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Universal canonical entropy for gravitating systems

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Abstract. The thermodynamics of general relativistic systems with boundary, obeying a Hamiltonian constraint in the bulk, is determined solely by the boundary quantum dynamics, and hence by the area spectrum. Assuming, for large area of the boundary, (a) an area spectrum as determined by non-perturbative canonical quantum general relativity (NCQGR), (b) an energy spectrum that bears a power law relation to the area spectrum, (c) an area law for the leading order microcanonical entropy, leading thermal fluctuation corrections to the canonical entropy are shown to be logarithmic in area with a universal coefficient. Since the microcanonical entropy also has universal logarithmic corrections to the area law (from quantum space-time fluctuations, as found earlier) the canonical entropy then has a universal form including logarithmic corrections to the area law. This form is shown to be independent of the index appearing in assumption (b). The index, however, is crucial in ascertaining the domain of validity of our approach based on thermal equilibrium.

Keywords. Entropy; canonical; microcanonical; black hole.

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1. Introduction

The asymptotically flat Schwarzschild space-time is well-known [1] to have a thermal instability: the Hawking temperature for a Schwarzschild black hole of mass M is given by $T \sim 1/M$ which implies that the specific heat $C \equiv \partial M/\partial T < 0$! The instability is attributed, within a standard canonical ensemble approach, to the superexponential growth of the density of states $\rho(M) \sim \exp M^2$ which results in the canonical partition function diverging for large M.

The problems with an approach based on an equilibrium canonical ensemble do not exist, at least for isolated spherically symmetric black holes, formulated as *isolated horizons* [2] of fixed horizon area; these can be consistently described in terms of an equilibrium *microcanonical* ensemble with fixed $\mathcal{A}$ (and hence disallowing thermal fluctuations of the energy M). For $\mathcal{A} \gg l^2_{\text{Planck}}$, it has been shown using loop quantum gravity [3], that all spherically symmetric four-dimensional isolated horizons possess a microcanonical entropy obeying the Bekenstein–Hawking area law (BHAL) [4,5]. Further, the microcanonical entropy has corrections to the BHAL due to quantum space-time fluctuations at fixed horizon area. These arise, in the limit of large $\mathcal{A}$, as an infinite series in inverse powers of horizon area beginning with a term logarithmic in the area [6], with completely finite coefficients,

$$S_{\mathrm{MC}} = S_{\mathrm{BH}} - \frac{3}{2}\log S_{\mathrm{BH}} + \mathrm{const.} + \mathcal{O}(S_{\mathrm{BH}}^{-1}), \tag{1}$$

where $S_{\mathrm{BH}} \equiv \mathcal{A}/4l_{\mathrm{Planck}}^2$.

On the other hand, asymptotically anti-de Sitter (adS) black holes with spherical symmetry are known [1] to be describable in terms of an equilibrium canonical ensemble, so long as the cosmological constant is large in magnitude. For this range of black hole parameters, to leading order in $\mathcal{A}$ the canonical entropy obeys the BHAL. As the magnitude of the cosmological constant is reduced, one approaches the so-called Hawking–Page phase transition to a 'phase' which exhibits the same thermal instability as mentioned above.

In this talk, we focus on the following:

- Is an understanding of the foregoing features of black hole entropy on some sort of a 'unified' basis possible? We shall argue, following [7] that it is indeed so, with some rather general assumptions.
- In addition to corrections (to the area law) due to fixed area quantum spacetime fluctuations computed using a microcanonical approach, can one compute corrections due to *thermal* fluctuations of horizon area within the canonical ensemble? Once again, the answer is in the affirmative with some caveats. The result found in [7], at least for the leading log area corrections, turns out to be *universal* in the sense that, just like the BHAL, it holds for all black holes independent of their parameters.

2. Canonical partition function: Holography?

Following [8], we start with the canonical partition function in the quantum case

$$Z_{\mathrm{C}}(\beta) \;=\; \mathrm{Tr}\;\exp -\beta\hat{H}. \tag{2}$$

Recall that in classical general relativity in the Hamiltonian formulation, the bulk Hamiltonian is a first class constraint, so that the entire Hamiltonian consists of the boundary contribution H_S on the constraint surface. In the quantum domain, the Hamiltonian operator can be written as

$$\hat{H} \;=\; \hat{H}_V \;+\; \hat{H}_S, \tag{3}$$

with the subscripts V and S signifying bulk and boundary terms respectively. The Hamiltonian constraint is then implemented by requiring

$$\hat{H}_V|\psi\rangle_V \;=\; 0 \tag{4}$$

for every physical state $|\psi\rangle_V$ in the bulk. Choose as basis for the Hamiltonian in (3) the states $|\psi\rangle_V \otimes |\chi\rangle_S$. This implies that the partition function may be factorized as

$$Z_{\rm C} \equiv {\rm Tr} \exp -\beta \hat{H} \\ = \underbrace{\dim \mathcal{H}_V}_{\text{indep. of } \beta.} \underbrace{{\rm Tr}_S \exp -\beta \hat{H}_S}_{\text{boundary}} . \tag{5}$$

