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Muonium/muonic hydrogen formation in atomic hydrogen

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Abstract. The muonium/muonic hydrogen atom formation in $\mu^{\pm}$–H collisions is investigated, using a two-state approximation in a time dependent formalism. It is found that muonium cross-section results are similar to the cross-section results obtained for positronium formation in e^{+}–H collision. Muonic hydrogen atom formation cross-sections in μ^{-}–H collision are found to be significant in a narrow range of energy (5 eV–25 eV).

Keywords. Muon; muonium; muonic hydrogen; two-state approximation, cross-sections.

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1. Introduction

Interaction of muons with matter has been extensively studied [1]. Muon is found to behave as a heavy electron/positron in matter [2]. When stopped by matter, muons are captured by nuclei. Positively charged muons (μ^{+}) may pick up electrons from the matter to form muonium. Hydrogen-like muonium is ideal for testing standard theory, fundamental symmetries in physics and for searching yet unknown interactions. Measurements of 1s–2s transition in muonium offers a determination of muon mass [3]. It has also led to better muon electron charge ratio. Negative muons (μ^{-}) may enter a hydrogen-like bound atomic state in the Coulomb field of nucleus to form the muonic atom [4]. The importance of muonic atom lies in the fact that the wave function of muon (μ^{-}) in all states and especially in the lowest s-state overlaps the nucleus more strongly than the normal electronic wave functions. Measurements of energy interval between various excited levels of muonic atom enable one to know about nuclear charge distributions. One can also determine charge and mass of muon more accurately. Initially, muonic atoms are found to be formed in higher excited states. However, they decay into ground states in a very short period (10^{-10}–10^{-14} s). Muons themselves have a short lifetime (2.197 μs) [2]. The decay of muons compete with muonium/muonic atom formation process. Muons interacting with atomic hydrogen may result in the formation of muonium or muonic hydrogen. Muonic hydrogen is a unique tool to study the low energy properties of the proton form factors [5]. The energy interval (2p–1s) in muonic hydrogen allows for the precise determination of proton charge and radius.

Extensive studies on muonium as well as muonic hydrogen have appeared in literature. In particular, elastic scattering of protons with muonic hydrogen has been studied [6]. Charge transfer studies with rare gas atoms have also been done [7–9]. Muonium atom has also been a topic of intense research. In particular, Janev and Belki [9] studied muonium formation in hydrogen using adiabatic approach for low energy and distorted wave Born approximation (DWBA) for high energy. Most of these studies use muonium as well as muonic hydrogen atom in their theoretical as well as experimental measurements. The objective of this paper is to provide muonium/muonic hydrogen formation cross-sections in ground state using a simple and reliable method.

The problem of muons interacting with hydrogen atoms is one of the simplest real three-body problems. In this problem all the interactions are well-known (Coulomb-type). The particles are distinguishable and so the collisions are free of exchange effects. This problem is similar to the problem of positrons (e^+) interacting with hydrogen (H) atom resulting in the formation of positronium. In muons, μ^+ interacting with H-atom forms muonium atom (μ^+e^-), and μ^- interacting with H-atom forms muonic hydrogen atom (μ^-p) bound state. Whereas for muonium formation to take place, a small threshold energy ($E > 0.12$ eV) is required, muonic hydrogen formation can take place even at zero energy of muons (negative).

