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On the dual symmetry between absorbing and amplifying random media

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Abstract. We re-examine the dual symmetry between absorbing and amplifying random media. By analysing the physically allowed choice of the sign of the square root to determine the complex wave vector in a medium, we draw a broad set of conclusions that enables us to resolve the apparent paradox of the dual symmetry and also to anticipate the large local electromagnetic field enhancements in amplifying random media.

Keywords. Random laser; Anderson localization; waves in disordered media.

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The bosonic nature of light that allows for the possibility of phase coherent amplification/absorption of light gives rise to a new class of problems involving the propagation of light in a spatially random but amplifying media (RAM). The phenomenon of mirrorless lasing in such media has been confirmed in several experiments [1–5]. There is evidence for two different regimes and associated mechanisms for the effect. One that explains the experimental results of gain narrowing and pulse shortening does not involve coherence: the multiple scattering causes light to behave diffusively and have long path lengths in the RAM and the effects are due to amplified spontaneous emission. This explanation is similar to the neutron gain in a nuclear reactor and was proposed by Letokhov in 1968 [6]. The second involves coherent feed-back mechanism arising either due to recurrent multiple scattering and the random cavity modes due to them [7,5], or the Anderson localization in such random media [8]. Although the output from the random laser has been demonstrated to have some coherence properties in recent experiments, the connection to Anderson localization is not completely understood.

In this paper, we will address a related issue. The transmission of light across a disordered slab decays exponentially with the thickness of the slab for large slab thickness, whether the medium is amplifying or absorbing [9,10]. This has been termed as a dual symmetry between absorbing and amplifying media, and is paradoxical in that, one would naively expect the amplification to increase the transmission exponentially. It has also been pointed out that such a symmetry is not unique to a disordered slab but is also found for a homogeneous slab of absorbing or amplifying medium [11,12]. In fact, this behaviour has even prompted Jiang *et al* [11] to question the very capability of the time independent Maxwell's equation to describe systems with linear amplification! However, the applicability of the time independent Maxwell's equation is justified on the grounds of analytic continuity. This dual symmetry also shows up in other contexts such as the distribution of dwell times for absorbing and amplifying random media being identical in the localized regime [13]. In this communication, we resolve this apparent paradox of the dual symmetry by considering the physical choice of the wave-vector in the complex plane. We show by arguments of analytic continuity that a localized mode remains localized regardless of absorption or amplification, and this is the origin of the dual symmetry.

Let us first examine the choice of the wave-vector in homogeneous media. The propagation of light in a medium is governed by the Maxwell's equations. For a time-harmonic plane wave, $\exp[i(\vec{k}\cdot\vec{r}-\omega t)]$, with an angular frequency ω, these reduce to

$$\vec{k}\times\vec{E}=\frac{\omega}{c}\mu\vec{H}, \tag{1}$$

$$\vec{k}\times\vec{H}=-\frac{\omega}{c}\epsilon\vec{E}, \tag{2}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields associated with the wave, and ϵ and μ are the complex dielectric permittivity and the complex magnetic permeability of the medium respectively. For complex ϵ, μ and $\vec{k}$, the wave becomes inhomogeneous. Writing $\epsilon=\epsilon'+i\epsilon''$, and $\mu=\mu'+i\mu''$, we note that the medium is absorptive if $\epsilon''>0$, $\mu''>0$, and amplifying if $\epsilon''<0$, $\mu''<0$. The mechanism for the absorption or amplification, of course, arises from the underlying atomic and molecular polarizabilities. In an isotropic medium, the Maxwell's equations require

$$|\vec{k}|^2=k_x^2+k_y^2+k_z^2=\epsilon\mu\frac{\omega^2}{c^2}. \tag{3}$$

To determine the wave-vector $\vec{k}$ in the medium, we have to carry out a square root operation. Obviously the choice of the sign of the square root will then have to be made so as to be consistent with the Maxwell's equations and causality.

