PRAMANA — journal of physics © Indian Academy of Sciences Vol. 63, No. 2 August 2004 pp. 381–385

Geometric formula for prism deflection

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Abstract. While studying neutron deflections produced by a magnetic prism, we have stumbled upon a simple 'geometric' formula. For a prism of refractive index n close to unity, the deflection simply equals the product of the refractive power $n-1$ and the base-to-height ratio of the prism, regardless of the apex angle. The base and height of the prism are measured respectively along and perpendicular to the direction of beam propagation within the prism. The geometric formula greatly simplifies the optimisation of prism parameters to suit any specific experiment.

Keywords. Prism optics; neutron refraction; X-ray refraction.

PACS Nos 03.75.Be; 41.50.+h; 42.25.Gy

Over three centuries ago, Isaac Newton used a glass prism to separate constituent colours in a beam of sunlight. Refraction of a wave at the boundary between two transparent media has since been thoroughly understood [1]. Snell's law, embodying conservation of the tangential component of the wave vector across the boundary, governs deflections produced by prisms of refractive index n. The refractive power, $n-1$, of most materials is of the order of unity for visible light, which therefore gets deflected through several degrees by a prism. X-rays and neutrons however, encounter refractive powers typically $\sim 10^{-5}$ or smaller in magnitude and undergo prism deflections in the arcsecond regime [2–7]. A simple proportionality between this deflection and the base-to-height ratio of the prism will presently be derived. It is surprising that despite the extensive use of prisms the world over to manipulate X-ray and neutron beam optics [2–6], this simple relation has escaped recognition for nearly a century, as far as we know.

The deflection δ produced by a prism (figure 1) equals the algebraic sum of angular deviations $\delta_1 = i - r$ and $\delta_2 = e - (A - r)$ due to refraction at the two interfaces between the prism and ambient vacuum. Thus,

$$\delta = i - r + e - (A - r) = i + e - A. \tag{1}$$

Here i, r and e denote angles of incidence, refraction and emergence respectively and A stands for the prism apex angle. In accordance with Snell's law,

$$\sin i / \sin r = n = \sin e / \sin(A - r). \tag{2}$$

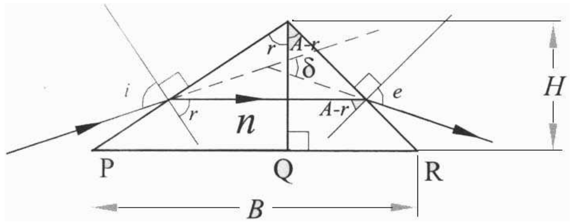

Figure 1. Prism deflection.

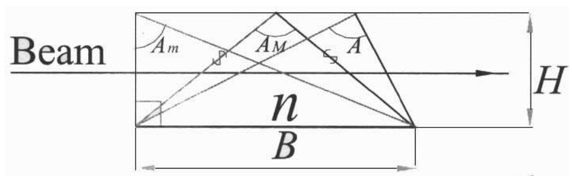

Figure 2. All these prisms effect the same deflection.

Hence the deflection

$$\delta = \arcsin(n \sin r) + \arcsin[n \sin(A - r)] - A, \tag{3}$$

depends critically on the apex angle of the prism for a given angle of incidence. In case $|n-1| \ll 1$, the deflection can be expressed as

$$\delta \approx (n-1)[\tan r + \tan(A-r)]. \tag{4}$$

Further, for the special case of beam propagation parallel to the prism base inside the prism,

$$\delta \approx (n-1)(PQ/H + QR/H) = (n-1)B/H, \tag{5}$$

B and H denoting the base and height of the prism respectively (figure 1). We thus have the striking result that for $|n-1| \ll 1$ and beam propagation parallel to the prism base, all prisms with the same B/H ratio produce the same beam deflection (figure 2) regardless of the apex angle A.

For a given base–height ratio, the prism apex angle has a maximum, viz.

$$A_{\mathrm{M}} = 2\arctan(B/2H), \tag{6}$$

for the isosceles prism (figure 2). This case is symmetric, viz. invariant under reversal of beam propagation, since

$$i = e = (A_{\mathrm{M}} + \delta)/2, \quad r = (A_{\mathrm{M}} - r) = A_{\mathrm{M}}/2, \quad \delta_1 = \delta_2 = \delta/2, \tag{7}$$

i.e. identical deflections occur at the two interfaces. Here $n = \sin[(A_{\mathrm{M}} + \delta)/2]/\sin(A_{\mathrm{M}}/2)$.