Muonium or muonic hydrogen formation is a charge-exchange (rearrangement) collision. To explain such collisions, a number of theories exist in literature which have limitations with regard to the energy of the projectile. For instance, at low energies, eigenfunction expansion (close coupling approximation) methods are employed and at higher energies Born expansion methods are used. Exhaustive review of these methods is given in refs [10–12]. Within the Born approximation, first Born approximation (FBA) is the simplest one. In Born approximation calculations, initial channel and final (rearrangement) channel are assumed to be orthogonal. An account of non-orthogonality of rearrangement channel is desirable. Bubelev and Madison [13] show that enforcing orthogonality in e^-–H scattering introduces spurious effects. An account of non-orthogonality was made initially by Band [14] for proton–hydrogen charge-exchange collision. Later, Kulhar and Shastry [15,16] gave a formulation for e^+–H charge-exchange problem. The formulation is based on a two-state approximation in a time-dependent quantum mechanical formalism. The non-orthogonality of the rearrangement channel is built into it. The two-state approximation (TSA) is also important for charge-exchange collisions because its application to p–H charge-exchange had led to results which were in better agreement with the experiment [14]. Now the experimental results for e^+–H charge-exchange are available [17,18]. The observed experimental results are found to be higher than the FBA results, a feature observed in TSA calculation for e^+–H charge-exchange problem also [15,16]. Since the results based on TSA are found to be reliable in intermediate and high energy region for proton–hydrogen charge-exchange collision, we can use this method for muonium and muonic hydrogen formation process also.

The energy requirements for the validity of TSA for collisions of the type shown by Band [14] were

$$3 + (1, 2) \rightarrow (3, 1) + 2,$$

$$k_{\rm i} > \{\mu_{\rm i}[V_1 + V_2]\}^{1/2},$$

$$k_{\rm f} > \{\mu_{\rm f}[V_1 + V_3]\}^{1/2},$$

where $V_{\rm i}$ is interaction potential between the jth and kth particle. The necessary conditions for the validity of TSA are the same as the criteria for the validity of FBA. Muon energy above 1.2 keV is necessary for the validity of TSA for muonium formation while muon (μ^-) energy greater than 2 eV is necessary for the validity of TSA for muonic hydrogen formation. Author expects the method to work when necessary conditions are met.

In §2 we discuss the theory and its application to the muonium and muonic hydrogen formation in the ground state. Section 3 contains results and discussion.

2. Theory

Muonium/muonic hydrogen formation is a rearrangement type of collision:

$$\mu^+ + \mathrm{H} \rightarrow (\mu^+ e^-) + p \tag{1}$$

$$\mu^- + \mathrm{H} \rightarrow (\mu^- p) + e^-. \tag{2}$$

Both processes (1) and (2) may be treated using same kinematics. Here

$$3 + (1,2) \rightarrow (3,1) + 2. \tag{3}$$

3 is the incident particle ($\mu^\pm$), while 2 is the outgoing particle (p or e^-). The complete Hamiltonian for the system is

$$H = -\frac{\nabla^2_{\tilde{R}_3}}{2\mu_{\rm i}} - \frac{\nabla^2_{\tilde{r}_{12}}}{2\mu_{12}} + V_1(r_{23}) + V_2(r_{31}) + V_3(r_{12}) \tag{4}$$

$$H = -\frac{\nabla^2_{\tilde{R}_2}}{2\mu_{\rm f}} - \frac{\nabla^2_{\tilde{r}_{31}}}{2\mu_{31}} + V_1(r_{23}) + V_2(r_{31}) + V_3(r_{12}). \tag{5}$$

Here $\tilde{r}_{ij}$ is the relative coordinate vector of the ith particle with respect to jth, $\tilde{R}_2$ and $\tilde{R}_3$ are coordinate vectors of 2 and 3 from centre of mass of (3,1) and (1,2) respectively. $\mu_{\rm i}$ and $\mu_{\rm f}$ are the reduced masses of the system in the initial and final states.

In the two-state approximation (TSA) one writes

$$\Psi(\tilde{R}, \tilde{r}, t) = b_{\rm i}(t)\varphi_{\rm i}(r_{12}) \exp[-i\tilde{k}_{\rm i} \cdot \tilde{R}_3]\mathrm{e}^{-iut} + a_{\rm f}(t)\varphi_{\rm f}(r_{31}) \exp[i\tilde{k}_{\rm f} \cdot \tilde{R}_2]\mathrm{e}^{-ivt} \tag{6}$$

with the initial conditions

$$|b_{\rm i}(t = -\infty)| = 1 \quad \text{and} \quad |\dot{a}_{\rm f}(t = \infty)| = 0. \tag{7}$$