For simplicity, let us consider an electromagnetic wave with a wave-vector $[k_x^{(1)},\ 0,\ k_z^{(1)}]$ to be incident from vacuum (medium-1) on the left ($z<0$) on a semi-infinite medium-2 ($z>0$) with an arbitrary value of ϵ and μ. Due to X-invariance, k_x is preserved across the interface. k_z in medium-2, however, has to be obtained from eq. (3) as

$$k_z=\pm\sqrt{\epsilon\mu\frac{\omega^2}{c^2}-k_x^2}, \tag{4}$$

where a physical choice has to be made for the sign of the square root. Let us consider non-magnetic media, $\mu=+1$. Now the waves in medium-2 could be propagating $[k_x^2<\mathrm{Re}(\epsilon\mu\omega^2/c^2)]$ or evanescent $[k_x^2>\mathrm{Re}(\epsilon\mu\omega^2/c^2)]$. This enables us

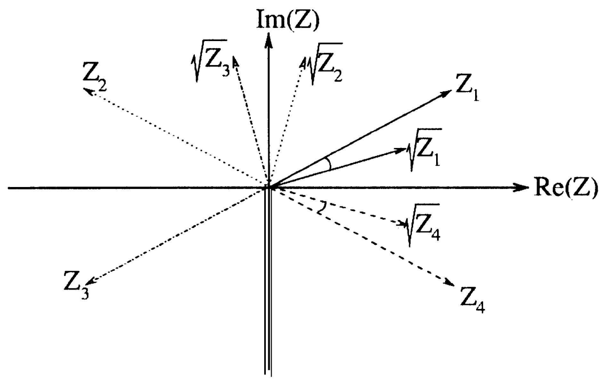

Figure 1. The Argand diagram for the complex wave-vectors $k_z = \sqrt{Z}$ for the four cases of absorption and amplification of propagating or evanescent waves. The branch cut in the complex plane is shown along the negative imaginary axis.

to divide the complex plane for $Z = k_z^2$ into the four quadrants shown in figure 1. Crucially, we note that there is a branch cut in the complex plane for $\sqrt{Z} = k_z$ and one cannot analytically continue the behaviour across this branch cut.

We will describe Z in each quadrant separately below. For the time being, we will consider that the imaginary part of Z (absorption/amplification) is small. The results for large absorption/amplification can always be obtained by analytic continuity because the scattering effects due to the imaginary parts can be neglected [14,15].

(i) $\mathrm{Re}(Z_1) > 0$, $\mathrm{Im}(Z_1) > 0$: This is the conventional case of a propagating wave in an absorbing medium. The wave decays in amplitude as it propagates in the medium. We have $0 < \mathrm{Arg}(Z_1) < \pi/2$ and $0 < \mathrm{Arg}(\sqrt{Z_1}) < \pi/4$.

(ii) $\mathrm{Re}(Z_4) > 0$, $\mathrm{Im}(Z_4) < 0$: This is the case of a propagating wave in an amplifying medium. The wave grows exponentially in amplitude as it propagates into the medium. We have $-\pi/2 < \mathrm{Arg}(Z_4) < 0$ and $-\pi/4 < \mathrm{Arg}(\sqrt{Z_4}) < 0$. Note that the phase for the square root rotates here in the opposite sense.

(iii) $\mathrm{Re}(Z_2) < 0$, $\mathrm{Im}(Z_2) > 0$: This is the case of evanescent wave in an absorbing medium. The decay of the wave is dominated by the evanescence (due to the large k_x), while $\mathrm{Im}(\epsilon)$ ensures absorption through a phase shift [16]. In this case, $\pi < \mathrm{Arg}(Z_2) < \pi/2$, and we choose $\pi/2 < \mathrm{Arg}(\sqrt{Z_2}) < \pi/4$. Then the Poynting vector points away from the source and decays to zero at infinity.

(iv) $\mathrm{Re}(Z_3) < 0$, $\mathrm{Im}(Z_3) < 0$: This is the final case of an evanescent wave in an amplifying medium and is not usually encountered. Clearly, the choice of an amplifying wave at a rate dominated by k_x is unphysical. The other choice for the square root would imply a decaying wave with the decay rate dominated by k_x and a phase change caused by $\mathrm{Im}\,\epsilon$, just as in case (iii). In this case, $\pi < \mathrm{Arg}(Z_3) < 3\pi/2$ and $\pi/2 < \mathrm{Arg}(\sqrt{Z_3}) < 3\pi/4$. But, $\mathrm{Im}(\epsilon)$ is negative and it results in a Poynting vector in the medium that points towards the source. This, however, does not violate causality as the Poynting

