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Statistical methods in nonlinear dynamics

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Abstract. Sensitivity to initial conditions in nonlinear dynamical systems leads to exponential divergence of trajectories that are initially arbitrarily close, and hence to unpredictability. Statistical methods have been found to be helpful in extracting useful information about such systems. In this paper, we review briefly some statistical methods employed in the study of deterministic and stochastic dynamical systems. These include power spectral analysis and aliasing, extreme value statistics and order statistics, recurrence time statistics, the characterization of intermittency in the Sinai disorder problem, random walk analysis of diffusion in the chaotic pendulum, and long-range correlations in stochastic sequences of symbols.

Keywords. Statistical methods; nonlinear dynamics; chaos; Lyapunov exponents; power spectrum; extreme value statistics.

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1. Introduction

The simplest problems in classical physics pertain to systems that exhibit regular motion. The corresponding dynamical equations can be solved analytically. Examples of such regular motion include simple harmonic motion and planetary motion in orbits given by conic sections. With the advent of high-speed digital computers, there is a growing interest in obtaining numerical solutions to differential equations that arise while modelling complex natural phenomena. These equations are generally nonlinear, and many of them exhibit chaotic solutions: trajectories which start out with nearly identical initial conditions diverge exponentially with time, giving rise to unpredictability. This kind of chaotic behaviour, arising from sensitive dependence on initial conditions, raises the following question: what sense can we make of such apparently random motion? Owing to the lack of predictability, a single trajectory by itself is not very interesting; however, a collection of these, viewed as a statistical ensemble, could provide a means by which we can extract useful information about the underlying nonlinear chaotic dynamics. Such an enterprise invariably calls for the extensive use of statistical methods in the study of deterministic systems.

On the other hand, statistical mechanics is built on the postulate of the equal *a priori* probabilities of the accessible microstates of an isolated macroscopic system. The dynamics of the microscopic constituents is governed by time-reversal invariant deterministic equations. Despite this, the macroscopic system exhibits irreversibility and possesses an arrow of time.

The burst of activity in the field of nonlinear dynamics in the 1970s raised hopes that chaos could perhaps provide a link between the time-reversal invariant microscopic dynamics and the time-directed macroscopic evolution. Although many serious efforts were made, no spectacular results emerged. Instead, chaos emerged as an independent and robust discipline on its own. The situation changed dramatically in the 1990s with the work of Evans *et al* [1], aimed at 'quantifying' and 'deriving' the second law of thermodynamics from deterministic microscopic dynamics. These efforts have considerably narrowed the gap between deterministic microscopic behaviour and stochastic macroscopic behaviour.

With these brief remarks on the connection between statistics and dynamics, we review in this paper a few of the statistical tools that are currently being employed in the study of nonlinear deterministic and stochastic dynamical systems. It is the long-term predictability that becomes impossible in chaotic systems; in stochastic system, of course, there is no definite predictability whatsoever, on short or long time-scales. Hence it is quite reasonable to expect that chaotic trajectories should exhibit some special features that distinguish them from stochastic ones. What are these features? We address this and related questions in this paper.

The first issue we deal with concerns the calculation of the power spectral density and the associated problem of aliasing; though aliasing is considered undesirable, we find that it helps to distinguish between a finite and discrete chaotic time series on the one hand, and a stochastic one, on the other hand. Section 2 is devoted to this topic. Next, in §3, we discuss the use of extreme value and order statistics in the study of nonlinear dynamical systems. Periodic orbits form a special class of orbits. The extreme value density exhibits a discontinuity at the maximum point of each periodic orbit. Chaotic systems contain periodic orbits that are unstable, hence the iteration rule cannot be used to obtain the other points of the periodic orbits. Order statistics, however, gives information on all the other points belonging to unstable periodic orbits. A general orbit can be characterized in terms of the frequency of its visits to a small neighborhood of its initial point. This aspect is investigated with the help of recurrence time statistics, described in §4. Next, in §5, we report studies on an iterated function system (IFS) describing the survival probability of a particle diffusing in a lattice with Sinai disorder and with absorption at origin. The dynamics of the survival probability exhibits intermittency. We also observe intermittency in a driven, damped, nonlinear pendulum. This motivates us to employ random walk arguments to model the time evolution of the system. The details are reported in §6. Random walk phenomenology is employed in the study of the dynamics of complex systems possessing long-range correlations. We show that coarse-graining of a Markov sequence of symbols leads to spurious long-range correlations. This work is reported in §7. Finally, in §8, we summarize the principal results presented in this review.

2. Power spectral analysis and aliasing

Power spectral analysis is one of the earliest tools used in the study of stationary stochastic processes and deterministic dynamical systems. Power spectra obtained from the time series of dynamical systems can be classified into the following types: The power spectrum of a periodic system has peaks corresponding to the associated frequencies and their harmonics. The power spectrum of a quasi-periodic system displays sharp lines corresponding to the constituent incommensurate frequencies and a complicated overtone spectrum. The spectral measure of a generic stationary stochastic process has an absolutely continuous part and a broad power spectrum. A chaotic system also exhibits broad-band structure. The power spectrum of a chaotic dynamical system is found to exhibit an exponential decay followed by a much slower (e.g., algebraic) decay [2–4]. There are two possible sources for the problems that arise in the power spectral analysis of discrete data, either obtained from experiment or generated numerically: (i) the data are sampled at finite time intervals; (ii) the data set is of finite length. The latter implies that the frequency information pertaining to the actual system is not completely available. A finite sampling time allows the computed power spectrum to lie in a domain governed by the Nyquist frequency; the actual frequencies are folded into this domain in a complicated way, making the interpretation somewhat difficult [5,6]. This problem is called aliasing, an issue which needs careful consideration while carrying out the analysis of time series data. In the course of our work on aliasing, we realized that the modifications to the power spectrum induced by a non-zero sampling time and a finite length of the time series are not undesirable features. Instead, they offer an easy means of distinguishing between a chaotic process and a generic stationary stochastic process. We now define the relevant quantities and proceed to discuss two illustrative special cases in which the issue of aliasing can be resolved analytically.

