PRAMANA
— journal of
physics

Vol. 62, No. 6
June 2004
pp. 1241–1254

Neutrino propagation in a weakly magnetized medium

SUSHAN KONAR[1] and SUBINOY DAS[2]
[1]Department of Physics and Meteorology, Indian Institute of Technology, Kharagpur 721 302, India
[2]Department of Physics, Columbia University, New York 10027, USA
E-mail: sushan@phy.iitkgp.ernet.in

MS received 9 December 2003; revised 23 March 2004; accepted 1 April 2004

Abstract. Neutrino–photon processes, forbidden in vacuum, can take place in the presence of a thermal medium and/or an external electro-magnetic field, mediated by the corresponding charged leptons (real or virtual). Such interactions affect the propagation of neutrinos through a magnetized plasma. We investigate the neutrino–photon absorptive processes, at the one-loop level, for massless neutrinos in a weakly magnetized plasma. We find that there is no correction to the absorptive part of the axial-vector–vector amplitude due to the presence of a magnetic field, to the linear order in the field strength.

Keywords. Neutrino propagation; medium; magnetic field.

PACS Nos 13.15.+g; 12.20.-m; 12.20.Ds

1. Introduction

Neutrino–photon reactions, forbidden (or highly suppressed) in vacuum, for example the plasmon decay ($\gamma \to \nu\nu$) or the Cerenkov process ($\nu \to \nu\gamma$) and the cross-processes, can be important in regions with very dense plasma and/or large-scale external magnetic fields such as encountered in the cosmological or the astrophysical context [1]. In the standard model, these ν–γ processes, appearing at the one-loop level, do not occur in vacuum because they are kinematically forbidden and also because the neutrinos do not couple to the photons at the tree-level. In the presence of a medium or a magnetic field, it is the charged particles (real plasma particles or virtual particles excited by an external field) running in the loop which, when integrated out, confer their electro-magnetic properties to the neutrino [2–4]. These processes also become kinematically allowed since the photon dispersion relation is modified in the presence of a medium and/or an external magnetic field opening up the phase space for such reactions to take place [3,5–9]. A thermal medium or/and an external magnetic field, thus, fulfils the dual purpose of inducing an effective neutrino–photon vertex and modifying the photon dispersion relation (see [4] and references therein for a detailed review).

The enhancement of ν–γ interactions by magnetic fields in an effective Lagrangian framework has been discussed in ref. [10]. It has also been shown that the ν–γ interaction in the presence of a thermal medium induces a small effective charge to the neutrino and that the neutrino electro-magnetic vertex is related to the photon self-energy in the medium [11,12]. Recently, this effective charge has been calculated, considering not only a thermal medium but also an external magnetic field, for neutrinos coupled to dynamical photons having $q_0 = 0$ and $|\vec{q}| \rightarrow 0$ [13]. In the weak field limit this effective charge acquired by a neutrino is, in fact, proportional to the field strength and also depends on the *direction* of the neutrino propagation with respect to the direction of the magnetic field.

Evidently, these processes modify the neutrino propagation through a magnetized medium. In order to accurately estimate the neutrino fluxes coming from regions pervaded by dense plasma and strong magnetic fields (various astrophysical objects, for example) it is extremely important to study, in particular, the absorptive processes. In view of this, we study the absorptive part of the one-loop polarization tensor of the ν–γ interaction, in the present work. It should be noted here that for most astrophysical systems, where the ν–γ processes acquire importance by virtue of large plasma density or the presence of magnetic fields (supernovae, newly born neutron stars or late stages of stellar evolution) the field strength is almost always smaller than the QED critical field. Therefore, the weak field limit ($e\mathcal{B} < m_e^2$, i.e, $\mathcal{B} \leq 10^{13}$ G) is appropriate for most astrophysical situations.

We find that the absorptive part *vanishes to linear order in* $\mathcal{B}$. This is quite significant because it contradicts the naive expectation and implies that the first non-trivial correction would come from the terms containing higher powers of the field strength. More importantly, it signifies that the absorptive processes do not acquire any spatial anisotropy due to the presence of a weak magnetic field. This is quite interesting because the magnitude of the effective charge of the neutrinos (coming from the real part of the one-loop amplitudes) is dependent on the relative direction of the neutrino momentum with respect to the magnetic field for any non-zero field strength. But no such effect is seen for the absorptive part.

The organization of the paper is as follows: In §2 we discuss the basics of neutrino–photon effective action and the fermion propagators in a magnetized medium. Section 3 contains the details of the calculation of the one-loop diagram and in §4 we consider the weak-field limit. Finally, we conclude with a discussion on the possible implications of our result in §5.

2. Formalism

The off-shell electro-magnetic vertex function Γ_μ is defined in such a way that, for on-shell neutrinos, the $\nu\nu\gamma$ amplitude is given by

$$\mathcal{M} = -i\bar{u}(k')\Gamma_\mu u(k)A^\mu(q), \tag{1}$$

where q, k, k' are the momentum carried by the photon and the neutrinos respectively and $q = k - k'$. Here, $u(k)$ is the neutrino wave function and A^μ stands for the electro-magnetic vector potential. In general, Γ_μ would depend on k, q, the characteristics of the medium and the external electro-magnetic field. We shall, in this work, consider neutrino momenta that are small compared to the masses of the W and Z bosons allowing us to neglect the momentum dependence in the W and Z propagators. This is equivalent to lowest-order $G_{\rm F}$ calculations and is justified for low-energy neutrinos and low temperatures and weak fields compared to the Fermi scale. Since, in this work we focus our attention on the possible astrophysical applications (notably in the context of the supernovae, the neutron stars or the late stages of stellar evolution), the characteristic temperatures or magnetic fields defined by the densities in such systems are much larger than those actually observed. Therefore, for the low-energy neutrinos the four-fermion interaction is given by the following effective Lagrangian:

$$\mathcal{L}_{\rm eff} = \frac{1}{\sqrt{2}} G_{\rm F} \overline{\nu} \gamma^\mu (1-\gamma_5) \nu \bar{l}_\nu \gamma_\mu (g_V - g_A \gamma_5) l_\nu \,, \tag{2}$$

where ν and l_ν are the neutrino and the corresponding lepton field respectively. For electron neutrinos,

$$g_V = 1 - (1 - 4\sin^2\theta_W)/2, \tag{3}$$

$$g_A = 1 - 1/2, \tag{4}$$

where the first terms in g_V and g_A are the contributions from the W exchange diagram and the second one from the Z exchange diagram. Then the amplitude effectively reduces to that of a purely photonic case with one of the photons replaced by the neutrino current, as seen in figure 1. Therefore, Γ_ν is given by

$$\Gamma_\nu = -\frac{1}{\sqrt{2}e} G_{\rm F} \gamma^\mu (1-\gamma_5) \left(g_V \Pi_{\mu\nu} - g_A \Pi^5_{\mu\nu} \right), \tag{5}$$

where $\Pi^5_{\mu\nu}$ represents the axial-vector–vector coupling and $\Pi_{\mu\nu}$ is the polarization tensor arising from figure 2. We have analysed the Lorentz tensor structure of $\Pi_{\mu\nu}$, taking into account all the available symmetry, in an earlier work [14]. Subsequently, a similar analysis has been carried out in [15] too. Since the tensorial structure of $\Pi^5_{\mu\nu}$ would be similar to that of $\Pi_{\mu\nu}$, we do not repeat the analysis here. It should be mentioned that the tensorial structure of $\Pi^5_{\mu\nu}$, in this context, has been discussed in detail in [7,16].

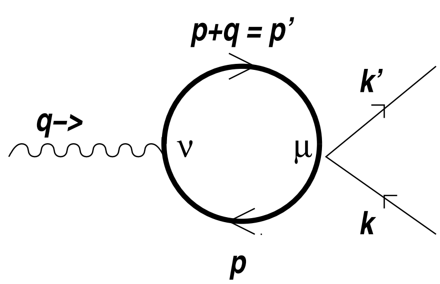

Figure 1. One-loop diagram for the effective electro-magnetic vertex of the neutrino in the limit of infinitely heavy W and Z masses.

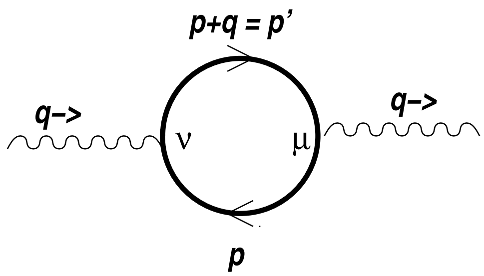

Figure 2. One-loop diagram for the vacuum polarization.

Because of the electro-magnetic current conservation, for the polarization tensor, we have the following gauge invariance condition:

$$q^{\mu}\Pi_{\mu\nu} = 0 = \Pi_{\mu\nu}q^{\nu}. \tag{6}$$

Same is true for the photon vertex of figure 1 and we have

$$\Pi^{5}_{\mu\nu}q^{\nu} = 0\,. \tag{7}$$

In an earlier paper [17] we have calculated the imaginary part of $\Pi^{\mu\nu}$ in a background medium in the presence of a uniform external magnetic field, in the weak-field limit, calculated at the one-loop level. We shall use the results of [17] here to obtain an expression for the total imaginary part of the effective neutrino current under equivalent conditions.

In order to calculate the absorptive processes in a thermal medium we use the real time formalism of the finite temperature field theory. The propagator acquires a matrix structure in this formalism and the off-diagonal elements provide the decay/production amplitudes. For the ease of calculation, we work with the 11-component of the propagator to find the imaginary part of the 11-component of the photon polarization tensor ($\Pi^{11}_{\mu\nu}$). This quantity, multiplied by appropriate factors, then gives the correct value of the imaginary part of the polarization tensor [18–21]. For notational brevity we shall suppress the 11-superscript for both the propagator and the polarization tensor in the rest of the paper. It should be mentioned here that we consider the effective photon–neutrino interaction coming from the imaginary part of the axial-vector–vector amplitude. Hence, like in the case of the polarization tensor, we work with the imaginary part of the 11-component of the axial-vector–vector amplitude.

The dominant contribution to $\Pi_{\mu\nu}$ and $\Pi^{5}_{\mu\nu}$ come from the electron lines in the loop. To evaluate this diagram we use the electron propagator within a thermal medium in the presence of a background electro-magnetic field. Rather than working with a completely general background field we specialize to the case of a constant (or slowly varying) magnetic field. Once this is assumed, the field can be taken in the z-direction without any further loss of generality. We denote the magnitude of this field by $\mathcal{B}$. Ignoring at first the presence of the medium, the electron propagator in such a field can be written down following Schwinger's approach [22–24]:

$$iS^{V}_{B}(p) = \int_{0}^{\infty} \mathrm{d}s\ \mathrm{e}^{\Phi(p,s)}C(p,s)\,, \tag{8}$$

where we have used the short-hands,

$$\Phi(p,s) \equiv is\left(p_\parallel^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s}\,p_\perp^2 - m^2\right) - \epsilon|s|, \tag{9}$$

$$C(p,s) \equiv \Big[(1 + i\sigma_z \tan e\mathcal{B}s)(\not{p}_\parallel + m) - (\sec^2 e\mathcal{B}s)\not{p}_\perp\Big] \tag{10}$$

and

$$\not{p}_\parallel = \gamma_0 p_0 - \gamma_3 p_3, \tag{11}$$

$$\not{p}_\perp = \gamma_1 p_1 + \gamma_2 p_2, \tag{12}$$

$$p_\parallel^2 = p_0^2 - p_3^2, \tag{13}$$

$$p_\perp^2 = p_1^2 + p_2^2. \tag{14}$$

Also, σ_z is given by

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5\,, \tag{15}$$

where the two forms are equivalent because of the definition of γ_5.

