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Canonical quantum gravity and consistent discretizations

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Abstract. This paper covers some developments in canonical quantum gravity that occurred since ICGC-2000, emphasizing the recently introduced consistent discretizations of general relativity.

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1. Introduction

Canonical quantum gravity is an attempt to apply the rules of quantum mechanics in their most traditional canonical form to general relativity. The first question that arises concerning this field of endeavor is: why? After all, attempts to canonically quantize general relativity were done in the 1960s, notably by DeWitt following earlier work by Dirac and Bergmann, and significant obstacles were found. Moreover, it is not expected that general relativity will be the ultimate theory of nature, and some proposals for such theories, like string theory, appear to offer advantages towards quantization. Our personal point of view on this issue is the following: there are good reasons why we have had so much trouble quantizing general relativity. It is the first theory we try to quantize that is invariant under diffeomorphisms (therefore it is a 'topological quantum field theory') that has local field-theoretic degrees of freedom. Some experience with topological theories was gained in the 1980s and 1990s, but the examples considered only had a finite number of degrees of freedom, i.e., they were really mechanical systems disguised as field theories. Now, it is expected that any ultimate theory of nature will be invariant under diffeomorphisms and will have local degrees of freedom. Therefore it is natural to first address these difficulties in the simplest possible scenario of a theory with some claim to describe certain aspects of nature, such as general relativity. What is learned will be of future use to tackle more elaborate theories, as for instance string theory. Therefore we do not see any conflict between having a group of people working on canonical quantum gravity while other groups study string theories or other theories that attempt to unify all interactions.

The aim of this article is to present a very brief and quick introduction to results since ICGC-2000. Due to time limitations we will only be able to cover a fraction of what has been achieved, and so we concentrate largely on issues which we had something to do with. In particular we will not be able to cover interesting work on the definition of semiclassical states by Ashtekar *et al* [1], Sahlmann and Thiemann [2] and Varadarajan [3]. I will not be able to discuss the work of Samuel on understanding the real Ashtekar variables four dimensionally [4], or the recent work on quasi-normal modes and the quantum of area [5]. Developments on quantum cosmology with the loop formulation are occurring rapidly and will be covered in the talk by Martin Bojowald. I will not cover attempts to define the path integral formulation (spin-foams) or Thiemann's 'Phoenix project' proposal [6].

2. Recent history

We start with a brief recent history of the field to put the current situation in perspective. Traditionally, canonical formulations of general relativity considered as canonical variables the metric on a spatial slice q^{ab} and a canonically conjugate momentum closely related to the extrinsic curvature of the slice. In 1985, Ashtekar [7] noted that one could use an alternative set of canonical variables. The variables consisted of a set of (densitized) triads $\tilde{E}^a_i$ and as canonically conjugated momentum, a (complex) $SO(3)$ connection A^i_a. The spatial metric (doubly densitized) can be easily reconstructed as $\tilde{\tilde{q}}^{ab} = \tilde{E}^a_i \tilde{E}^b_i$. In terms of these variables, the canonical theory has the usual constraints of canonical gravity and an additional Gauss law. This is appealing since it suggests that the phase space of general relativity can be viewed as a subspace of the phase space of Yang–Mills theories. The usual constraints of canonical gravity adopt rather simple expressions in terms of these variables, and they are given by polynomial expressions. This initially raised hopes that the quantization could proceed in a better way with this formulation than with the usual one.

The new variables immediately change the perspective on quantization. In particular, the most natural quantum representation to consider is to pick wave functions that are functionals of the connection $\Psi[A]$. The Gauss law constraint (promoted to a quantum operator) implies that these functions have to be gauge invariant functions of the connection. Rovelli and Smolin [8] noted that, one could use as a basis of functions of the connection, the traces of holonomies of the connection along loops. This technique had been used in the context of Yang–Mills theories by Gambini and Trias [9]. The coefficients of the expansion of a wave function in this basis are purely geometrical functions of the loops, and contain the same information as the original wave functions. These coefficients are the wave functions in the 'loop representation'. This representation is particularly attractive at the time of dealing with another of the constraints of canonical gravity: the diffeomorphism constraint. This constraint implies that the wave functions should be functions of loops that are invariant under diffeomorphisms. Such functions have been studied by mathematicians for some time, and they are called knot invariants. This implies a significant improvement over the use of the traditional variables, since there the diffeomorphism constraint did not encounter such a natural encoding in terms of the wave functions.