2.1 *Power spectrum: Definition*

Let $x(t)$ be one of the state variables describing a chaotic dynamical system. An experimental (or numerical) realization of $x(t)$, however, is represented as a time series, $\{x_j = x(t = j\tau),\ j = 0, \ldots, N-1\}$, where τ is the sampling interval. If $X(f, N, \tau)$ is the discrete Fourier transform of $\{x_j\}$, the power spectrum is defined by

$$P(f, N, \tau) = \frac{\tau}{N} \left\langle |X(f, N, \tau)|^2 \right\rangle . \tag{1}$$

Let $C(f)$ be the power spectrum of the continuous variable $x(t)$. The computed power spectrum $P(f, N, \tau)$ tends to the true power spectrum $C(f)$ in the limit $\tau \to 0$ and $N \to \infty$. A non-zero value of τ results in the aliasing of $P(f, N, \tau)$: frequencies greater than the Nyquist frequency $f_m = 1/2\tau$ are folded into the region $-f_m \leq f \leq f_m$. A finite value of N gives rise to certain finite-size effects. It can be shown [7,8] that the computed power spectrum and the true power spectrum are related by

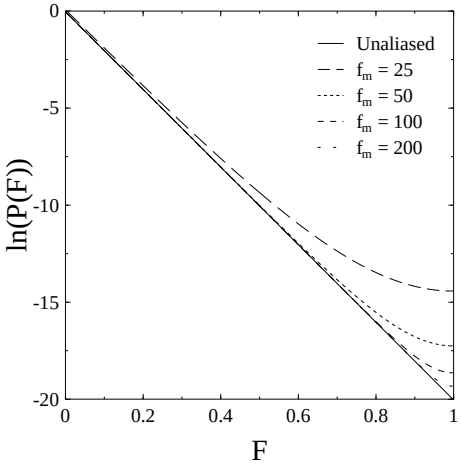

Figure 1. Aliased exponential decay of sampled data with different Nyquist frequencies. As f_m increases, the power spectrum tends to the true one.

$$P(f, N, \tau) = \sum_{m=-\infty}^{\infty} C(f + \tau^{-1} m) + R(f, N, \tau), \tag{2}$$

where the correction $R(f, N, \tau)$ vanishes in the limit $N \to \infty$. The first term on the right-hand side in eq. (2) represents $\lim_{N\to\infty} P(f, N, \tau)$, while the second represents finite-size effects.

2.2 *Aliased exponential decay*

Consider a power spectrum with an exponential dependence on f, given by $C(f) = a \exp(-bf)$. The aliased version of $C(f)$, denoted by $P_{\rm a}(f)$, can be obtained for large N from eq. (2) according to [7,8]

$$P_{\rm a}(F) = a \, \frac{\cosh\left(bf_m(1-F)\right)}{\sinh(bf_m)}, \tag{3}$$

where $F = f/f_m$ is the scaled frequency. The discrepancy between the aliased and true power spectra, expressed as a function of F, gets confined to a region closer and closer to the end-points as τ decreases (see figure 1).

2.3 *Aliasing of power-law decay*

Now consider the case of a true power spectrum with a power-law decay, given by $C(f) = A/f^2$. It is possible to obtain [7,8] an expression for the corresponding aliased spectrum $P_{\rm a}(f)$. This is given by

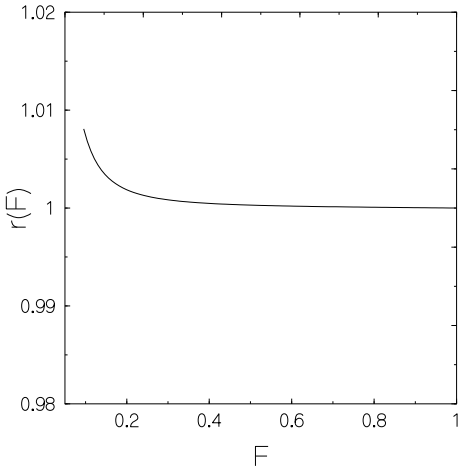

Figure 2. Characterization of the power spectral density at high frequencies. $r(F) = [P(f, N, \tau)/(P(f_m, N, \tau)]\, P_a(F)$. $r(F) \to 1$ as $F \to 1$, showing that the true power spectrum exhibits a power-law tail in accordance with eq. (4).

$$P_{\mathrm{a}}(F) = \frac{A\pi^2}{4f_m^2}\mathrm{cosec}^2\left(\frac{\pi F}{2}\right). \tag{4}$$

In terms of the unscaled frequency f, we have $P_{\mathrm{a}}(f) = A\,[\pi\tau/\sin(\pi\tau f)]^2$, which tends to $C(f)$ in the limit $\tau \to 0$. Figure 2 shows that the power spectrum of the Lorenz oscillator in the high-frequency regime indeed exhibits a power-law decay. Thus, exponential behaviour and power-law behaviour will manifest themselves in the computed power spectrum according to eqs (3) and (4), respectively. Further, we observe that the power-law tail at high frequencies is in itself a finite-size effect, vanishing in the limit $N \to \infty$. These features of the power spectrum yield information on chaos in a scalar time series; they help to distinguish a chaotic time series from a random one.

3. Extreme value statistics and order statistics

The extreme value in a set of realizations sampled independently and randomly from a distribution is a random variable; the statistics of the extreme value depends on the sample size and the distribution from which the sample is drawn. The aim of the theory of extreme value statistics is to study the distribution of the observed extremes arising in a sample of a given size. The classical theory was developed by Fisher and Tippet [9], Gnedenko [10] and Gumbel [11]. For a recent review see [12].

Consider an n-point set of independent and identically distributed random variables with range (a, b), and let $M_n = \max(X_0, X_1, \ldots, X_{n-1})$. Let $F(x) = \mathrm{Prob}(X \leq x)$ denote the cumulative distribution of X; the cumulative distribution of M_n is then given by

$$F_n(x) = \text{Prob}(X_0 \leq x, X_1 \leq x, \ldots, X_{n-1} \leq x) = [F(x)]^n \,. \tag{5}$$

For example, if the density of X is uniform in the interval $[0, 1]$, the distribution of M_n is $F(x) = x^n$.