Of course in the range of integration indicated in eq. (8) s is never negative and hence $|s|$ equals s. It should be mentioned here that we follow the notation adopted in our previous papers [13,17] to ensure continuity. In the presence of a background medium, the above propagator is modified to [25]:

$$iS(p) = iS_B^V(p) - \eta_{\rm F}(p)\left[iS_B^V(p) - i\overline{S}_B^V(p)\right]\,, \tag{16}$$

where

$$\overline{S}_B^V(p) \equiv \gamma_0 S_B^{V\dagger}(p)\gamma_0 \tag{17}$$

for a fermion propagator and $\eta_{\rm F}(p)$ contains the distribution function for the fermions and the anti-fermions:

$$\eta_{\rm F}(p) = \Theta(p\cdot u) f_{\rm F}(p,\mu,\beta) + \Theta(-p\cdot u) f_F(-p,-\mu,\beta)\,. \tag{18}$$

Here, $f_{\rm F}$ denotes the Fermi–Dirac distribution function:

$$f_{\rm F}(p,\mu,\beta) = \frac{1}{{\rm e}^{\beta(p\cdot u-\mu)} + 1}\,, \tag{19}$$

and Θ is the step function. Rewriting eq. (16) in the form

$$iS(p) = iS_{\rm re} + iS_{\rm im} \tag{20}$$

we recognize

$$S_{\rm re} = \frac{1}{2}\left[S_B^V(p) + \overline{S}_B^V(p)\right], \tag{21}$$

$$S_{\rm im} = (1/2 - \eta_{\rm F}(p))\left[S_B^V(p) - \overline{S}_B^V(p)\right], \tag{22}$$

where the subscripts 're' and 'im' refer to the real and imaginary parts of the propagator. Using the form of $S_B^V(p)$ in eq. (8) we obtain the imaginary part to be

$$iS_{\rm im} = (1/2 - \eta_{\rm F}(p)) \int_{-\infty}^{\infty} {\rm d}s\; {\rm e}^{\Phi(p,s)} C(p,s), \tag{23}$$

with $\Phi(p,s)$ and $C(p,s)$ defined by eqs (9) and (10).

3. Calculation of the one-loop diagram

$\Pi_{\mu\nu}(q, \mathcal{B})$ *in odd powers of* $\mathcal{B}$ — The amplitude of the one-loop diagram of figure 2 can be written as

$$i\Pi_{\mu\nu}(q) = -\int \frac{{\rm d}^4 p}{(2\pi)^4} (ie)^2 \;{\rm tr}\; [\gamma_\mu \, iS(p) \gamma_\nu \, iS(p')] \;, \tag{24}$$

where, for the sake of notational simplicity, we have used

$$p' = p + q \,. \tag{25}$$

The minus sign on the right side is for a closed fermion loop and $S(p)$ is the propagator given by eq. (16). This implies that the absorptive part of the polarization tensor is given by

$$\Pi^{11}_{\mu\nu}(q) = -ie^2 \int \frac{{\rm d}^4 p}{(2\pi)^4} \;{\rm tr}\; [\gamma_\mu \, iS_{\rm im}(p) \gamma_\nu \, iS_{\rm im}(p')] \,, \tag{26}$$

and, the gauge invariant contribution to the absorptive part of the vacuum polarization tensor which is odd in $\mathcal{B}$ is given by [17]

$$\begin{aligned}\Pi_{\mu\nu}(q,\beta)^{\rm O} = &-4ie^2 \varepsilon_{\mu\nu\alpha_\parallel\beta} q^\beta \int \frac{{\rm d}^4 p}{(2\pi)^4} X(\beta,q,p) \int_{-\infty}^{\infty} {\rm d}s \; {\rm e}^{\Phi(p,s)} \int_{-\infty}^{\infty} {\rm d}s' \; {\rm e}^{\Phi(p',s')} \\ &\times \left[p^{\widetilde{\alpha}_\parallel} \tan e\mathcal{B}s + p'^{\widetilde{\alpha}_\parallel} \tan e\mathcal{B}s' - \frac{\tan e\mathcal{B}s \; \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')} \; (p+p')^{\widetilde{\alpha}_\parallel} \right], \end{aligned} \tag{27}$$

where we have defined

$$X(\beta, q, p) = (1/2 - \eta_{\rm F}(p))\,(1/2 - \eta_{\rm F}(p'))\,. \tag{28}$$

$\Pi^5_{\mu\nu}(k, \mathcal{B})$ *in odd powers of* $\mathcal{B}$ — The amplitude of the one-loop diagram of figure 1 can be written as

$$\Pi^5_{\mu\nu}(q) = -ie^2 \int \frac{{\rm d}^4 p}{(2\pi)^4} \;{\rm tr}\; [\gamma_\mu \, \gamma_5 \, iS(p) \gamma_\nu \, iS(p')] \;. \tag{29}$$

Using eq. (23) we find that the absorptive part of the polarization tensor is given by

$$\Pi^5_{\mu\nu}(q) = -ie^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, X(\beta,q,p) \int_{-\infty}^{\infty} \mathrm{d}s\; \mathrm{e}^{\Phi(p,s)} \int_{-\infty}^{\infty} \mathrm{d}s'\; \mathrm{e}^{\Phi(p',s')} \times \mathrm{tr}\left[\gamma_\mu\,\gamma_5\, C(p,s)\gamma_\nu\, C(p',s')\right]. \tag{30}$$