Rovelli and Smolin [10] also noted that one could overcome a technical problem of the loop representation due to the fact that the basis of loops is really an overcomplete basis (which implies that certain relations exist between the coefficients in the expansion). They noted that if one used a mathematical construction known as spin networks, introduced by Penrose in the 1960s, one could label in a simple way the independent elements of the basis of loops. Spin networks are diagrams made of lines that intersect at vertices. Each line carries a holonomy in a certain representation of the group and the lines are 'tied together' at intersections using invariant tensors in the group. Rovelli and Smolin [11] noted that one could construct simple quantum operators associated to the area of a surface and the volume of a region of space and these operators, acting on spin network states had a simple action with discrete eigenvalues.

The remaining constraint of canonical gravity is called the Hamiltonian constraint. This constraint does not have a simple geometrical interpretation and has to be implemented as a quantum operator and studied. For years, this implementation remained elusive. Most of the results in canonical gravity (with the notable exception of the calculation of black hole entropies [7]) were confined to 'kinematical' results, i.e. results of considering states that are not necessarily annihilated by the Hamiltonian constraint.

Ashtekar and Lewandowski [12] were able to define a rigorous integration theory in the space of functions of a connection modulo gauge transformations, that acquires a particularly simple form when presented in terms of spin networks. This allowed to show rigorously that the spectrum of the area and volume are quantized [13].

An implementation of the Hamiltonian constraint was finally found by Thiemann [14]. He noted certain classical identities that allowed to write in an elegant way the singly-densitized Hamiltonian constraint. A density weight one operator is the only one that is expected to have an action on spin network states, the reason being that in the absence of an *a priori* background metric, the only naturally defined density is of weight one, namely the Dirac delta function. In addition, Thiemann presented a recipe for handling the Hamiltonian constraint of the Lorentzian theory without using complex variables (the initial presentation of the new variables required the introduction of a complex-valued $SU(2)$ connection in order for the Hamiltonian constraint to be polynomial).

Thiemann's result is remarkable in many ways. To begin with, it provides the first non-trivial, consistent, finite and well-defined theory of quantum gravity. Moreover, the action of the Hamiltonian constraint on a spin network state is relatively simple: it acts at vertices by adding a line connecting two of the incoming lines (it produces a term for every pair of incoming lines) and multiplying times a factor that depends on the valences of the lines. The big open question at the moment is if Thiemann's Hamiltonian constraint really captures the correct dynamics of general relativity. It could be the case that it provides a well-defined theory but that it does not correspond to a desirable quantization of general relativity.

There are some remarkable achievements already of Thiemann's theory. In particular, Bojowald particularized the theory to homogeneous cosmologies (unlike the usual quantum cosmology in which one imposes homogeneity classically and later quantizes, Bojowald carries out his quantization by using results inspired in the full theory; this construction has been formalized recently [15]), and has found very attractive results. But there are also some worries about Thiemann's theory. In particular, the action of the Hamiltonian constraint appears prima facie to be too simple. Since it adds a line connecting two other lines, it generates new vertices in the spin network that are planar. In particular this implies that if one acts on a bra state with the operator, it removes a line connecting two planar vertices. Now if one considers a bra state based on a spin network of arbitrary complexity, but lacking planar vertices, the Hamiltonian constraint annihilates it. This seems to imply that there are too many solutions. This was observed by Thiemann as well in $2+1$ dimensions [16]. There the extra states can be eliminated by a careful choice of inner product as not normalizable. It is questionable if a similar construction will be feasible in $3+1$ dimensions.

To try to understand Thiemann's theory, it is worthwhile to examin the spirit of the construction a bit. In order to regularize the Hamiltonian constraint, Thiemann first discretizes the theory on a lattice and then quantizes the theory. It is therefore of interest to analyse what kind of theory is one actually quantizing on the lattice. This led us to consider the subject of consistent discretizations.

3. Consistent discretizations

Discretizations are very commonly used as a tool to treat field theories. Classically, when one wishes to solve the equations of a theory on a computer, one replaces the continuum equations by discrete approximations to be solved numerically. At the level of quantization, lattices have been used to regularize the infinities that plague field theories. This has been a very successful approach for treating Yang–Mills theories.