Thus, the relevance of the bulk physics seems rather limited due to the constraint (4). The partition function further reduces to

$$Z_{\rm C}(\beta) \ = \ \dim \mathcal{H}_V \ Z_S(\beta), \tag{6}$$

where $\mathcal{H}_V$ is the space of bulk states $|\psi\rangle$ and Z_S is the 'boundary' partition function given by

$$Z_S(\beta) \ = \ {\rm Tr}_S \ \exp -\beta \hat{H}_S. \tag{7}$$

Since we are considering situations where, in addition to the boundary at asymptopia, there is also an inner boundary at the black hole horizon, quantum fluctuations of this boundary lead to black hole thermodynamics. The factorization in eq. (6) manifests in the canonical entropy as the appearance of an additive constant proportional to dim $\mathcal{H}_V$. Since thermodynamic entropy is defined only up to an additive constant, we may argue that the bulk states do not play any role in black-hole thermodynamics. This may be thought of as the origin of a weaker version of the holographic hypothesis [9].

For our purpose, it is more convenient to rewrite (7) as

$$Z_{\rm C}(\beta) = \sum_{n\in\mathcal{Z}} \underbrace{g\left(E_S(\mathcal{A}(n))\right)}_{\text{degeneracy}} \ \exp -\beta E_S(\mathcal{A}(n)), \tag{8}$$

where we have made the assumptions that (a) the energy is a function of the area of the horizon $\mathcal{A}$ and (b) this area is quantized. The first assumption (a) basically originates from the idea in the last paragraph that black-hole thermodynamics ensues solely from the boundary states whose energy ought to be a function of some property of the boundary-like area. The second assumption (b) is actually explicitly provable in theories like NCQGR as we now briefly digress to explain.

3. Spin network basis in NCQGR

The basic canonical degrees of freedom in NCQGR are holonomies of a distributional $SU(2)$ connection and fluxes of the densitized triad conjugate to this connection. The Gauss law (local $SU(2)$ invariance) and momentum (spatial diffeomorphism) constraints are realized as self-adjoint operators constructed out of these variables. States annihilated by these constraint operators span the kinematical Hilbert space. Particularly convenient bases for this kinematical Hilbert space are the spin network bases. In any of these bases, a ('spinet') state is described in terms of *links* $l_1, \ldots, l_n$ carrying spins ($SU(2)$ irreducible representations) $j_1, \ldots, j_n$ and *vertices* carrying invariant $SU(2)$ tensors ('inter-twiners'). A particularly important property of such bases is that geometrical observables like area operator are diagonal in this basis with *discrete spectrum.* An internal boundary of a space-time

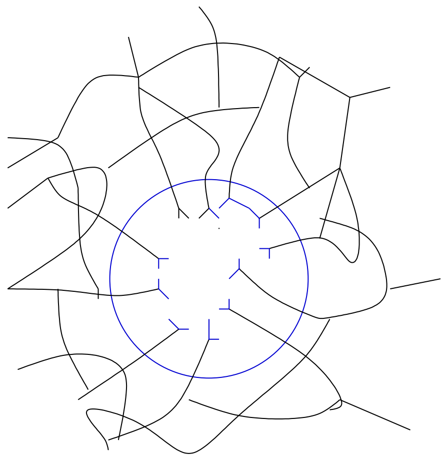

Figure 1. Internal boundary (horizon) pierced by spinet links.

like a horizon appears in this kinematical description as a punctured $\mathcal{S}^2$ with each puncture having a deficit angle $\theta = \theta(j_i), i = 1, \ldots, p$, as shown in figure 1.

For macroscopically large boundary areas $\mathcal{A} \gg l_{\text{Planck}}^2$, the area spectrum is dominated by $j_i = 1/2, \forall i = 1, \ldots, p,\ p \gg 1$. This is the situation when the deficit angles at each puncture takes its smallest non-trivial value, so that a classical horizon emerges. This implies that

$$\mathcal{A}(p) \ \sim \ p \ l_{\text{Planck}}^2, \ p \gg 1. \tag{9}$$

This completes our digression on NCQGR.