Conservation of energy is in-built in the formalism and will require

$$\beta + \frac{k_{\rm i}^2}{2\mu_{\rm i}} = \alpha + \frac{k_{\rm f}^2}{2\mu_{\rm f}} \quad (u = \nu) \tag{8}$$

β and α are the binding energies in the initial and final stages, i.e., of hydrogen and muonic atom. Using an approach similar to [15], one obtains the following expression for the differential cross-section:

$$\frac{{\rm d}\sigma}{{\rm d}\Omega}(a_0^2) = (2\pi)^4 \mu_{\rm i}\mu_{\rm f} \sum_{\{\rm i\}\{\rm f\}} |M_{\rm fi}|^2, \tag{9}$$

where $M_{\rm fi}$ is the matrix element in the TSA.

$$M_{\rm fi} = \frac{h_{\rm fi}^3 - s_{\rm fi} h_{\rm ii}^3}{1 - |s_{\rm fi}|^2}, \tag{10}$$

where $h_{\rm fi}^3$ is the matrix element in the FBA while remaining terms in eq. (10) give contribution due to non-orthogonality of the rearrangement channel.

Now consider the application to the process of muonium formation in μ^+–H collision (the formation of muonic hydrogen with μ^- as incident particle is similar). In our case $m_3 = m_{\mu^\pm} = m_\mu$, $m_2 = m_p = M$ and $m_1 = m_{e^-} = m$, here $\mu^\pm, p, e^-$ denote muons, proton and electron respectively.

The potentials are Coulombic and are (atomic units)

$$V_1 = \frac{1}{r_{23}}, \quad V_2 = -\frac{1}{r_{31}} \quad \text{and} \quad V_3 = -\frac{1}{r_{12}}. \tag{11}$$

The expression for $h_{\rm fi}^3$, $h_{\rm ii}^3$ and $s_{\rm fi}$ are as follows:

$$h_{\rm fi}^3 = \frac{1}{(2\pi)^3}\left[-\left(\frac{C^2}{2\mu_{13}} + \alpha\right)\varphi_{\rm f}^*(\tilde{C})\varphi_{\rm i}(\tilde{B}) + \frac{1}{2\pi^2}\int \frac{{\rm d}^3k}{k^2}\varphi_{\rm f}^*(\tilde{C} - \tilde{k})\varphi_{\rm i}(\tilde{B} - \tilde{k})\right], \tag{12}$$

$$h_{\rm ii}^3 = \frac{1}{4\pi^2} \tag{13}$$

and

$$s_{\rm fi} = s_{\rm if}^* = \frac{1}{(2\pi)^3}\varphi_{\rm f}^*(\tilde{C})\varphi_{\rm f}^*(\tilde{B}), \tag{14}$$

where

$$\tilde{B} = \tilde{k}_{\rm f} - \tilde{k}_{\rm i}\left(1 - \frac{m}{m + M}\right) \tag{15}$$

and

$$\tilde{C} = \tilde{k}_{\rm f}\left(1 - \frac{m}{m + m_\mu}\right) - \tilde{k}_{\rm i}. \tag{16}$$

The Fourier transform for ground states of hydrogen $\varphi_{\rm i}(\tilde{p})$ and muonium atom $\varphi_{\rm f}(\tilde{p})$ are easily evaluated as

$$\varphi_{\rm i}(\tilde{p}) = \frac{8\sqrt{\pi}}{[p^2+1]^2}, \tag{17}$$

$$\varphi_{\rm f}(\tilde{p}) = \frac{8\sqrt{\pi}}{\sqrt{a_\mu^5}\left[p^2+\frac{1}{a_\mu^2}\right]^2}, \tag{18}$$

where a_μ is the radius of the muonic atom in its ground state ($a_\mu = \hbar^2/\mu e^2$).

The integral appearing in eq. (12) is of the type shown in [19] and an analytic expression for this can be written (see [15]).