Further, if we constrain δ_1 and δ_2 to have the same sign, the prism apex angle will also have a minimum value given by

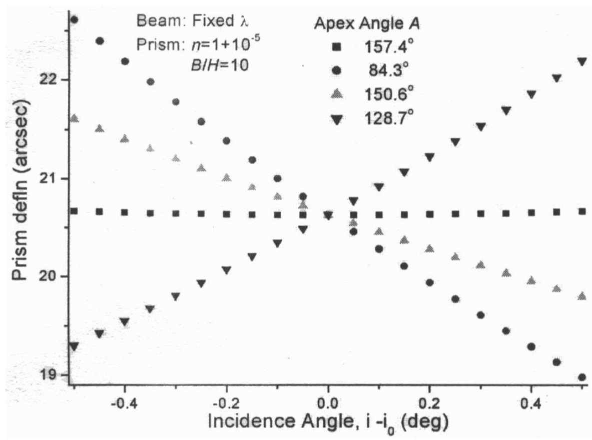

Figure 3. Variation of δ with i for a perfectly monochromatic beam.

$$A_{\mathrm{m}} = \arctan\,(B/H), \tag{8}$$

when the prism has a right angle at the base (figure 2). Here the beam suffers a null deflection at the right-angled interface and the full deflection δ at the other interface, so that $n = \sin(A_{\mathrm{m}} + \delta)/\sin A_{\mathrm{m}}$. Thus all prisms with a constant base-to-height ratio (figure 2) yield the same net beam deflection δ, which occurs at a single interface for the minimum apex angle A_{m} (8), but comprises deflections at both interfaces as A increases. Finally, the two deflections δ_1 and δ_2 become equal (7) at the maximum apex angle A_{M}. If the angle of incidence i varies near this condition, the corresponding variation of the deflection δ depends markedly on the apex angle. For the maximum apex angle A_{M}, the δ variation is minimised since the first order variation of δ with i vanishes. As the apex angle reduces, the δ variation becomes more pronounced. Variations of the beam deflection due to the angular divergence of an incident beam with a fixed wavelength λ_0 is depicted in figure 3 for prisms with $B/H = 10$, $n = 1.00001$ and four apex angles ranging from $A_{\mathrm{m}} = 84.3°$ to $A_{\mathrm{M}} = 157.4°$.

These considerations apply for X-rays and neutrons passing through a material prism and for neutrons of either spin-state traversing a triangular region permeated by a magnetic field. It is customary to Bragg-reflect neutrons or X-rays from a perfect crystal to obtain a monochromatic beam, wherein neutrons or X-rays of each wavelength λ emerge within a few arcsecond wide fan centred at the corresponding Bragg angle $\theta_{\mathrm{B}}(\lambda)$. When the mean wavelength λ_0 in such a monochromatic beam subtends the correct incidence angle i_0 at the prism, another wavelength λ becomes incident at $i(\lambda) = i_0 \pm [\theta_{\mathrm{B}}(\lambda) - \theta_{\mathrm{B}}(\lambda_0)]$. The variation of δ with λ in such a situation is displayed in figure 4 for a beam of $\lambda_0 = 1.73$ Å from a perfect crystal $\{1\,1\,1\}$ silicon monochromator incident on a prism with $B/H = 10$ and λ^2-proportional refractive power, for five apex angles between A_{m} and A_{M}. The δ variation is minimum for the optimum apex angle $A_0 = 139.9°$ given by

$$\tan A_0 = (B/H)/[\cot^2\theta_{\mathrm{B}0} - (B/2H)^2 + 1], \tag{9}$$

where $\theta_{\mathrm{B}0} = \theta_{\mathrm{B}}(\lambda_0)$. A_0 increases from 0 for $\theta_{\mathrm{B}0} \sim 0$ to A_{M} as $\theta_{\mathrm{B}0}$ approaches 90°.