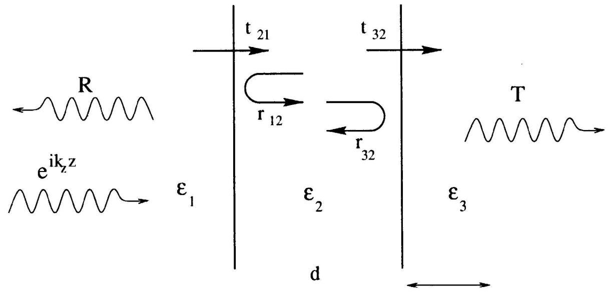

Figure 2. Schematic for transmission through a dielectric amplifying or absorbing slab. The notation for the partial refletion and transmission (Fresnel) coefficients are indicated.

vector/energy flow decays exponentially to zero at infinity and no information flows in from the infinities. This counter-intuitive behaviour does *not* imply that source has turned into a sink – rather it indicates that there would be a large (infinitely large for unsaturated linear gain) accumulation of energy density or intense local field enhancements near a source. Our choice of the wave-vector also ensures that the near-field of a source cannot be probed at infinities merely by making the intervening medium amplifying.

This can also be understood in terms of the fundamental bosonic property of light – photons in the localized mode stimulate the amplifying medium to emit more photons into the same localized mode. In other words, the near-field modes of a source remain evanescent even inside an amplifying medium and do not affect the far-field. In summary, the branch-cut in the complex plane for the square root has to be chosen along the negative imaginary axis. Hence the results for a propagating wave in an amplifying medium cannot be analytically continued across for an evanescent wave in an amplifying medium.

Before we move on to discuss the phenomenon in random media, let us examine the transmission through a homogeneous slab of absorbing or amplifying media, shown in figure 2. The transmission through the slab can be written in terms of a series for multiple scattering in between the interfaces:

$$\begin{aligned} T &= t_{21}t_{32}\mathrm{e}^{ik_z d} + t_{21}r_{32}r_{12}t_{32}\mathrm{e}^{3ik_z d} + t_{21}r_{32}r_{12}r_{32}r_{12}t_{32}\mathrm{e}^{5ik_z d} + \cdots \\ &= \frac{t_{21}t_{32}\mathrm{e}^{ik_z d}}{1 - r_{32}r_{12}\mathrm{e}^{2ik_z d}}, \end{aligned} \tag{5}$$

where t_{jk} and r_{jk} are the partial transmission and reflection Fresnel coefficients for light (obtained by matching the tangential components of E and H) across the interface between media-(j) and (k) where $j, k = 1, 2, 3$ (see figure 2) and k_z is the complex wave-vector in medium-2. For an incident propagating wave, from our discussion above, we note that $\mathrm{Im}(k_z) > 0$ for an absorbing slab and $\mathrm{Im}(k_z) < 0$ for an amplifying slab. For an absorbing slab, the transmission is seen to decay exponentially at all length scales. For a slab of an amplifying medium, although the transmission increases exponentially for small slab thickness, it begins to decay exponentially for larger thicknesses when $|r_{32}r_{12}\mathrm{e}^{2ik_z d}| > 1$. The exponentially growing term in the numerator, $e^{ik_z d}$, is overwhelmed by the exponential in the denominator, $e^{2ik_z d}$. Thus there is a critical length (l_c) above which the transmission always decreases. This symmetrical behaviour of the absorbing and amplifying slab was pointed out in refs [11,12].

The geometric series in eq. (5) for the transmission converges uniformly so long as $|r_{32}r_{12}e^{2ik_z d}| < 1$. For $|r_{32}r_{12}e^{2ik_z d}| > 1$, provided that the sum is carried out to include all infinite terms, the sum is exact and the correctness of the answer survives the formal divergence of the series, a result that can be understood through arguments of analytic continuation [17]. A similar divergence of the multiple scattering series arises in the recent context of the perfect lens and evanescent waves [18]. Only for certain resonant conditions, i.e., when $\mathrm{Arg}(r_{32}r_{12}e^{2ik_z d}) = 2n\pi$, does the sum of the series pass through its poles and these correspond to the laser oscillations of the system. In all other cases the wave does not couple to the eigenmodes of the slab and for even infinitesimal amounts of absorption or amplification the wave decays with distance inside the slab.