3.1 *Extreme value statistics of one-dimensional chaotic systems*

Extreme value statistics is formulated for a one-dimensional chaotic system as follows [13]: The n-time probability density that the $n-1$ values of the record following the state x_0 are $x_1, x_2, \ldots, x_{n-1}$ is given by

$$\rho(x_0, x_1, \ldots, x_{n-1}) = \rho_s(x_0) \prod_{j=1}^{n-1} \delta\left(x_j - f^j(x_0)\right), \tag{6}$$

where $\rho_s(x)$ is the invariant density. Let $M_n = \max\{x_0, f(x_0), \ldots, f^{n-1}(x_0)\}$. The cumulative distribution of extreme values is given by $F_n = \text{Prob}(M_n \leq x)$, which can be written as

$$F_n(x) = \int_0^x \mathrm{d}x_0 \rho_s(x_0) \prod_{j=1}^{n-1} \int_0^x \mathrm{d}x_j \, \delta\big(x_j - f^j(x_0)\big). \tag{7}$$

The extreme value density, denoted by $\rho_n(x)$, is the derivative of $F_n(x)$. We have [8,13]

$$\begin{aligned} \rho_n(x) = {} & \rho_s(x) \prod_{j=1}^{n-1} \Theta\left(x - f^j(x)\right) \\ & + \sum_{i=1}^{n-1} \sum_{\alpha=1}^{I_i} A_{i\alpha}(x) \Theta\left(x - g_{i\alpha}(x)\right) \prod_{j \neq i} \Theta\left(x - f^j\left(g_{i\alpha}(x)\right)\right), \end{aligned} \tag{8}$$

where

$$A_{i\alpha}(x) = \rho_s\big(g_{i\alpha}(x)\big) / \left|\mathrm{d}f^i(z)/\mathrm{d}z\right|_{z=g_{i\alpha}} , \tag{9}$$

$\{g_{i\alpha}(x)\}$ being the set of pre-images of x with respect to f^i: namely, $\{g_{i\alpha}(x) | f^i(g_{i\alpha}(x)) = x\}$, $\alpha = 1, \ldots, I_i$ where I_i is the number of pre-images. From eqs (7) and (8), it can be shown that the extreme value density $\rho_n(x)$ is discontinuous on a set of points belonging to the unstable periodic points of the map. Although the set of points on which the extreme value density is discontinuous belongs to the set of periodic points, they do not exhaust all the periodic orbits. In an n-point data set, periodic points of all orders up to $(n-1)$ will be present. For a chaotic system, owing to the fact that the topological entropy is positive, there will be more than one cycle of a given order. The extreme value density picks up a discontinuity corresponding to the largest periodic point of each such cycle. This is illustrated by the example of the logistic map at fully developed chaos, $x_{n+1} = 4x_n(1 - x_n)$. Table 3. shows the periodic orbits of all orders up to four for

Table 1. Periodic orbits of all orders up to 4 for the logistic map.

Period	Orbits
1	0.7500
2	0.3455 $\longrightarrow$ 0.9045
3	0.1170 $\longrightarrow$ 0.4132 $\longrightarrow$ 0.9698
	0.1883 $\longrightarrow$ 0.6113 $\longrightarrow$ 0.9505
4	0.0338 $\longrightarrow$ 0.1305 $\longrightarrow$ 0.4539 $\longrightarrow$ 0.9915
	0.0432 $\longrightarrow$ 0.1654 $\longrightarrow$ 0.5523 $\longrightarrow$ 0.9891
	0.2771 $\longrightarrow$ 0.8013 $\longrightarrow$ 0.6368 $\longrightarrow$ 0.9251

the logistic map. The locations of the 5-point data set shown in figure 3 coincide exactly with the largest periodic point in each cycle (see table 3). More generally, the extreme value density pertaining to a continuous map with a continuous invariant density is discontinuous on a set of points which correspond to the maxima of various periodic orbits. For numerical computation of the extreme value density, the invariant density is not required in closed form; its existence is necessary and sufficient to render meaning to the averaging procedure.

3.2 *Unstable periodic orbits and order statistics*

As shown earlier, extreme value statistics is useful for locating unstable periodic orbits; but it does not locate all the points of these unstable periodic orbits. It is

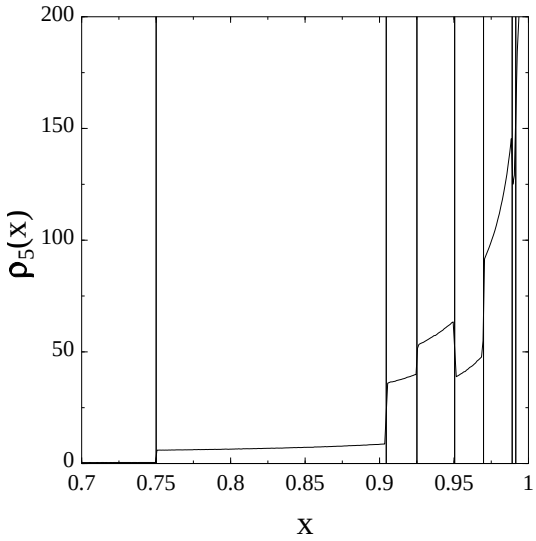

Figure 3. Discontinuities of the invariant density for the 5-point data sets for the logistic map. The discontinuities are a subset of the periodic orbits of the logistic map (see table 1).