4. Fluctuation effects on canonical entropy

Going back to eq. (8), we can now rewrite the partition function as an integral, using the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} f(n) \ = \ \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x \ \exp(-2\pi i m x) f(x). \tag{10}$$

For macroscopically large horizon areas $\mathcal{A}(p)$, $x \gg 1$, so that the summation on the rhs of (10) is dominated by the contribution of the $m = 0$ term. In this approximation, we have

$$\begin{aligned} Z_{\mathrm{C}} \ &\simeq \ \int_{-\infty}^{\infty} \mathrm{d}x \ g(E(A(x))) \ \exp -\beta E(A(x)) \\ &= \ \int \mathrm{d}E \ \exp\left[S_{\mathrm{MC}}(E) \ - \ \log\left|\frac{\mathrm{d}E}{\mathrm{d}x}\right| \ - \ \beta E \right], \end{aligned} \tag{11}$$

where $S_{\mathrm{MC}} \equiv \log g(E)$ is the microcanonical entropy.

Now, in equilibrium statistical mechanics, there is an inherent ambiguity in the definition of the microcanonical entropy, since it may also be defined as $\tilde{S}_{\rm MC} \equiv \log\rho(E)$, where $\rho(E)$ is the density of states. The relation between these two definitions involves the 'Jacobian' factor $|{\rm d}E/{\rm d}x|^{-1}$,

$$\tilde{S}_{\rm MC} \;=\; S_{\rm MC} \;-\; \log\left|\frac{{\rm d}E}{{\rm d}x}\right|. \tag{12}$$

Clearly, this ambiguity is irrelevant if one is interested only in the leading order BHAL. However, if one is interested in logarithmic corrections to BHAL as we are, this difference is crucial and must be taken into account.

We next proceed to evaluate the partition function in eq. (11) using the saddle point approximation around the point $E = M$, where M is to be identified with the (classical) mass of the boundary (horizon). Integrating over the Gaussian fluctuations around the saddle point, and dropping higher order terms, we get

$$Z_{\rm C} \simeq \exp\left\{S_{\rm MC}(M) - \beta M - \log\left|\frac{{\rm d}E}{{\rm d}x}\right|_{E=M}\right\}\cdot\left[\frac{\pi}{-S''_{\rm MC}(M)}\right]^{1/2} . \tag{13}$$

Using $S_{\rm C} = \log Z_{\rm C} + \beta M$, we obtain for the canonical entropy

$$S_{\rm C} = S_{\rm MC}(M) \underbrace{-\frac{1}{2}\log(-\Delta)}_{\delta_{\rm th}S_{\rm C}}, \tag{14}$$

where

$$\Delta \;\equiv\; \frac{{\rm d}^2 S_{\rm MC}}{{\rm d}E^2}\;\left(\frac{{\rm d}E}{{\rm d}x}\right)^2\bigg|_{E=M}. \tag{15}$$

Equation (14) exhibits the equivalence of the microcanonical and canonical entropies, exactly as one expects when thermal fluctuation corrections are ignored. A few elementary manipulations on (15) yield

$$\Delta \;=\; \left[\frac{{\rm d}^2 S_{\rm MC}}{{\rm d}\mathcal{A}^2} \;-\; \left(\frac{{\rm d}S_{\rm MC}}{{\rm d}\mathcal{A}}\right)\underbrace{\frac{{\rm d}^2E/{\rm d}\mathcal{A}^2}{{\rm d}E/{\rm d}\mathcal{A}}}_{\text{non-univ.}}\right]\;\left(\frac{{\rm d}\mathcal{A}}{{\rm d}x}\right)^2\bigg|_{E=M}. \tag{16}$$

Here, we observe that the microcanonical entropy obeys the BHAL *universally*, i.e., independent of the horizon parameters, and may even have universal logarithmic corrections in the horizon area. However, the factors in the rhs of (16) underbraced 'non-univ.' depend explicitly on the area dependence of the energy and is hence a function of the horizon parameters.

We now make the following assumptions (for large area):

- Assume $E(\mathcal{A}) \;=\; \text{const.}\ \mathcal{A}^n$.
- Assume $S_{\rm MC}(\mathcal{A}) \;\sim\; \mathcal{A}$.

Recall also that $\mathcal{A} \sim x$ for $x \gg 1$ (large area). Substitution in eq. (16) now leads to the following simple formula

$$\delta_{\rm th} S_{\rm C} \;=\; \frac{1}{2} \log S_{\rm BH} \;-\; \frac{1}{2} \log(n-1) \;+\; {\rm const.} \tag{17}$$

The (leading logarithmic) thermal fluctuation correction to the canonical entropy of a space-time with an inner boundary is universal, independent of n; it is also insensitive (for large areas) to the negative coefficient of the log(area) correction in the microcanonical entropy due to quantum space-time fluctuations. We note *en passant* that the contribution due to $\mathrm{d}E/\mathrm{d}x$ was first included in the canonical ensemble in ref. [10], although we have not included it in this paper on these grounds. An earlier paper by us [11] which delineated the contribution of fixed-area quantum space-time fluctuations to the BHAL as distinct from thermal fluctuation effects missed out this 'Jacobian' term. Similar to this is the case of ref. [12] which also uses the saddle point approximation to express the microcanonical entropy in terms of the canonical entropy [12a].