Discretizing general relativity is subtler than what one initially thinks. Consider a $3+1$ decomposition of the Einstein equations. One has 12 variables to solve for (the six components of the spatial metric and the six components of the extrinsic curvature). Yet, there are 16 equations to be solved, six evolution equations for the metric, six for the extrinsic curvature and four constraints. In the continuum, we know that these 16 equations are compatible, i.e., one can find 12 functions that satisfy them. However, when one discretizes the equations, the resulting system of algebraic equations is in general incompatible. This is well-known, for instance, in numerical relativity. The usual attitude there is to ignore the constraints and solve the 12 evolution equations (this scheme is called 'free evolution'). The expectation is that in the limit in which the lattice is infinitely refined, the constraints will also be satisfied if one satisfied them initially. The situation is more involved if one is interested in discretizing the theory in order to quantize it. There, one needs to take into account all equations. In particular, in the continuum the constraints form an algebra. If one discretizes the theory, the discrete version of the constraints will in many instances fail to close an algebra. Theories with constraints that do not form algebras imply the existence of more constraints which usually makes them inconsistent. For instance, it might be the case that there are no wave functions that can be annihilated simultaneously by all constraints. One can ask the question if this is not happening in the construction that Thiemann works out. To our knowledge, this issue has not been probed. What is clear is that discretizing relativity in order to quantize it will require some further thinking.

The new proposal we have put forward [17], called consistent discretization, is that, in order to make the discrete equations consistent, the lapse and the shift need to be considered as some of the variables to be solved for. Then one has 16 equations and 16 unknowns. This might appear surprising since our intuition from the continuum is that the lapse and the shift are freely specifiable. But we need to acknowledge that the discrete theory is a different theory, which may approximate the continuum theory in some circumstances, but nevertheless is different and may have important differences even at the conceptual level. This is true of any discretization proposal, not only ours.

We have constructed a canonical approach for theories discretized in the consistent scheme [18]. The basic idea is that one does not construct a Legendré transform and a Hamiltonian starting from the discretized Lagrangian picture. The reason for this is that the Hamiltonian is a generator of infinitesimal time evolutions, and in a discrete theory, there is no concept of infinitesimal. What plays the role of a Hamiltonian is a canonical transformation that implements the finite time evolution from discrete instant n to $n+1$. The canonical transformation is generated by the Lagrangian viewed as a type I canonical transformation generating functional. The theory is then quantized by implementing the canonical transformation as a unitary evolution operator.

4. Examples

We have applied this discretization scheme to perform a discretization (at a classical level) of BF theory and Yang–Mills theories [18]. In the case of BF theories this provides the first direct discretization scheme on a lattice that is known for such theories. In the case of Yang–Mills theories it reproduces known results. We have also studied the application of the discretization scheme in simple cosmological models. We find that the discretized models approximate general relativity well and avoid the singularity [19]. More interestingly, they may provide a mechanism for explaining the value of fundamental constants [20]. When the discrete models tunnel through the singularity, the value of the lapse gets modified and therefore the 'lattice spacing' before and after is different. Since in the lattice gauge theories the spacing is related to the 'dressed' values of the fundamental constants, this provides a mechanism for fundamental constants to change when tunneling through a singularity, as required in Smolin's [21] 'life of the cosmos' scenario.

It is quite remarkable that the discrete models work at all. When one solves for the lapse and the shift one is solving non-linear coupled algebraic equations. It could have happened that the solutions were complex. It could have happened that there were many possible 'branches' of solutions. It could have happened that the lapse turned negative. Although all these situations are possible given certain choices of initial data, it is remarkable that it appears that one can choose initial data for which pathologies are avoided and the discrete theory approximates the continuum theory in a controlled fashion.

We are currently exploring the Gowdy models with this approach, initially at a classical level only. Here the problem is considerably more complex than in cosmological models. The equations to be solved for the lapse and the shift become a coupled system that couples all points in the spatial discretization of the lattice. The problem can only be treated numerically. We have written a FORTRAN code to solve the system using iterative techniques (considerable care needs to be exercise since the system becomes almost singular at certain points in phase space) and results are encouraging. In the end, the credibility of the whole approach will hinge upon us producing several examples of situations of interest where the discrete theories approximate continuum GR well.

5. Several conceptual advantages

The fact that in the consistent discrete theories one solves the constraints to determine the value of the Lagrange multipliers has rather remarkable implications. The presence of the constraints is one of the most significant sources of conceptual problems in canonical quantum gravity. The fact that we approximate the continuum theory (which has constraints) with a discrete theory that is constraint free allows us to bypass in the discrete theory many of the conceptual problems of canonical quantum gravity. One of the main problems we can deal with is the 'problem of time'. This problem has generated a large amount of controversy and has several aspects to it. We cannot cover everything here, the definitive treatise on the subject is the paper by Kuchař [22].