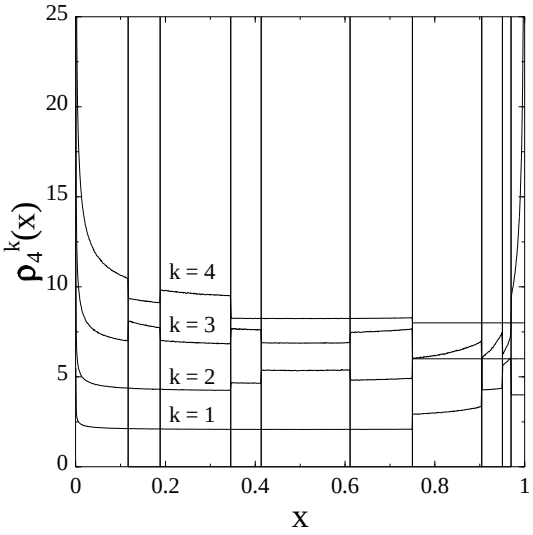

Figure 4. The order densities of 4-point sequence of iterates $\rho_4^k(x)$ of the logistic map at fully-developed chaos, for $1 \leq k \leq 4$. The vertical lines indicate the locations of unstable periodic points. The points of discontinuity of the order densities coincide with the unstable periodic points, and exhaust all the periodic points of order less than four.

precisely in this context that order statistics becomes useful; employing the latter, we can extract all the unstable periodic points of all orders up to n. Let $f(x)$: $[a, b] \rightarrow [a, b]$ be a continuous, one-dimensional chaotic map with an invariant density $\rho_s(x)$. Let $\{x_0, x_1 = f(x_0), \ldots, x_{n-1} = f^{n-1}(x_0)\}$ be an n-point set, and M_n^k the kth largest member of this set. Based on an analysis along similar lines to that for the extreme value density, the order density $\rho_n^k(x)$ for any $n\,(k \leq n)$ is, in general, discontinuous on a certain set of points. Our central result is the observation that the extreme value density is discontinuous on the set of points which is the union of discontinuous points corresponding to all $k \leq n$. Let O_n denote the set of interior periodic points (i.e., excluding the end-points a, b) of all orders strictly less than n. Formally, we have $O_n = \bigcup_{k=1}^{n} S_n^k$, $\forall n$. Figure 4 shows the discontinuities of all orders up to four for the logistic map. The vertical lines correspond to periodic points of orders less than four (see table 1). Thus, the order statistics captures all the unstable periodic points.

4. Recurrence time statistics

The phase trajectories in a chaotic system are in general aperiodic, but restricted to a certain region of phase space (e.g., a strange attractor). They can come arbitrarily close to points they have already visited. One of the interesting questions then is the mean recurrence time, i.e., the average time between successive returns of a representative orbit to a specified finite region (or cell) in the phase space.

Recurrence is an important concept in statistical mechanics that is useful, for instance, in establishing the connection between irreversible macroscopic behaviour and microscopic behaviour that is, in principle, reversible.

We assume that the phase space Γ is endowed with an invariant measure μ under the one-parameter family of ergodic transformations T_τ, such that $\mu(\Gamma) = 1$ and $0 < \mu(C) < 1$ for any small region or cell $C \subset \Gamma$. Here τ denotes the time step in the case of discrete-time dynamics. (For continuous time dynamics, τ may be regarded as the interval between successive samplings, and may be taken to be as small as possible.) The distribution of the time of recurrence to C is the conditional probability [14]

$$F(C, 0 \,|\, C, n\tau) = \mathrm{Prob}\big(X_0 \in C \,|\, X_\tau \notin C, \ldots, X_{(n-1)\tau} \notin C, X_{n\tau} \in C\big), \tag{10}$$

suitably averaged over the initial point X_0. If $\rho(x)$ denotes the invariant density associated with the measure μ, $F(C, 0|C, n\tau)$ can be written in the explicit form [15]

$$F(C, 0 \,|\, C, n\tau) = \frac{1}{\mu(C)} \int_C \mathrm{d}X_0 \, \rho(X_0) \int_{\overline{C}} \mathrm{d}X_1 \ldots \int_{\overline{C}} \mathrm{d}X_{n-1} \times \int_C \mathrm{d}X_n \prod_{r=1}^{n} \delta\left(X_{r\tau} - T_{r\tau} X_0\right), \tag{11}$$

where $\overline{C}$ is the complement of C in Γ. For ergodic transformations, the first moment of F, which is the mean recurrence time $\langle n \rangle_\tau$ for a discrete-time system, is found to be [14]

$$\langle n \rangle_\tau = \frac{\tau}{\mu(C)}. \tag{12}$$

Equation (12) implies that $\langle n \rangle_\tau$ must vanish as $\tau \to 0$, which is not true in general. To circumvent this, Smoluchowski [16] modified the formula as

$$\langle n \rangle_\tau = \frac{\tau[1 - \mu(C)]}{\mu(C) - \mu(C_1)}, \tag{13}$$

where $\mu(C_1)$ is the measure of the set of points that start in C and remain in that cell without leaving it at time τ: $C_1 = \{X \in C,\ T_\tau X \in C\}$. Thus

$$\langle n \rangle_\tau = \tau \frac{1 - \mu(C)}{\mu(C) - \mu(C \cap C_{-\tau})}, \tag{14}$$

where $C_{-\tau}$ is the pre-image of C. Equation (14) may be re-expressed in the form

$$\langle n \rangle_\tau = \frac{[1 - \mu(C)]}{Q_\tau},$$
$$Q_\tau = \frac{1}{\tau} \int_C \mathrm{d}X_0 \, \rho(X_0) \left[1 - \int_C \mathrm{d}X \ \delta(X - T_\tau X_0)\right]. \tag{15}$$

This formula can be applied to recurrences in deterministic as well as stochastic evolution [17]. In the case of continuous time dynamics, one must pass to the limit $\tau \to 0$ to obtain the mean time of recurrence to any given sub-set C of the phase space.

The statistics of recurrence times has been investigated in detail for quasi-periodic motion, one-dimensional chaotic maps, and for chaotic flows [15,17–20]. A number of interesting features emerge, such as the relatively large dispersion of the recurrence time (about its mean value) in most cases. Most notable is the case of intermittent chaos, in which the recurrence time distribution changes to a power law, leading to a divergence of the variance of the recurrence time under suitable circumstances. Concomitantly, successive recurrences become correlated events in such cases, in marked contrast to the statistical independence of these for purely hyperbolic systems.