Notice that the phase factors appearing in eq. (30) are even in $\mathcal{B}$. Thus, we need consider only the odd terms from the traces. Performing the traces, the expression, odd in powers of $\mathcal{B}$, comes out to be

$$\Pi^5_{\mu\nu}(q)^{\mathrm{O}} = -4e^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, X(\beta,q,p) \int_{-\infty}^{\infty} \mathrm{d}s\; \mathrm{e}^{\Phi(p,s)} \times \int_{-\infty}^{\infty} \mathrm{d}s'\; \mathrm{e}^{\Phi(p',s')}\, R_{\mu\nu}, \tag{31}$$

where

$$\begin{aligned} R_{\mu\nu} &= \varepsilon_{\mu\nu 12}\, m^2 \left(\tan e\mathcal{B}s + \tan e\mathcal{B}s'\right) \\ &\quad + \left(g_{\mu\widetilde{\alpha}_\parallel}\, g_{\nu\beta_\parallel} + g_{\mu\beta_\parallel}\, g_{\nu\widetilde{\alpha}_\parallel}\right) p^{\alpha_\parallel}\, p^{\beta_\parallel} \left(\tan e\mathcal{B}s + \tan e\mathcal{B}s'\right) \\ &\quad - \left(g_{\mu\widetilde{\alpha}_\parallel}\, g_{\nu\beta_\perp} + g_{\mu\beta_\perp}\, g_{\nu\widetilde{\alpha}_\parallel}\right) p^{\alpha_\parallel}\, p'^{\beta_\perp}\, \tan e\mathcal{B}s\, \sec^2 e\mathcal{B}s' \\ &\quad - \left(g_{\mu\widetilde{\alpha}_\parallel}\, g_{\nu\beta_\perp} + g_{\mu\beta_\perp}\, g_{\nu\widetilde{\alpha}_\parallel}\right) p^{\beta_\perp}\, p'^{\alpha_\parallel}\, \tan e\mathcal{B}s'\, \sec^2 e\mathcal{B}s \\ &\quad + \left(g_{\mu\widetilde{\alpha}_\parallel}\, g_{\nu\beta_\parallel} + g_{\mu\beta_\parallel}\, g_{\nu\widetilde{\alpha}_\parallel} - g_{\mu\nu} g_{\widetilde{\alpha}_\parallel\beta_\parallel}\right) \left(p^{\alpha_\parallel}\, q^{\beta_\parallel}\, \tan e\mathcal{B}s + q^{\alpha_\parallel}\, p^{\beta_\parallel}\, \tan e\mathcal{B}s'\right). \end{aligned} \tag{32}$$

In writing this expression, we have used the notation $g_{\mu\widetilde{\alpha}_\parallel}$, for example. This signifies that $\widetilde{\alpha}_\parallel$ is an index which can take only the 'parallel' indices, i.e., 0 and 3, and is moreover different from the index α appearing elsewhere in the expression. Now, since we perform the calculations in the rest frame of the medium where $p\cdot u = p_0$ the distribution function does not depend on the spatial components of p. In the last two terms of eq. (32), the integral over the transverse components of p has the following generic structure:

$$\int \mathrm{d}^2p_\perp\; \mathrm{e}^{\Phi(p,s)}\mathrm{e}^{\Phi(p',s')}\left(p^{\beta_\perp} \text{ or } p'^{\beta_\perp}\right). \tag{33}$$

Notice that,

$$\frac{\partial}{\partial p_{\beta_\perp}}\left[\mathrm{e}^{\Phi(p,s)}\mathrm{e}^{\Phi(p',s')}\right] = \left(\tan e\mathcal{B}s\; p^{\beta_\perp} + \tan e\mathcal{B}s'\; p'^{\beta_\perp}\right)\frac{2i}{e\mathcal{B}}\mathrm{e}^{\Phi(p,s)}\mathrm{e}^{\Phi(p',s')}. \tag{34}$$

However, this expression, being a total derivative, should integrate to zero. Thus we obtain that

$$\tan e\mathcal{B}s\; p^{\beta_\perp} \stackrel{\circ}{=} -\tan e\mathcal{B}s'\; p'^{\beta_\perp}, \tag{35}$$

where the sign '$\stackrel{\circ}{=}$' means that the expressions on both sides of it, though not necessarily equal algebraically, yield the same integral. This gives

$$p^{\beta_\perp} \stackrel{\circ}{=} -\frac{\tan e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'}\, q^{\beta_\perp}\,,$$
$$p'^{\beta_\perp} \stackrel{\circ}{=} \frac{\tan e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'}\, q^{\beta_\perp}\,. \qquad (36)$$

Also, using the definition of the exponential factor $\Phi(p,s)$ from eq. (9), we notice that

$$m^2 \tan e\mathcal{B}s\; \mathrm{e}^{\Phi(p,s)}\, \mathrm{e}^{\Phi(p',s')}$$
$$= \tan e\mathcal{B}s \left\{ i\frac{\mathrm{d}}{\mathrm{d}s'} + (p'^2_\parallel - \sec^2 e\mathcal{B}s'\; p'^2_\perp) \right\} \mathrm{e}^{\Phi(p,s)}\, \mathrm{e}^{\Phi(p',s')}. \qquad (37)$$

Moreover, taking another derivative with respect to $p^{\alpha_\perp}$ of eq. (34) we obtain, from the fact that this derivative should also vanish on p integration,

$$p^2_\perp \stackrel{\circ}{=} \frac{1}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} \left[-ie\mathcal{B} + \frac{\tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'}\, q^2_\perp \right] \qquad (38)$$

and

$$p'^2_\perp \stackrel{\circ}{=} \frac{1}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} \left[-ie\mathcal{B} + \frac{\tan^2 e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'}\, q^2_\perp \right]. \qquad (39)$$