Let us now examine a random amplifying medium. We will assume disorder in the real part of ϵ while assuming the amplification/absorbing to be spatially uniform, as is the case with most other studies. The case of spatially varying absorption/amplification is dealt with in [15]. The presence of disorder localizes some of the modes in the system. For large enough disorder and large slab thickness, all the modes can be localized, at least in the lower (1 and 2) dimensions. Localized modes are characterized by an exponentially decaying envelope with an asymptotic form $\sim f(\vec{r})\exp(-r/\xi_l)$, where ξ_l is the localization length. As we have argued before, the presence of small absorption or amplification will not change this decaying nature of the mode. It will, however, influence whether the wave dissipates into the medium if the medium is absorptive, or grows in amplitude (local field enhancements) by drawing energy from the medium if it is amplifying.

For a wave incident from vacuum on to a random medium, the average for the transmission can be shown to decay exponentially for large sample length $L \gg \xi_l$ as [19]

$$\langle \ln T \rangle = (-1/\xi_l)L, \tag{6}$$

where the average is over all the disorder configurations. Our above arguments suggest that in the localized regime we should not expect the exponential decay of the transmission with length of the sample to change and become exponential growth even if the medium is amplifying. Paasschens *et al* [9] mathematically derive this behaviour using the relation $S(\epsilon'')S^{\dagger}(-\epsilon'') = 1$ for the scattering matrix. This conclusion is also substantiated by the recent calculations of Vanneste and Sebbah [20] that the presence of amplification does not alter the mode structure of the localized modes in the random medium for weak amplification. For larger amplification levels, we expect that the scattering due to the mismatch in the imaginary parts of the dielectric constant will also contribute to localization [15]. Then, the rate of the exponential decay would be different for an active random medium and has been shown to be [9,12]

$$\langle \ln T \rangle = (-1/l_a - 1/\xi_l)L, \tag{7}$$

where l_a is the absorption or the amplification length and is independent of the sign of $\mathrm{Im}(\epsilon)$. Our analysis also suggests that there would be large local field enhancements for the localized modes, which will only be limited by gain saturation in the medium. This behaviour is also found by Jiang and Soukoulis when they used a rate equation model that involves gain saturation in their numerical simulations [21]. For sample length smaller than a localization length $L \ll \xi_1$, when the modes are not localized, the transmission through the sample will increase exponentially with L. This is the diffusive regime of the random laser.

Overall, the transmission through a RAM has a non-monotonic behaviour with respect to the length of the sample. It initially would increase with length up to about a localization length and decay exponentially beyond. This is also found in the simulations of Paaschens *et al* [9] and Joshi and Jayannavar [10]. Finally we should also note that there is a possibility of extended modes for certain disorder configurations, the so-called Azbel resonances [22]. These would give rise to large fluctuations in the transmission for even a passive medium, and particularly so for a random amplifying medium. In fact, these fluctuations could be extremal [23], in that the transmission through a channel would be exponentially small or exponentially large in such random amplifying media. This leads to the result that all transmission moments through the system diverge [12] due to the non-self averaging nature.

A form of the symmetry between absorption and amplification also manifests in the equations for the reflection obtained by the method of invariant imbedding [19,24]. For a one-channel one-dimensional conductor of length L the equation for the reflection is

$$\frac{\mathrm{d}R(L)}{\mathrm{d}L} = 2ikR(L) + \frac{ik}{2}\eta(L)[1+R(L)]^2, \tag{8}$$

where $\eta(L) = \sqrt{\epsilon\mu}$ is the refractive index. It is immediately seen that the equation is invariant under the transformation $R(L) \to 1/R^*(L)$ and $\eta(L) \to \eta^*(L)$ [25]. Thus, the reflection for amplifying systems can be deduced from the reflectivity for the corresponding absorbing system with the same magnitude of $\mathrm{Im}(\eta)$. This symmetry holds for the multi-channel case also.

We conclude that the dual symmetry between absorption and amplification is correct in that the transmission across a slab always decays at length scales larger than the localization length. This is mandated by the analytic continuity for the complex wave-vector due to which localized modes remain localized regardless of absorption or amplification. For homogeneous or weakly scattering media also, the dual symmetry holds true, but at length scales greater than a critical length determined by the amplification factor.

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