4.1 *Fully-developed chaos*: *The example of the tent map*

As a simple illustration [15] of the behaviour of the recurrence time for a hyperbolic system, consider the case of the tent map $X_{n+1} = 1 - |2X_n - 1| X_n \in [0, 1]$, which has a uniform invariant density $\rho(X) = 1$. Consider the partition of the phase space (the unit interval) into K equal-sized cells, the jth cell being given by

$$C_j = [(j-1)/K, j/K,), \quad j = 1, \ldots, K. \tag{16}$$

The partition is Markov for every K, since all boundary points either belong to the unstable periodic points of the map, or are pre-images of periodic points. Let $F(C, 0 \,|\, C, n\tau)$ be the probability of first return of the representative point to the cell C at time $n\tau$, as defined in eq. (10). To find this probability, one first calculates $P(j, 0 \,|\, k, n)$, the n-step conditional probability, for a transition from cell C_j to cell C_k, as

$$P(j, 0|k, n) = \frac{\int_{C_j} \mathrm{d}Y \int_{C_k} \mathrm{d}X \delta\big(X - f^{(n)}(Y)\big)\rho(Y)}{\int_{C_j} \mathrm{d}Y\, \rho(Y)}. \tag{17}$$

The invariant density is a constant (unity) for the map under consideration. It can be verified explicitly that the coarse-grained dynamics is a Markov chain in this case. Therefore $P(j, 0 \,|\, k, n) = (W^n)_{jk}$ where W, the transition matrix of the Markov chain, can be evaluated in closed form. The recurrence time distribution is then related to $P(j, 0 \,|\, k, n)$ via the renewal equation [15]

$$P(j, 0|j, n) = \sum_{n'=1}^{n} F(j, 0 \,|\, j, n') P(j, 0 \,|\, j, n - n'). \tag{18}$$

Using discrete Laplace transforms, a closed form expression can then be obtained for the (transform of) the recurrence time distribution, and its moments computed. The mean time of recurrence in cell C_j is found to be

$$\langle n_{jj} \rangle = \frac{1}{\mu(C_j)} = K, \quad \forall\, j, \tag{19}$$

which is in agreement with the general result, eq. (12).

As an example, consider the case $K = 3$, i.e., the unit interval is coarse-grained into $C_1 = [0\,,\,1/3)$, $C_2 = [1/3\,,\,2/3)$, $C_3 = [2/3\,,\,1)$. It can then be shown that [15]

$$F(1,0\,|\,1,n) = \left[1-(-1)^n\right]2^{-(n+3)/2}, \tag{20}$$

while

$$F(2,0\,|\,2,n) = F(3,0\,|\,3,n) = (1-\delta_{n,1})2^{-(n-1)}. \tag{21}$$

The mean recurrence time to any cell is

$$\langle n_{jj}\rangle = \sum_n n\,F(j,0\,|\,j,n) = 3 \tag{22}$$

for $j = 1, 2$ or 3, as expected. But the corresponding variances are quite different, being given by

$$\mathrm{Var}\,n_{11} = 8, \quad \mathrm{Var}\,n_{22} = \mathrm{Var}\,n_{33} = 2.$$

The scatter of recurrence times around the mean value is also seen to be quite large.

As the map under consideration is uniformly hyperbolic, the rigorous results [21] that exist for such systems are applicable. In particular, recurrence time distributions are typically exponentially decaying functions of time, as illustrated by the forms obtained above for $F(1,0\,|\,1,n)$, $F(2,0\,|\,2,n)$, etc. Successive recurrences to any cell are mutually independent random variables, and the complete statistics of multiple recurrences can be determined, using standard techniques from the theory of Markov chains. When the chaotic dynamics is intermittent, however, very interesting and significant departures from the foregoing manifest themselves. For details, see [15,19,20].

4.2 *Chaotic dynamics in continuous-time systems*

The study of recurrence time statistics of continuous-time systems raises an important question: Does the mean recurrence time exist in the limit of continuous sampling? A well-defined expression for the mean recurrence time exists in the limit $\tau \to 0$ for regular motion as well as chaotic motion, provided the probability distribution fulfills certain smoothness properties which are satisfied, among others, by SRB type measures [17,22]. For diffusion processes, resolution-dependent mean recurrence times can be defined which tend to zero when $\tau \to 0$. The mean recurrence time also exhibits a scaling behaviour that links the recurrence times to the phase space cell dimensions and to the intrinsic time-scales of the system. It appears that recurrence time statistics in the context of Hamiltonian systems, where strong and weak ergodic behaviours are intertwined in phase space, would help in resolving some of the questions that have arisen in the context of entropy, work and heat fluctuations [1].

In continuous-time dissipative chaotic systems [18], unlike the case of fully chaotic one-dimensional maps, the attractor occupies a fractal set of zero phase space

volume. The invariant measure of a cell of linear size ϵ can be estimated to be $\mu(C) \simeq \epsilon^d$, where d is the fractal dimension of the attractor. The denominator in the expression for mean recurrence time in eq. (13), $\mu(C) - \mu(C \cap C_{-\tau})$, may be estimated to scale like $\epsilon^{(d-\delta)}$ for some δ $(0 < \delta \leq 1)$. Hence the mean recurrence time is expected to scale according to $\langle t \rangle \sim \epsilon^{-(d-\delta)}$. This has been verified [18] for the Lorenz attractor for cell sizes that are not too small.

5. Intermittency in the Sinai lattice

In this section, we turn to the study of the intermittent behaviour of a random dynamical system. The model considered is that of Sinai disorder [23]: a random walk on a one-dimensional lattice with quenched random site-dependent jump probabilities, such that there is no net bias in either direction. Jumps to the right and left are governed by the probabilities $p_i \sim \exp(-U_i^+/k_{\rm B}T)$ and $q_i \sim \exp(-U_i^-/k_{\rm B}T)$. The local potential barrier ΔU_i is defined by $\Delta U_i/k_{\rm B}T = (U_i^+ - U_i^-)/k_{\rm B}T \sim \ln(q_i/p_i)$. If the configuration average of this quantity over the entire lattice vanishes, then there is no net bias. The precise conditions specified by Sinai are

$$\langle \ln(q/p) \rangle = 0, \quad \left\langle \ln^2 (q/p) \right\rangle < \infty. \tag{23}$$

Sinai [23] showed that the diffusion is ultra-slow in such a lattice: the mean squared displacement diverges like $(\ln t)^4$. The survival probability x_i for reaching site $(i+1)$ from site i is given by the recursion relation [24–27]

$$x_i = \frac{p_i}{1 - q_i x_{i-1}} \quad (i = 1, 2, \ldots), \quad \text{with } x_0 = \frac{p_0}{1 - q_0} < 1. \tag{24}$$

