured by Lyapunov exponents. However, these exponents are well defined only asymptotically and are not unique for finite time calculations [6–9]. In such cases bred vectors are the proper tool to use and in this paper we provided the relation between predictability and the value of the BV dimension.

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