Therefore, incorporating eqs (37) and (38) in eq. (32) we finally have

$$R_{\mu\nu} = R_{\mu_\perp\nu} + R_{\mu_\parallel\nu}\,. \qquad (40)$$

In writing this we have defined

$$R_{\mu_\perp\nu} = g_{\mu_\perp\nu} g_{\widetilde{\alpha_\parallel}\beta_\parallel} (p^{\alpha_\parallel} q^{\beta_\parallel} \tan e\mathcal{B}s + q^{\alpha_\parallel} p^{\beta_\parallel} \tan e\mathcal{B}s')$$
$$- g_{\mu\beta_\perp}\, g_{\nu\widetilde{\alpha_\parallel}}\, q^{\beta_\perp}\, p^{\alpha_\parallel} (\tan e\mathcal{B}s - \tan e\mathcal{B}s')$$
$$+ g_{\mu\beta_\perp}\, g_{\nu\widetilde{\alpha_\parallel}}\, q^{\beta_\perp}\, q^{\alpha_\parallel} \frac{\tan^2 e\mathcal{B}s' \sec^2 e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'}\,. \qquad (41)$$

It is evident that $R_{\mu_\parallel\nu}\, q^\nu = 0$ in accordance with eq. (7). On the other hand, we have

$$R_{\mu_\parallel\nu} = R^{\mathrm{A}}_{\mu_\parallel\nu} + R^{\mathrm{B}}_{\mu_\parallel\nu}\,, \qquad (42)$$

such that

$$R^{\mathrm{A}}_{\mu_\parallel\nu} = \varepsilon_{\mu\nu12} (\tan e\mathcal{B}s + \tan e\mathcal{B}s') p^2_\parallel$$
$$+ (g_{\mu\widetilde{\alpha_\parallel}}\, g_{\nu\beta_\parallel} + g_{\mu\beta_\parallel}\, g_{\nu\widetilde{\alpha_\parallel}})\, p^{\alpha_\parallel} p^{\beta_\parallel} (\tan e\mathcal{B}s + \tan e\mathcal{B}s')$$
$$+ g_{\mu\widetilde{\alpha_\parallel}}\, g_{\nu\beta_\parallel}\, p^{\alpha_\parallel} q^{\beta_\parallel} \tan e\mathcal{B}s + (g_{\mu\widetilde{\alpha_\parallel}}\, g_{\nu\beta_\parallel} - g_{\mu_\parallel\nu} g_{\widetilde{\alpha_\parallel}\beta_\parallel})\, q^{\alpha_\parallel} p^{\beta_\parallel} \tan e\mathcal{B}s'$$
$$- g_{\mu\widetilde{\alpha_\parallel}}\, g_{\nu\beta_\perp}\, q^{\beta_\perp} p^{\alpha_\parallel} (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \qquad (43)$$

and

$$R^{\mathrm{B}}_{\mu_\parallel \nu} = \left(g_{\mu\beta_\parallel}\, g_{\nu\widetilde{\alpha}_\parallel} - g_{\mu_\parallel \nu} g_{\widetilde{\alpha}_\parallel \beta_\parallel} \right) p^{\alpha_\parallel}\, q^{\beta_\parallel}\ \tan e\mathcal{B}s + g_{\mu\beta_\parallel}\, g_{\nu\widetilde{\alpha}_\parallel}\, q^{\alpha_\parallel}\, p^{\beta_\parallel}\ \tan e\mathcal{B}s'$$
$$+ g_{\mu\widetilde{\alpha}_\parallel}\, g_{\nu\beta_\perp}\, q^{\beta_\perp}\, q^{\alpha_\parallel}\, \frac{\tan^2 e\mathcal{B}s' \sec^2 e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} - \varepsilon_{\mu\nu 12}\, \frac{\sec^2 e\mathcal{B}s \tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} q_\perp^2\, . \tag{44}$$

Again, it is obvious that $R^{\mathrm{B}}_{\mu_\parallel \nu} q^\nu = 0$. In its present form, $R^{\mathrm{B}}_{\mu_\parallel \nu}$ does not render itself to obvious gauge-invariance. However, the theory dictates that $R^{\mathrm{A}}_{\mu_\parallel \nu} q^\nu$ should vanish and we have shown in Appendix A that it is indeed so. Therefore, the complete gauge-invariant expression for $R_{\mu\nu}$ is given by

$$\begin{aligned}
R_{\mu\nu} = {} & \varepsilon_{\mu\nu 12}\, (\tan e\mathcal{B}s + \tan e\mathcal{B}s') p_\parallel^2 \\
& + \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\parallel} + g_{\mu\beta_\parallel}\, \varepsilon_{\nu\alpha_\parallel 12} \right) p^{\alpha_\parallel}\, p^{\beta_\parallel}\, (\tan e\mathcal{B}s + \tan e\mathcal{B}s') \\
& + \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\parallel} + g_{\mu\beta_\parallel}\, \varepsilon_{\nu\alpha_\parallel 12} - g_{\mu\nu}\, \varepsilon_{\alpha_\parallel \beta_\parallel 12} \right) \\
& \times \left(p^{\alpha_\parallel}\, q^{\beta_\parallel}\ \tan e\mathcal{B}s + q^{\alpha_\parallel}\, p^{\beta_\parallel}\ \tan e\mathcal{B}s' \right) \\
& - \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\perp} + g_{\mu\beta_\perp}\, \varepsilon_{\nu\alpha_\parallel 12} \right) q^{\beta_\perp}\, p^{\alpha_\parallel}\, (\tan e\mathcal{B}s - \tan e\mathcal{B}s') \\
& + \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\perp} + g_{\mu\beta_\perp}\, \varepsilon_{\nu\alpha_\parallel 12} \right) q^{\beta_\perp}\, q^{\alpha_\parallel}\, \frac{\tan^2 e\mathcal{B}s' \sec^2 e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} \\
& - \varepsilon_{\mu\nu 12}\, \frac{\sec^2 e\mathcal{B}s \tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} q_\perp^2\, ,
\end{aligned} \tag{45}$$