3. Results and discussion

We have computed differential cross-sections (figure 1) for muonium formation in its ground state for an energy of 2.5 keV at several angles (θ). Unlike the case of Ps formation [14], the differential cross-section is highly peaked in forward direction because muon is heavier than positron. Similar feature was also obtained in proton–hydrogen charge-exchange problem [14]. The effect of non-orthogonality of rearrangement channel is to enhance the differential cross-section in the forward direction, a feature observed in the Ps formation process also [15]. Due to the cancellation effect of attractive and repulsive parts of interaction a zero in the differential cross-section appears. For angles beyond this angle TSA results are lower in comparison to FBA results. Similar features were observed in the Ps formation process also.

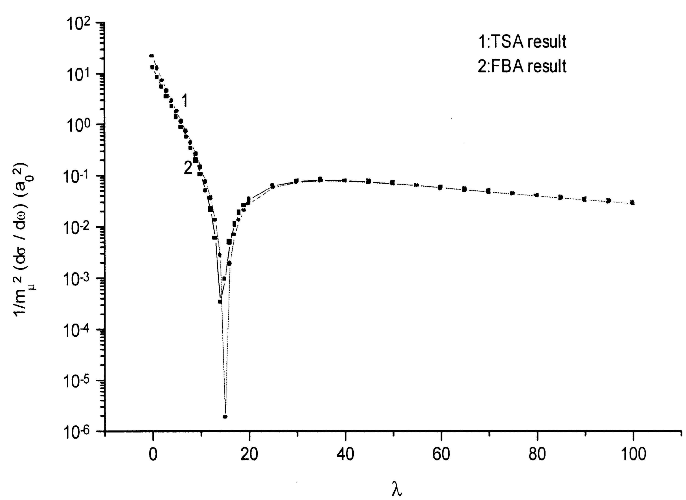

Figure 1. Differential cross-sections for muonium formation in μ^+–H collision at an energy of 2.5 keV as a function of λ ($\lambda = (1-\cos\theta)\times 10^6$).

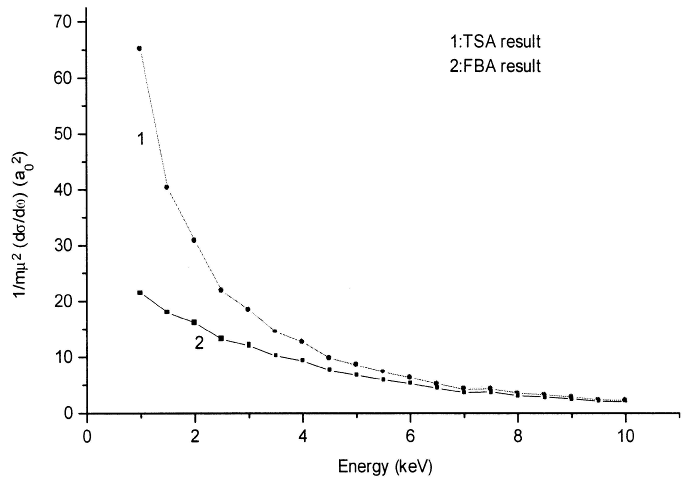

Figure 2. Forward differential cross-sections for muonium formation μ^+–H collision as a function of muon energy.

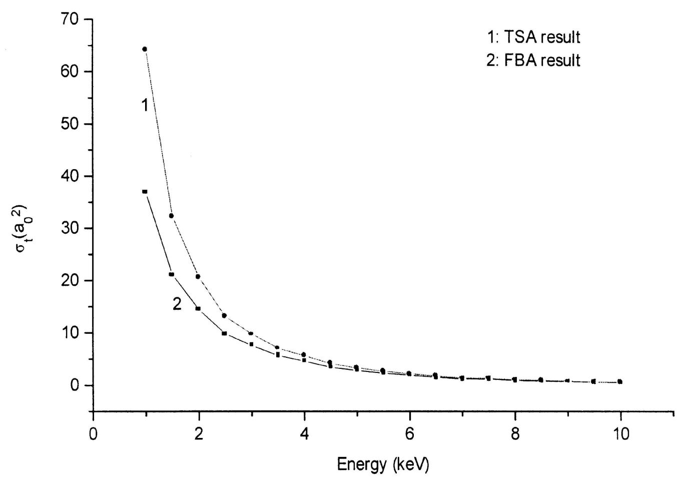

Figure 3. Total cross-sections for muonium formation in μ^+–H collision for different energies of muon.