We have assumed that x_0 is less than unity, to allow for partial absorption at the origin with probability $(1 - p_0)$. A simple model of Sinai disorder is binary disorder, where p_i (equivalently, q_i) can only take two values, λ and $(1 - \lambda)$, the quantity $\epsilon = \lambda - \frac{1}{2}$ being a measure of the strength of the disorder. The recursion relation can be re-written in the form

$$x_i = \eta_i f(x_{i-1}) + (1 - \eta_i) g(x_{i-1}) \tag{25}$$

in terms of the two maps

$$f(x) = \frac{\lambda}{1 - (1 - \lambda)x} \quad \text{and } g(x) = \frac{1 - \lambda}{1 - \lambda x}, \tag{26}$$

where $\{\eta_i\}$ are independent and identically distributed random variables: $\eta = 0$ with probability P, and $\eta = 1$ with probability $(1 - P)$. Equations (26) constitute an iterated function system (IFS) [24–27]. The Sinai problem corresponds to the unbiased case $P = \frac{1}{2}$, in which the mean-square displacement has the well-known behaviour $\langle x^2(t) \rangle \sim (\ln t)^4$. The map f has a fixed point at $x = 1$, while the map g has two fixed points, at $x = X_1 = (1 - 2\epsilon)/(1 + 2\epsilon) < 1$ and at $x = 1$. The dynamics is restricted to the region $(X_1\,, 1)$. The IFS is shown in figure 5a. Random iterations over the maps f and g with equal probability $(P = \frac{1}{2})$ exhibit

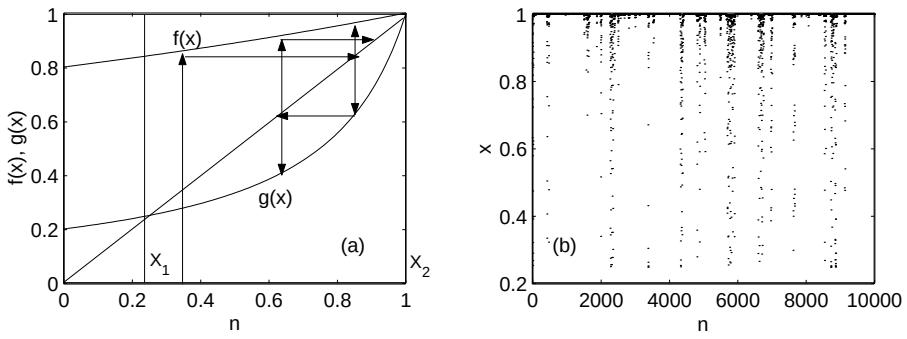

Figure 5. (**a**) Maps $f(x)$ and $g(x)$ for the IFS; (**b**) intermittency of the IFS for binary Sinai disorder with $\epsilon = 0.3$

intermittency: the representative point persists for long times near the upper fixed point at unity (the laminar regions), with occasional bursts that take it away from this neighbourhood, as seen in figure 5b. It has been conjectured that the Sinai conditions are necessary for the emergence of intermittency [24]. However, though $P \neq \frac{1}{2}$ implies that the first of the Sinai conditions is not met, we find [26] that the IFS continues to exhibit intermittency when $P > \frac{1}{2}$. As P is increased, the bursts become less frequent, and the laminar regions become longer. The intermittency can be characterized in terms of the average laminar length $\langle l \rangle$, which is defined as the number of iterations during which the trajectory stays within a specified distance from unity, as a function of P. Our results show that the average laminar length exhibits a power-law divergence of the form $\langle l \rangle \sim (1 - P)^{-\alpha}$.

6. Intermittency and diffusion in the driven and damped pendulum

The driven pendulum, being a paradigmatic model that arises in a variety of physical problems is an extensively studied nonlinear system. The pendulum equation is

$$\ddot{\theta} + \gamma \dot{\theta} + \sin \theta = \epsilon \sin \omega t, \tag{27}$$

where θ is the phase angle, γ is the damping coefficient, ϵ is the strength of the forcing, and ω is the forcing frequency.

Equation (27) is solved numerically and the trajectories are strobed once per drive cycle. The solutions show complex behaviour and exhibit chaos in certain parametric regions [28–31]. We have studied diffusion and intermittency in the pendulum equation near two bifurcation points across a periodic window in ω. Figure 6 shows the evolution of θ and $\dot{\theta}$ for ω near one of the bifurcation points. In the laminar regions of $\dot{\theta}$, there is Newtonian motion of θ, while there is hardly any net motion in θ during the bursts in $\dot{\theta}$. It is known that close to a bifurcation point ω_c, the diffusion coefficient D (defined as the ratio of the mean squared displacement to the time) exhibits a power-law divergence, $D \sim |\omega - \omega_c|^{-\alpha}$.

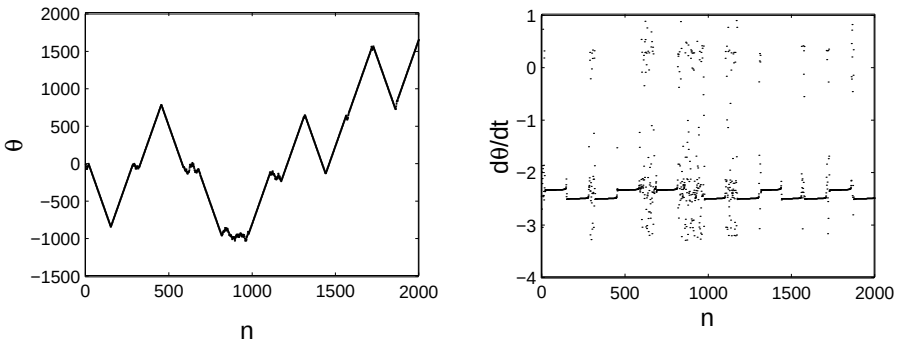

Figure 6. θ (left) and $\dot{\theta}$ (right) as functions of the Strobe time n for the driven, damped pendulum.