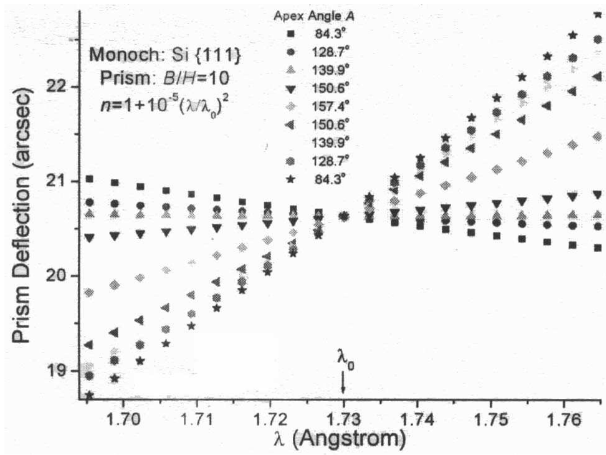

Figure 4. δ-variation for a neutron beam from a perfect crystal monochromator.

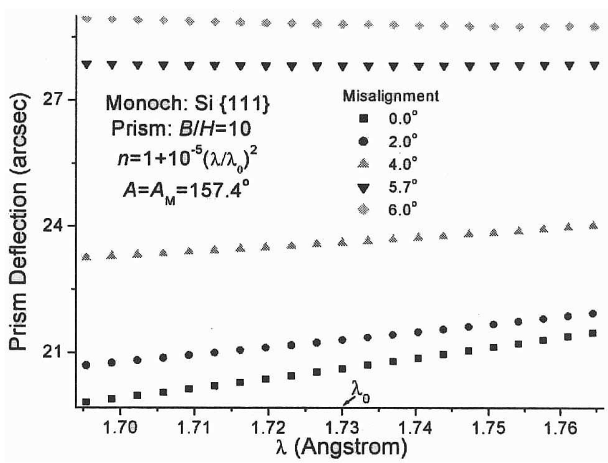

Figure 5. Minimising δ variation by misaligning isosceles prism.

The δ variation can also be minimised using the isosceles prism ($A = A_{\mathrm{M}}$), misaligned by $\chi = 5.7°$ (figure 5) obtained from the relation

$$\begin{aligned}\tan\chi &= \{[\tan^2\theta_{\mathrm{B0}} + \sin^2 A_{\mathrm{M}}]^{1/2} - \tan\theta_{\mathrm{B0}}\}/[2\sin^2(A_{\mathrm{M}}/2)] \\ &= \cot(A_{\mathrm{M}}/2)\tan\{[\arctan(\sin A_{\mathrm{M}}\cot\theta_{\mathrm{B0}})]/2\}. \qquad (10)\end{aligned}$$

However, here the base and height of a prism are measured respectively along and perpendicular to the direction of beam propagation within the prism. Hence a misaligned isosceles prism works like an asymmetric prism of an enhanced base–height ratio ($\approx \tan(A_{\mathrm{M}}/2+\chi) + \tan(A_{\mathrm{M}}/2-\chi)) = 13.5$, increasing the deflection $\delta(\lambda_0)$ in the same proportion from 20.6 to 27.8 arcsec. As θ_{B0} rises from $\sim 0°$ to $\sim 90°$, the misalignment χ falls from $(180 - A_{\mathrm{M}})/2$ to $0°$.

It has hitherto been customary for experimenters to design a prism with the largest practicable apex angle and to vary the incidence angle to produce a desired large deflection of a neutron (or X-ray) beam. Relation (5) derived here delineates effects of the base–height ratio and apex angle of a prism, thus dramatically simplifying the prism design. A desired neutron (or X-ray) beam deflection in any experiment can now be attained with a prism of a given refractive power just by selecting the proper base–height ratio. Any δ-profile in wavelength required in a specific application can then be tailored with the appropriate choice of the apex angle of the prism (figure 4).

To summarise, in the domain of low refractive powers, we have presented an amazingly simple equality of the optical deflection produced by a prism to the product of the refractive power and the base-to-height ratio of the prism.

Acknowledgement

The authors thank Sohrab Abbas for preparing neat schematic sketches.

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