To simplify the discussion of the problem of time, let us consider an aspect of quantum mechanics that most people find unsatisfactory perhaps from the first time they encounter the theory as undergraduates. It is the fact that in the Schrödinger equation, the variables x and t play very different roles. The variable x is a quantum operator of which we can, for instance, compute its expectation value, or its uncertainty. In contrast t is assumed to be a continuous external parameter. One is expected to have a clock that behaves perfectly classically and is completely external to the system under study. Of course, such a construction can only be an approximation. There is no such thing as a perfect classical clock and in many circumstances (for instance quantum cosmology) there is no 'external clock' to the system of interest. How is one to do quantum mechanics in such circumstances? The answer is: 'relationally'. One could envision promoting all variables of a system to quantum operators, and choosing one of them to play the role of a 'clock'. Let us call such a variable as t (it could be, for instance the angular position of the hands of a real clock, or it could be something else). One could then compute conditional probabilities for other variables to take certain values x_0 when the 'clock' variable takes the value t_0. If the variable we chose as our 'clock' does correspond to a variable that is behaving classically as a clock, then the conditional probabilities will approximate well the probabilities computed in the ordinary Schrödinger theory. If one picked a 'crazy time' then the conditional probabilities are still well-defined, but they do not approximate any Schrödinger theory well. If there is no variable that can be considered a good classical clock, Schrödinger's quantum mechanics does not make sense and the relational quantum mechanics is therefore a generalization of Schrödinger's quantum mechanics.

Relational quantum mechanics therefore appears well-suited as a technique to use in quantizing general relativity, particularly in cosmological situations where there is no externally defined 'classical time'. Page and Wootters [23] advocated this. Unfortunately, there are technical problems when one attempts the construction in detail for general relativity. The problem arises when one wishes to promote the variables to quantum operators. Which variables to choose? In principle, the only variables that make sense physically are those that have vanishing Poisson brackets (or quantum mechanically vanishing commutators) with the constraints. But since the Hamiltonian is one of the constraints, then such variables are 'perennials', i.e. constants of motion, and one cannot reasonably expect any of them to play the role of a 'clock'. One could avoid this problem by considering variables that do not have vanishing Poisson brackets with the constraints. But this causes problems. Quantum mechanically one wishes to consider quantum states that are annihilated by the constraints. Variables that do not commute with the constraints as quantum operators map out of the space of states that solve the constraints. The end result of this, as discussed in detail by Kuchař [22], is that the propagators constructed with the relational approach do not propagate.

Notice that all the problems are due to the presence of the constraints. In our discrete theory, since there are no constraints, there is no obstruction to constructing the relational picture. We have discussed this in detail in [24].

Of great interest is the fact that the resulting relational theory will never entirely coincide with a Schrödinger picture. In particular, since no clock is perfectly classical, pure states do not remain pure forever in this quantization, but slowly decohere into mixed states. We have estimated the magnitude of this effect, and it is proportional to $\omega^2 T_{\rm Planck} T$ where ω is the frequency associated with the spread in energy levels of the system under study, $T_{\rm Planck}$ is Planck's time, and T is the time that the system lives. The effect is very small. Only for systems that have rather large energy spreads (Bose Einstein condensates are a possible example) the effect may be close to observability. With current technologies, the condensates do not have enough atoms to achieve the energy spreads of interest, but it might not be unfeasible as technology improves to observe the effect [25].

The fact that a pure state evolves into a mixed state opens other interesting possibilities, connected with the black hole information puzzle. This puzzle is related to the fact that one could consider a pure quantum state that collapses into a black hole. The latter will start evaporating due to Hawking radiation until eventually it disappears. What one is left with at the end of the day appears to be the outgoing radiation, which is in a mixed state. Therefore a pure state appears to have evolved into a mixed state. There is a vast literature discussing this issue (see for instance [26] for a short review). Possible solutions proposed include that the black hole may not disappear entirely or that some mechanism may allow pure states to evolve into a mixed state. But we have just discussed that the relational discrete quantum gravity predicts such decoherence! We have estimated that the decoherence is fast enough to turn the pure state into a mixed one before the black hole can evaporate completely, at least if one considers black holes larger than a few hundred Planck masses (for smaller holes the evaporation picture is not accurate anyway) [27]. The result is quite remarkable, since the decoherence effect, as we pointed out before, is quite small. It is large enough to avoid the information puzzle in black holes, even if one considers smaller and smaller black holes which evaporate faster since they also have larger energy spreads and therefore the decoherence effect operates faster.

6. Summary

Analysing the problems of the dynamics of loop quantum gravity led us to develop a way to discretize general relativity consistently. Surprisingly, the consistent discretizations not only approximate general relativity well in several situations, but also allow to handle several of the hard conceptual problems of canonical quantum gravity. What is now needed is to demonstrate that the range of situations in which the discrete theory approximates general relativity well is convincingly large enough to consider its quantization as a route for the quantization of general relativity.

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