We now establish a relationship between the diffusion coefficient and the average laminar length. The average laminar length is known to increase as $\omega \to \omega_c$. The motion during any long time interval can be treated as a sum of the concatenated Newtonian motions during the laminar phases, ignoring the relatively small displacements during the bursts. Since the switching times between the laminar regions and the bursts are random, the motion can be treated as a random walk [31]. The net motion in θ during n drive cycles is then $\theta(n) = \sum_{i=1}^{N} \zeta_i(l_i)$, where N is the number of laminar regions between time 0 and time n. The variables ζ_i are independent random displacements which take positive and negative values with equal probability. The mean squared displacement is given by $\langle \theta^2(n) \rangle = N \langle \zeta^2 \rangle$. The mean squared displacement per step is $\langle \zeta^2 \rangle \sim a \langle l \rangle^2$, where a, the average velocity in the laminar regions, takes values of the same magnitude and opposite signs, and $\langle l \rangle$ stands for the average laminar length. The problem of determining the mean squared displacement reduces to determining the mean squared laminar length. The laminar length has an exponential distribution, as shown numerically in [31]. The mean squared laminar length is then given by $\langle l^2 \rangle = 2 \langle l \rangle^2$. In time n, the number of laminar lengths is $N = n/\langle l \rangle$. Hence $\langle \theta^2(n) \rangle \sim n \langle l \rangle$, which shows that the diffusion coefficient is proportional to the mean laminar length.

7. Dynamics of complex systems: Anomalous fluctuations

Long-range correlations are of interest in a wide variety of situations, including heart-beat intervals, stock market fluctuations and other phenomena in econophysics, firing statistics in neuron models, and DNA sequences [32,33]. An approach to the analysis of long-range correlations is to view them as having been generated from a random walk model [34].

A random walk can be generated as a sum of the displacements $U(i) = \pm 1$, where $U(i)$ denotes the ith 'step'. The sum $Y(l) = \sum_{i=1}^{l} U(i)$ may be viewed as the net displacement after l steps. A quantity of interest in long-range correlations is the mean squared fluctuation $F^2(l)$, defined as

$$F^2(l) = \left\langle \left[Y(l_0+l) - Y(l_0)\right]^2 \right\rangle - \left\langle \left[Y(l_0+l) - Y(l_0)\right] \right\rangle^2, \tag{28}$$

where $\langle \cdots \rangle$ denotes the average over the realizations of the random walk and over the initial position l_0 of the walk. In terms of the autocorrelation function $C(l) = \langle U(l_0)U(l_0+l)\rangle - \langle U(l_0)\rangle^2$, an equivalent definition is $F^2(l) = \sum_{i=1}^{l}\sum_{j=1}^{l} C(j-i)$. Three types of behaviour can be distinguished:

(i) If the base sequence is random, e.g., if $U(i)\,(i = 1, 2, \ldots)$ are independent and identically distributed random variables with $P\big(U(i) = +1\big) = P\big(U(i) = -1\big) = \frac{1}{2}$, then $F(l) \sim l^{1/2}$, as expected of a simple random walk.

(ii) If $C(l) \sim \exp(-l/R)$, implying correlations up to some length R (as in a Markov chain, for instance), then $F(l) \sim l^{\alpha}$ $(\alpha > \frac{1}{2})$ for $l < R$, and $F(l) \sim l^{1/2}$ for $l \gg R$. We shall return to Markov chains shortly.

(iii) If there exists no characteristic length scale, and $C(l)$ decays asymptotically as a power law $l^{-\gamma}$, with a corresponding power spectral density $\Phi(\omega) \sim \omega^{-\beta}$, then $F(l) \sim l^{\alpha}$, with $\alpha \neq \frac{1}{2}$. The exponent α is related to γ and β according to $\alpha = (2-\gamma)/2 = (1+\beta)/2$.

Consider a sequence of binary symbols, say 0 and 1. Let g_0 and $g_1 = (1 - g_0)$ denote the respective *a priori* probabilities of the occurrence of 0 and 1. Let $t_i \in \{0, 1\}$ denote the symbol in the ith site in the chain, and let $T_{L,i}$ denote the string of L symbols preceding the ith site in the chain. Thus $T_{L,i} = \{t_{i-L},\, t_{i-L+1},\, \ldots,\, t_{i-1}\}$. Given the chain till the site $(i-1)$, the aim is to extend the chain by one unit by placing a symbol at the ith location. With what probabilities should the symbol 0 or 1 be placed in the ith site?

Let $p\,(t_i = 0\,|\,T_{L,i}) \equiv p_0$ and $p\,(t_i = 1\,|\,T_{L,i}) \equiv p_1$ denote these probabilities, *conditioned* by the occurrence of the sequence $T_{L,i}$. The Markov chain model is specified by the assumption that p_0 is of the form

$$p_0 = \widehat{g}_0(T_{L,i})g_0 + \widehat{g}_1(T_{L,i})g_1, \tag{29}$$

where $\widehat{g}_t(T_{L,i})$ is the frequency with which the symbol t (= 0 or 1) has occurred in $T_{L,i}$. It then follows that

$$p_1 = p(t_i = 1|T_{L,i}) = p(t_i = 0|T'_{L,i}), \tag{30}$$

where the sequence $T'_{L,i}$ is obtained from $T_{L,i}$ by adding 1 to each $t_j \in T_{L,i}$, the addition being done modulo 2. Therefore

$$p_1 = \widehat{g}_0(T'_{L.i})g_0 + \widehat{g}_1(T'_{L.i})g_1 = \widehat{g}_1(T_{L,i})g_0 + \widehat{g}_0(T_{L,i})g_1. \tag{31}$$

We thus have

$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} \widehat{g}_0 & \widehat{g}_1 \\ \widehat{g}_1 & \widehat{g}_0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}, \tag{32}$$

where $\widehat{g}_0$ and $\widehat{g}_1$ written without any arguments correspond to the original sequence $T_{L,i}$.