where we have used the identity

$$g_{\mu\widetilde{\alpha}_\parallel}\, a^{\alpha_\parallel} = \varepsilon_{\mu\alpha_\parallel 12}\, a^{\alpha_\parallel}\, , \tag{46}$$

valid for any vector a^α.

4. The weak field limit

Retaining terms up to $O(\mathcal{B})$ in eq. (45) we have

$$\begin{aligned}
\Pi^5_{\mu\nu}(q)(O(\mathcal{B})) = {} & -4e^3\mathcal{B} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ X(\beta, q, p) \int_{-\infty}^{\infty} \mathrm{d}s\ \mathrm{e}^{\Phi(p,s)} \int_{-\infty}^{\infty} \mathrm{d}s'\ \mathrm{e}^{\Phi(p',s')} \\
& \times \Bigg[\varepsilon_{\mu\nu 12}\, (s+s') p_\parallel^2 + \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\parallel} + g_{\mu\beta_\parallel}\, \varepsilon_{\nu\alpha_\parallel 12} \right) p^{\alpha_\parallel}\, p^{\beta_\parallel} (s+s') \\
& + \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\parallel} + g_{\mu\beta_\parallel}\, \varepsilon_{\nu\alpha_\parallel 12} - g_{\mu\nu}\, \varepsilon_{\alpha_\parallel \beta_\parallel 12} \right) \left(p^{\alpha_\parallel}\, q^{\beta_\parallel}\, s + q^{\alpha_\parallel}\, p^{\beta_\parallel}\, s' \right) \\
& - \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\perp} + g_{\mu\beta_\perp}\, \varepsilon_{\nu\alpha_\parallel 12} \right) q^{\beta_\perp}\, p^{\alpha_\parallel}\, (s - s') \\
& + \left(\varepsilon_{\mu\alpha_\parallel 12}\, g_{\nu\beta_\perp} + g_{\mu\beta_\perp}\, \varepsilon_{\nu\alpha_\parallel 12} \right) q^{\beta_\perp}\, q^{\alpha_\parallel}\, \frac{{s'}^2}{s+s'} - \varepsilon_{\mu\nu 12}\, q_\perp^2\, \frac{{s'}^2}{s+s'} \Bigg] .
\end{aligned} \tag{47}$$

This entire expression *vanishes* upon integration, as has been shown in Appendix B. Therefore, to $O(\mathcal{B})$ the electro-magnetic vertex function is simply

$$\Gamma_\nu = -\frac{1}{\sqrt{2}e} G_{\rm F} \gamma^\mu (1-\gamma_5)\, g_V \Pi_{\mu\nu}(O(\mathcal{B}))\,, \tag{48}$$

where $\Pi_{\mu\nu}(O(\mathcal{B}))$ is given by

$$\Pi_{\mu\nu}(O(\mathcal{B})) = -\,4ie^3 \mathcal{B} \varepsilon_{\mu\nu\alpha_\parallel\beta} q^\beta \int \frac{{\rm d}^4 p}{(2\pi)^4}\, X(\beta,q,p) \int_{-\infty}^{\infty} {\rm d}s\; {\rm e}^{\Phi(p,s)} \times \int_{-\infty}^{\infty} {\rm d}s'\; {\rm e}^{\Phi(p',s')} p^{\widetilde{\alpha}_\parallel} \left(s + s' - \frac{2ss'}{s+s'} \right). \tag{49}$$

For a detailed discussion on the properties of $\Pi_{\mu\nu}(O(\mathcal{B}))$ in various background media, see [17].

In this context, it would be worthwhile to compare our results with the case of a non-magnetic thermal plasma. Following the formulation in §2 we find that the absorptive part of the one-loop polarization tensor (the axial-vector–vector interaction) for a non-magnetic thermal plasma is given by eq. (30) where

$$iS_{\rm im} = (1/2 - \eta_{\rm F}(p)) \int_{-\infty}^{\infty} {\rm d}s\; {\rm e}^{\Phi_0(p,s)} C_0(p,s) \tag{50}$$

with

$$\Phi_0(p,s) = is\left(p^2 - m^2\right) - \epsilon|s|, \tag{51}$$

$$C_0(p,s) = \not{p} + m\,. \tag{52}$$

With the above definitions, we find that

$$\Pi^5_{\mu\nu}(q,\mathcal{B}=0) = -16i\pi^2\, e^2\, \varepsilon_{\mu\nu\alpha\beta}\, q^\beta \times \int \frac{{\rm d}^4 p}{(2\pi)^4}\, p^\alpha\, X(\beta,q,p)\, \delta(p^2-m^2)\, \delta(p'^2 - m^2)\,. \tag{53}$$

Therefore, the absorptive part of the polarization tensor, for a non-magnetic thermal plasma, is given by eq. (53). Evidently, the same is true for a weakly magnetized plasma, to linear order in the strength of the magnetic field.

5. Conclusion

In this work, we have considered massless neutrinos. However, recent observations indicate that the neutrinos may have mass. Nevertheless, our present treatment can be modified for massive neutrinos following the method adopted in [11].