Recalling that there is at least 'circumstantial' evidence that the microcanonical entropy has a 'universal' form [13–15], identical to that obtained in ref. [6] quoted in eq. (1), the total canonical entropy, including both the 'finite-size' logarithmic corrections due to quantum space-time fluctuations and the thermal energy (area) fluctuations, is given by

$$S_{\rm C} = S_{\rm BH} - \log S_{\rm BH} - \frac{1}{2} \log(n-1) \;+\; {\rm const.} \;+\; \cdots. \tag{18}$$

Clearly, the nature of the quantum and thermal fluctuation corrections preserves the desired property of superadditivity for the canonical entropy.

Two remarks are in order at this point: first of all, the form of the $S_{\rm MC}$ quoted in eq. (1) arises from the assumption that the residual gauge subgroup of local Lorentz ($SL(2,C)$) invariance, which survives on the Cauchy slice of the isolated horizon, is assumed to be $SU(2)$. The log correction found in [6] originates from counting the boundary states of an $SU(2)$ Chern–Simons theory, where the boundary has the topology of a punctured S^2. On the other hand, it has been argued [2] that the boundary conditions appropriate to a non-rotating isolated horizon leave only a compact $U(1)$ subgroup of this $SU(2)$ on the Cauchy slice of the isolated horizon. If so, the $-(3/2) \log S_{\rm BH}$ in (1) is replaced by $-(1/2) \log S_{\rm BH}$ [8,16,17], and consequently there is *no* log(area) correction to the canonical entropy, as the thermal fluctuation correction precisely cancels the quantum space-time fluctuation correction. This is reminiscent of the phenomenon of 'non-renormalization' that often occurs in certain quantum field theories. This cancellation is universal in the sense that it should hold for all non-rotating isolated horizons and presumably for rotating ones as well.

The second remark pertains to the role of the index n appearing in the assumption regarding the power law relation between the boundary energy and boundary area. Observe that the area dependence of the correction term is quite independent of n. However, n has a crucial role to play: it determines the range of validity of the saddle point approximation used to evaluate $Z_{\rm C}$. For $n > 1$, both $S_{\rm C}$ and the canonical Gibbs free energy remain real, implying that the saddle point has been correctly found. For $n < 1$, however, both the canonical entropy and the free energy acquire an imaginary piece, signifying a *breakdown* of the saddle point approximation. Forcing the saddle point at the value M implies that this is a point of *unstable* equilibrium. Thus, for example, using the formulas worked out in [11,12], we obtain

- $n = 2$ for the non-rotating 2+1-dimensional BTZ black hole so long as the horizon radius $r_{\mathrm{H}} > (-\Lambda)^{-1/2}$, signifying the validity of the calculation in this case;
- $n = 3/2$ for the four-dimensional adS Schwarzschild black hole, also indicating that the formula is reliable for this case in the same range of r_{H}. But,
- $n = 1/2$ for the four-dimensional asymptotically flat Schwarzschild black hole, delineating the thermal instability mentioned at the very outset.

The thermal instability for $n < 1$ is generic for *all* asymptotically flat (and also dS) black holes at large area, including the extremal limits. On the other hand, for the adS black holes, so long as one stays away from the Hawking–Page phase transition to the adS gas phase, there is no thermal instability [1].

5. Conclusions

The canonical entropy of gravitating systems with horizons has a universal correction to the area law due to thermal fluctuations, in the form of $1/2\ \log \mathcal{A}$, provided certain very general assumptions are made about the relation between the energy and the area of the boundary. Within these assumptions, asymptotically flat black holes display an unstable thermal equilibrium exactly as expected on general grounds. Inclusion of the finite-size quantum corrections, in a microcanonical ensemble corresponding to fixed boundary area, either leads to a net logarithmic correction to the area law for canonical entropy obeying superadditivity, if the gauge group on the boundary is $SU(2)$, or to *no* net logarithmic correction at all, if the gauge group is compact $U(1)$.

The agenda for future work, then, is the investigation of the assumptions made, within the framework a theory of quantum gravitation like NCQGR. One expects to have a better understanding of the thermal instability encountered for asymptotically flat (and also dS) black holes from such an exploration.

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