We may write $g_{0,1} = \frac{1}{2}(1 \pm \mu)$, where $-1 \leq \mu \leq +1$. Let $F^2(L)$ denote the fluctuations in the number of times the symbol 0 occurs in a string of length L. The above model can be solved analytically [35–37], to yield the results

$$F(L) \sim \begin{cases} L^{1/2} & \text{for } \mu \leq \mu_c = \frac{1}{2} \\ \\ L^{\mu} & \text{for } \mu > \mu_c \,. \end{cases} \tag{33}$$

Hence there is a transition from diffusive to super-diffusive behaviour for $\mu > \mu_c = \frac{1}{2}$.

Now consider a ternary sequence of symbols, $\{0, 1, 2\}$. Analogously, the probabilities p_0 , p_1 , p_2 are governed by the matrix equation

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \widehat{g}_0 & \widehat{g}_1 & \widehat{g}_2 \\ \widehat{g}_1 & \widehat{g}_2 & \widehat{g}_0 \\ \widehat{g}_2 & \widehat{g}_0 & \widehat{g}_1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}, \tag{34}$$

where $g_t = \frac{1}{3}(1 + \mu_t)$, $t \in \{0, 1, 2\}$, and $-1 \leq \mu_t \leq 2$. Since $g_0 + g_1 + g_2 = 1$, there are only two independent parameters in the problem, μ_0 and μ_1 (say). This ternary sequence model can also be solved analytically [37], to yield

$$F(L) \sim \begin{cases} L^{1/2} & \text{for } \Gamma \leq \frac{1}{2} \\ \\ L^{\Gamma} & \text{for } \Gamma > \frac{1}{2} \,, \end{cases} \tag{35}$$

where $\Gamma = [(\mu_0^2 + \mu_1^2 + \mu_0\mu_1)/3]^{1/2}$. The parameter space in the (μ_0, μ_1) plane is the triangular region bounded by the axes and the line $L_1 : \mu_0 + \mu_1 = 1$, as shown in figure 7. The existence of a critical value $\Gamma_c = \frac{1}{2}$ implies that the parameter space is divided into two regions, corresponding respectively to diffusive and super-diffusive behaviour. The former is the region inside the ellipse $\Gamma = \frac{1}{2}$ or $\mu_0^2 + \mu_1^2 + \mu_0\mu_1 = \frac{3}{4}$, while the latter is the region of the triangle lying outside this ellipse.

Suppose we now coarse grain the ternary sequence (called T for brevity) to a binary one (called B_T). The coarse graining is carried out as follows: The symbols 0 and 1 of T are both re-labelled as the symbol 0 of B_T; the symbol 2 of T is re-labelled as the symbol 1 of B_T. Then the *a priori* probabilities for the occurrence of 0 and 1 in B_T will be $g_0(B_T) = \frac{1}{3}(2 + \mu_0(T) + \mu_1(T))$ and $g_1(B_T) = \frac{1}{3}(1 + \mu_2(T))$, respectively. The condition for non-trivial correlations in B_T then turns out to be $\mu_0(T) + \mu_1(T) > \frac{1}{4}$. In other words, the parameter region between the two parallel lines L_1 and L_2 in figure 7 leads to long-range correlations in B_T. This is so because the (single independent) parameter $\mu(B_T)$ of the coarse-grained binary sequence is related to $\mu_0(T)$ and $\mu_1(T)$ according to $\mu(B_T) = \frac{1}{3}[1 + 2(\mu_0(T) + \mu_1(T))]$. This implies a correspondence between the range $\frac{1}{4} \leq \mu_0(T) + \mu_1(T) \leq 1$ of T on the one hand, and the range $\frac{1}{2} \leq \mu(B_T) \leq 1$ of B on the other. Every line parallel to L_1 in figure 7 is mapped to a point in the range $\frac{1}{2} \leq \mu \leq 1$, and *vice versa*. The shaded part of the ellipse of figure 7 belongs to the diffusive region of T; but it gets mapped to the super-diffusive region of B_T. Conversely, the fact that the correlation parameter of B_T has a value greater than $\frac{1}{2}$ does not necessarily mean that the parent ternary sequence T also has long-range correlations. The important finding is that the process of coarse graining of a symbolic sequence can, in principle, give rise to spurious long-range correlations. For further details we refer to [37].

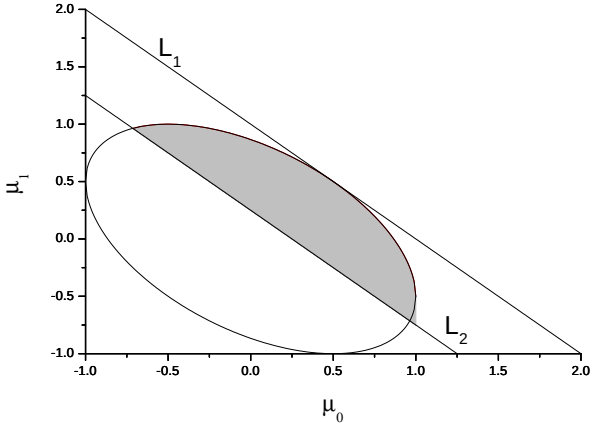

Figure 7. Phase diagram for the ternary sequence. L_1 and L_2 label the lines $\mu_0 + \mu_1 = 1$ and $\mu_0 + \mu_1 = \frac{1}{4}$, respectively. The regions inside and outside the ellipse correspond to the diffusive and super-diffusive regimes, respectively. The shaded region, though diffusive for the ternary sequence, is super-diffusive when the ternary sequence is coarse-grained into a binary sequence.

8. Concluding remarks

We have reviewed briefly in this paper some of the statistical tools often employed in the study of nonlinear dynamics and stochastic systems. These include numerical estimation of the power spectral density; extreme value, order and recurrence statistics; iterated function systems; and Markov chain and random walk methods. We find that the size-dependence of the tail of the spectral density provides a handle to distinguish between a chaotic time series and a stochastic one; extreme value statistics and order statistics can help extract information on unstable periodic orbits; we have described intermittent behaviour in an IFS derived from the condition for Sinai disorder, and shown that this condition is not necessary for the emergence of intermittency. In the context of a chaotic pendulum, we have shown using a random walk model that the diffusion constant is proportional to the mean laminar length. Finally, in our investigation of the dynamics of symbol sequences, we have found that coarse graining can, in principle, introduce spurious long-range correlations.

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