It is important to note that the correction to the absorptive part of the axial-vector–vector amplitude due to the presence of a magnetic field is zero to the linear order in the field strength compared to the case of a non-magnetic thermal plasma. Therefore, it indicates that the absorptive part of the axial-vector–vector amplitude is not enhanced, to the first order, by the presence of a magnetic field contrary to a naive expectation. It should be emphasized that this result is valid for the case of a weak magnetic field. However, as has been mentioned before, the magnetic field in most astrophysical systems are well below the critical field value. Therefore, our result would be pertinent to such situations.

It also needs to be emphasized that unlike in the case of the real part (which gives the effective charge of the neutrinos) where a magnetic field breaks the isotropy of space there is no such introduction of a preferential direction in the case of absorption. Therefore, even though the effective charge of the neutrinos picks out the direction of the external magnetic field, to $O(\mathcal{B})$ the absorption processes do not see any direction dependence.

Appendix A: Proof of gauge invariance

In order to show the gauge invariance of $\Pi^5{\mu_\|}\nu$ let us consider the case of $\mu_\| = 3$, first. Then we shall have

$$\Pi^5_{3\nu}(q)^{\mathrm{A}}\, q^\nu = -4e^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, X(\beta, q, p) \int_{-\infty}^{\infty} \mathrm{d}s\, \mathrm{e}^{\Phi(p,s)} \int_{-\infty}^{\infty} \mathrm{d}s'\, \mathrm{e}^{\Phi(p',s')}$$
$$\times [p^0\, (q_\|^2 + 2p\cdot q)\, (\tan e\mathcal{B}s + \tan e\mathcal{B}s') - p^0\, q_\perp^2\, (\tan e\mathcal{B}s - \tan e\mathcal{B}s')\,]. \tag{A1}$$

Now, from the definition of Φ, it follows that, apart from the small convergence factors,

$$\frac{i}{e\mathcal{B}}\, (\Phi(p,s) + \Phi(p',s')) = \left(p'^2_\| + p_\|^2 - 2m^2\right)\xi - \left(p'^2_\| - p_\|^2\right)\zeta$$
$$- p'^2_\perp \tan(\xi - \zeta) - p_\perp^2 \tan(\xi + \zeta)\,, \tag{A2}$$

where we have defined the parameters

$$\xi = \frac{1}{2} e\mathcal{B}(s + s')\,,$$
$$\zeta = \frac{1}{2} e\mathcal{B}(s - s'). \tag{A3}$$

Thus,

$$\left(p'^2_\| - p_\|^2\right) \mathrm{e}^{\Phi(p,s)+\Phi(p',s')} = \left(ie\mathcal{B}\frac{\mathrm{d}}{\mathrm{d}\zeta} + p'^2_\perp \sec^2(\xi - \zeta) - p_\perp^2 \sec^2(\xi + \zeta)\right)$$
$$\times \mathrm{e}^{\Phi(p,s)+\Phi(p',s')}. \tag{A4}$$

It should be noted that $q_\|^2 + 2q\cdot p_\| = {p'}^2_\| - p_\|^2$. Hence,

$$\Pi^5_{3\nu}(q)^{\rm A}\, q^\nu = -4e^2 \int \frac{{\rm d}^4p}{(2\pi)^4}\; X(\beta,k,p)\, p^0 \int_{-\infty}^{\infty} {\rm d}s \int_{-\infty}^{\infty} {\rm d}s' \times\Bigg\{ (\tan e\mathcal{B}s + \tan e\mathcal{B}s')\left(ie\mathcal{B}\frac{\rm d}{{\rm d}\zeta} + p'^2_\perp \sec^2(\xi-\zeta) - p^2_\perp \sec^2(\xi+\zeta)\right) - k^2_\perp\,(\tan e\mathcal{B}s - \tan e\mathcal{B}s')\Bigg\} {\rm e}^{\Phi(p,s)}\,{\rm e}^{\Phi(p',s')}\,. \tag{A5}$$

Using eqs (38) and (39) this can be further modified to

$$\Pi^5_{3\nu}(q)^{\rm A}\, q^\nu = -\,4e^2 \int \frac{{\rm d}^4p}{(2\pi)^4}\; X(\beta,k,p)\, p^0 \int_{-\infty}^{\infty} {\rm d}s \int_{-\infty}^{\infty} {\rm d}s' \times(\tan e\mathcal{B}s + \tan e\mathcal{B}s')\left(\frac{\rm d}{{\rm d}\zeta} + \sec^2 e\mathcal{B}s - \sec^2 e\mathcal{B}s'\right){\rm e}^{\Phi(p,s)}{\rm e}^{\Phi(p',s')} -4e^2 q^2_\perp \int \frac{{\rm d}^4p}{(2\pi)^4}\; X(\beta,k,p)\, p^0 \int_{-\infty}^{\infty} {\rm d}s\; {\rm e}^{\Phi(p,s)} \int_{-\infty}^{\infty} {\rm d}s'\; {\rm e}^{\Phi(p',s')} \left[\frac{\sec^2 e\mathcal{B}s'\, \tan^2 e\mathcal{B}s}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} - \frac{\sec^2 e\mathcal{B}s\, \tan^2 e\mathcal{B}s'}{\tan e\mathcal{B}s + \tan e\mathcal{B}s'} - (\tan e\mathcal{B}s - \tan e\mathcal{B}s')\right] \tag{A6}$$

which vanishes identically. Hence, $\Pi^5_{3\nu}(q)^{\rm A}$ satisfies eq. (7). The gauge invariance for $\Pi^5_{0\nu}(q)^{\rm A}$ can be shown in a similar fashion.