We have computed forward differential and total cross-sections for muonium formation in the energy region 1 keV–10 keV (see figures 2 and 3). Table 1 shows the calculated values of forward differential and total cross-sections in the energy range (1 keV –10 keV). It is seen that the effect of non-orthogonality is significant in low and intermediate energy region. For instance, at an energy of 8 keV, the TSA result for total cross-section is 11.5% higher than the FBA result while at 10 keV it is 9.65% higher. At higher energies the non-orthogonality effect diminishes (a feature observed in Ps formation also). The forward differential cross-section for muonium formation is shown to exhibit a small peak near 7.5 keV, a feature which may be attributed to the mass difference in the incident particle. For positive muons with energies of the order of 1 keV or less, validity conditions for TSA do not hold and at such energies the results of TSA may not be reliable. At lower energies polarization effects are also important and must be taken into account.

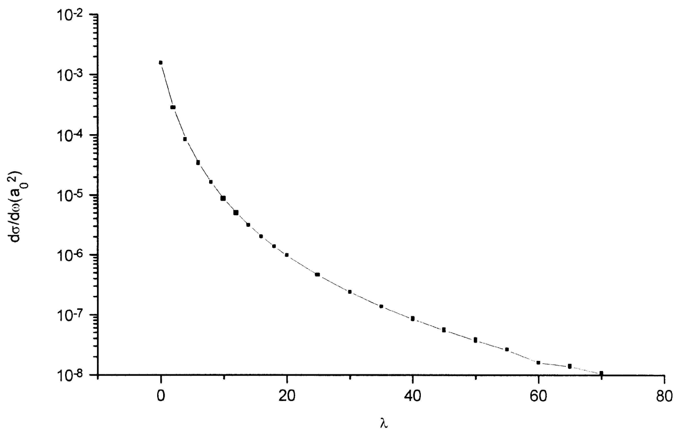

Figure 4. Differential cross-sections for muonic hydrogen formation in μ^-–H collision at an energy of 15 eV as a function of λ ($\lambda = (1 - \cos\theta) \times 10^3$).

Table 1. Energy of muon (keV) vs. forward differential and total cross-sections (a_0^2) for muonium.

E (keV)	$\frac{1}{m_\mu^2}\left(\frac{d\sigma}{d\Omega}\right)_{\rm FBA}$	$\frac{1}{m_\mu^2}\left(\frac{d\sigma}{d\Omega}\right)_{\rm TSA}$	$\sigma^{\rm t}_{\rm FBA}$	$\sigma^{\rm t}_{\rm TSA}$
1	21.529	65.101	36.835	64.286
1.5	17.958	40.298	21.130	32.298
2	16.111	30.899	14.442	20.640
2.5	13.184	21.794	9.806	13.091
3	12.015	18.429	7.557	9.766
3.5	10.168	14.482	5.651	7.052
4	9.278	12.651	4.586	5.611
4.5	7.549	9.794	3.444	4.109
5	6.758	8.523	2.833	3.333
5.5	5.929	7.294	2.315	2.689
6	5.214	6.288	1.918	2.197
6.5	4.415	5.226	1.540	1.752
7	3.611	4.206	1.213	1.366
7.5	3.741	4.332	1.169	1.315
8	3.078	3.522	0.934	1.042
8.5	2.843	3.229	0.825	0.917
9	2.535	2.859	0.710	0.785
9.5	2.113	2.366	0.579	0.637
10	2.028	2.260	0.534	0.585

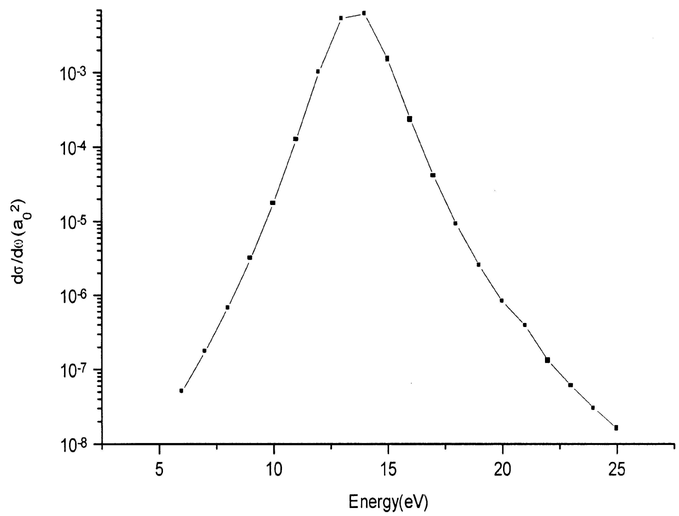

Figure 5. Forward differential cross-sections for muonic hydrogen formation in μ^-–H collision as a function of muon energy.

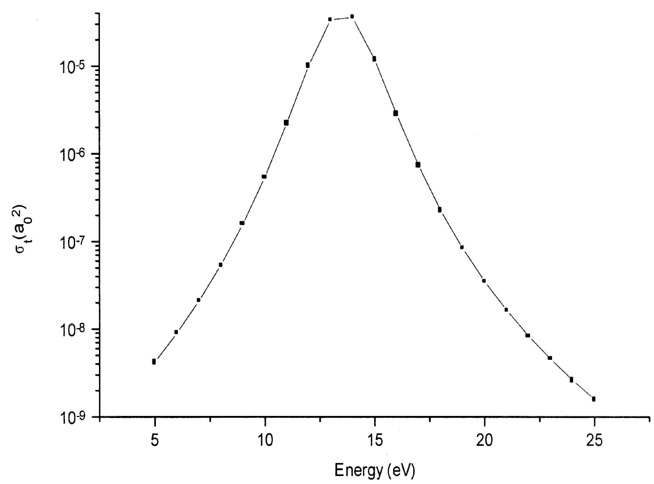

Figure 6. Total cross-section for muonic hydrogen formation in μ^-–H collision for different energies of muon.

Computation has also been done for the process of muonic hydrogen formation in the ground state for μ^-–H charge-exchange collision. Here, some interesting results were obtained (see figures 4–6). It may be noticed that significant cross-section results are obtained only in a narrow region (5 eV–25 eV) of energy. The differential cross-sections (figures 4,5) and total cross-section (figure 6) are practically same for FBA and TSA. It shows that the contribution of non-orthogonality is insignificant in μ^-–H charge-exchange collision. The cross-section (differential as well as total) results obtained are unlike the case of e^+–H and p–H charge exchange collisions [14,15]. Both forward differential as well as total cross-section results show a resonance-like behaviour. In total cross-section the peak appears at 13.5 eV near the ionization energy of H-atom. It may be mentioned that this feature appears in Ps formation process too (see Majumdar and Rajagopal [20]). It may be pointed out that in one of the first experimental results on e^+–H charge-exchange collision [17], the peak for Ps formation was observed at 13.6 ± 2.8 eV, near the ionization energy of H-atom. At low energies, target polarization plays an important role and must be taken into account. At higher energies, beside the formation into ground state, muonium and muonic hydrogen formation into higher excited states can take place. Thus total formation cross-section should include contribution of higher excited states. An estimate of contribution of excited states can be made using scaling laws postulated by Jackson and Schiff [21]. A study of formation cross-section for higher excited states is contemplated. It may be added that the results obtained in TSA will be closer to experimental results as energy is increased and at very high energy TSA results will merge into FBA results.

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