Appendix B: Evaluation of the integrals

From the definition of Φ it follows that

$$\Phi(p,s) + \Phi(p',s') = it\,(p^2 + p\cdot q + q^2/2 - m^2 - i\epsilon_1) + it'\,(p\cdot q + q^2/2 - i\epsilon_2)\,, \tag{B1}$$

where $t = s + s'$ and $t' = s' - s$. Using this, eq. (47) can be rewritten in the following form:

$$\Pi^5_{\mu\nu}(q) = -4e^3\mathcal{B} \int \frac{{\rm d}^4p}{(2\pi)^4}\; X(\beta,q,p) \times \int_{-\infty}^{\infty} {\rm d}t \int_{-\infty}^{\infty} {\rm d}t' \left(A\,t + B\,t' + C\,\frac{t'^2}{t}\right) {\rm e}^{\Phi(p,s)}{\rm e}^{\Phi(p',s')}, \tag{B2}$$

where A, B, C are functions of p and q. Now,

$$\int_{-\infty}^{\infty} {\rm d}t\, t\, {\rm e}^{it(p^2+p\cdot q+q^2/2-m^2-i\epsilon_1)} = 4\delta(p^2 + p\cdot q + q^2/2 - m^2)\,(p^2 + p\cdot q + q^2/2 - m^2) \times((p^2 + p\cdot q + q^2/2 - m^2)^2 + \epsilon_1^2)^{-1}, \tag{B3}$$

where we have used the identity:

$$\frac{1}{a \pm i\epsilon} = \mathcal{P}(a) \mp i\pi\delta(a)\,, \tag{B4}$$

$\mathcal{P}$ being the principal value and the integral:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x\, \mathrm{e}^{i(a-ib)x} = \frac{4iab}{(a^2+b^2)^2}\,, \tag{B5}$$

for $\mathrm{Re}(b) > |\mathrm{Im}(a)|$ [26]. Therefore,

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t'\, t\, \mathrm{e}^{\Phi(p,s)}\, \mathrm{e}^{\Phi(p',s')} \\ &= \delta(p^2 + p\cdot q + q^2/2 - m^2)\, \frac{2(p^2 + p\cdot q + q^2/2 - m^2)}{(p^2 + p\cdot q + q^2/2 - m^2)^2 + (\epsilon_1)^2} \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}t'\, \mathrm{e}^{it'(p\cdot q + q^2/2 - i\epsilon_2)}. \end{aligned} \tag{B6}$$

Since the numerator and the argument of the delta function are the same, this integral vanishes upon p-integration, provided we take the limit $\epsilon_1 \to 0^+$ later. It could be similarly argued that the second term in eq. (B2) vanishes upon p-integration. In case of the third term, an integration by parts for the t'-integral renders it to the form of eq. (B5) and the above argument then can be followed through to show that this also vanishes upon p-integration.

Acknowledgments

The authors would like to thank Palash B Pal for helpful discussions and Kaushik Bhattacharya for pointing out some mistakes in the manuscript. In addition, they thank IUCAA, Pune, where SK was a post-doctoral fellow and IIT, Kanpur, where SD was an undergraduate student at the time of preparation of this manuscript.

References

[1] G G Raffelt, *Stars as laboratories for fundamental physics* (University of Chicago Press, 1996)
[2] L L DeRaad, K A Milton and N D Hari-Dass, *Phys. Rev.* **D14**, 3326 (1976)
[3] J C Olivo, J F Nieves and P B Pal, *Phys. Rev.* **D40**, 3679 (1989)
[4] A N Ioannisian and G G Raffelt, *Phys. Rev.* **D55**, 7038 (1997)
[5] S J Hardy and D B Melrose, *Publ. Astron. Soc. Aus.* **13**, 144 (1996)
[6] V N Oraevsky, V B Semikoz and Y A Smorodinsky, *J. Exp. Theor. Phys. Lett.* **43**, 709 (1986)
[7] R Shaisultanov, *Phys. Rev.* **D62**, 113005 (2000)
[8] H Gies and R Shaisultanov, *Phys. Rev.* **D63**, 73003 (2000)
[9] H Gies and R Shaisultanov, *Phys. Lett.* **B480**, 129 (2000)

[10] R Shaisultanov, *Phys. Rev. Lett.* **80**, 1586 (1998)
[11] J F Nieves and P B Pal, *Phys. Rev.* **D49**, 1398 (1994)
[12] T Altherr and P Salati, *Nucl. Phys.* **B421**, 662 (1994)
[13] K Bhattacharya, A K Ganguly and S Konar, *Phys. Rev.* **D65**, 013007 (2002)
[14] A K Ganguly, S Konar and P B Pal, *Phys. Rev.* **D60**, 105014 (1999)
[15] J C D'Olivo, J F Nieves and S Sahu, *Phys. Rev.* **D67**, 025018 (2003)
[16] C Schubert, *Nucl. Phys.* **B585**, 429 (2000)
[17] A K Ganguly and S Konar, *Phys. Rev.* **D63**, 065001 (2001)
[18] A Das, *Finite temperature field theory* (World Scientific, Singapore, 1997)
[19] R Kobes, *Phys. Rev.* **D42**, 562 (1990)
[20] J F Nieves, *Phys. Rev.* **D42**, 4123 (1990)
[21] N P Landsman and C G Weert, *Phys. Rep.* **145**, 141 (1987)
[22] J Schwinger, *Phys. Rev.* **82**, 664 (1951)
[23] W Y Tsai, *Phys. Rev.* **D10**, 1342, 2699 (1974)
[24] W Dittrich, *Phys. Rev.* **D19**, 2385 (1979)
[25] P Elmfors, D Grasso and G Raffelt, *Nucl. Phys.* **B479**, 3 (1996)
[26] I S Gradshteyn and I M Ryzhik, *Table of integrals, series and products* (Academic Press, London, 2000)