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Conformally flat tilted Bianchi Type-V cosmological models in general relativity

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Abstract. We have investigated two conformally flat tilted Bianchi Type-V cosmological models in general relativity. To get a determinate solution, we have assumed a supplementary condition $A = B^n$ between metric potentials where n is a constant. The behaviour of the model for $n = 2$ is discussed in detail. Various physical and geometrical aspects of the models are also discussed.

Keywords. Conformally flat; tilted; Bianchi V cosmological.

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1. Introduction

The study of Bianchi Type-V cosmological models create more interest as these models contain isotropic special cases and allow arbitrarily small anisotropy levels at any instant of cosmic time. This property makes them suitable as model of our universe. Also Bianchi Type-V models are more complicated than the simplest Bianchi type models (e.g. the Einstein tensor has off-diagonal terms so that it is more natural to include tilt and heat conduction) while at the same time, these are simple generalization of the negative curvature of FRW models. The study of Bianchi Type-V models is more actively pursued, as in these models, the Einstein field equation becomes more tractable [1–15]. In all these studies, it may be observed that matter distribution has always been characterized by either a perfect fluid or a perfect fluid with electromagnetic field. Roy and Prasad [16] have investigated some LRS Bianchi Type-V cosmological models of local embedding class one filled with perfect fluid with heat conduction and radiation. Bianchi Type-V cosmological models have also been investigated by Farnsworth [17], Maartens and Nel [18], Wainwright *et al* [19], Collins [20]. Conformally flat non-static Bianchi Type-I spherically symmetric and plane symmetric cosmological models have been investigated by Roy and Bali [21,22].

In this paper, we have investigated two conformally flat tilted Bianchi Type-V cosmological models in general relativity. To get a determinate solution, we have also assumed a supplementary condition $A = B^n$ between metric potentials. As a special case, the model for $n = 2$, is also discussed. The various physical and geometrical aspects of the models are also discussed.

2. The field equations

We consider the Bianchi Type-V metric in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2\mathrm{d}x^2 + B^2\mathrm{e}^{2x}\mathrm{d}y^2 + C^2\mathrm{e}^{2x}\mathrm{d}z^2, \tag{2.1}$$

where A, B and C are functions of t alone.

The energy–momentum tensor for perfect fluid distribution with heat conduction is taken into the form given by Ellis [23] as

$$T_i^j = (\varepsilon + p)v_i v^j + pg_i^j + q_i v^j + v_i q^j \tag{2.2}$$

together with

$$g_{ij}v^i v^j = -1, \tag{2.3}$$

$$q_i q^i > 0, \tag{2.4}$$

$$q_i v^i = 0, \tag{2.5}$$

where p is the pressure, ε the density, and q^i the heat conduction vector orthogonal to v^i.

The fluid flow vector v^i has the component $(\sinh\lambda/A,\ 0,\ 0,\ \cosh\lambda)$ and λ is the tilt angle.

Using the units in which $c = G = 1$, the Einstein field equation

$$R_i^j - \frac{1}{2}Rg_i^j = -8\pi T_i^j \tag{2.6}$$

for the line-element (2.1) when $B = C$ leads to

$$\frac{2B_{44}}{B} + \frac{B_4^2}{B^2} - \frac{1}{A^2} = -8\pi[(\varepsilon + p)\sinh^2\lambda + p + 2Aq'\sinh\lambda] \tag{2.7}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{1}{A^2} = -8\pi p \tag{2.8}$$

$$\frac{2A_4 B_4}{AB} + \frac{B_4^2}{B^2} - \frac{3}{A^2} = -8\pi[-(\varepsilon + p)\cosh^2\lambda + p - 2Aq'\sinh\lambda] \tag{2.9}$$

$$\frac{2A_4}{A} - \frac{2B_4}{B} = -8\pi[(\varepsilon + p)A\sinh\lambda\cosh\lambda + A^2 q'(\cosh\lambda + \sinh\lambda\tanh\lambda], \tag{2.10}$$

where the suffix 4 after A, B denotes ordinary differentiation with respect to t.

Equations (2.7)–(2.10) are four equations in six unknowns A, B, ε, p, q and λ. For the complete determination of these quantities, we assume two extra conditions. First we assume that the space-time is conformally flat which leads to

$$C_{2323} = 0 \tag{2.11}$$

and secondly, we assume

$$A = B^n, \tag{2.12}$$

where

$$C_{2323} = \frac{1}{3}\left[\frac{A_{44}}{A} - \frac{B_{44}}{B} - \frac{A_4 B_4}{AB} + \frac{B_4^2}{B^2}\right]. \tag{2.13}$$

From (2.11) and (2.13), we have

$$\frac{A_{44}}{A} - \frac{B_{44}}{B} - \frac{A_4 B_4}{AB} + \frac{B_4^2}{B^2} = 0. \tag{2.14}$$

Equations (2.12) and (2.14) lead to

$$\frac{B_{44}}{B} + (n-1)\frac{B_4^2}{B^2} = 0. \tag{2.15}$$

From eqs (2.7), (2.9) and (2.12), we have

$$\frac{B_{44}}{B} - n\frac{B_4^2}{B^2} + \frac{1}{B^{2n}} = -4\pi[(\varepsilon + p)\cosh 2\lambda + 4B^n q' \sinh\lambda] \tag{2.16}$$

and

$$\frac{B_{44}}{B} + (n+1)\frac{B_4^2}{B^2} - \frac{2}{B^{2n}} = 4\pi(\varepsilon - p). \tag{2.17}$$

Equations (2.8), (2.17) and (2.11) lead to

$$(n - n^2 + 1)\frac{B_4^2}{B^2} - n\frac{B_{44}}{B} - \frac{1}{B^{2n}} = 4\pi(\varepsilon + p). \tag{2.18}$$

Equations (2.10) and (2.12) lead to

$$-16\pi q' B^n \sinh\lambda = \frac{2(n-1)}{B^n}\ \frac{B_4}{B}\tanh 2\lambda + \{4\pi(\varepsilon + p)\}\frac{\sinh^2 2\lambda}{\cosh 2\lambda}. \tag{2.19}$$

From eqs (2.16) and (2.19), we have

$$\frac{B_{44}}{B} - n\frac{B_4^2}{B^2} + \frac{1}{B^{2n}} = -\frac{4\pi(\varepsilon + p)}{\cosh 2\lambda} + \frac{2(n-1)}{B^n}\tanh 2\lambda. \tag{2.20}$$

Equations (2.18) and (2.20) lead to

$$\begin{aligned}\frac{B_{44}}{B} - n\frac{B_4^2}{B^2} + \frac{1}{B^{2n}} &= n\frac{B_{44}}{B} + (n^2 - n - 1)\frac{B_4^2}{B^2} \\ &\quad + \frac{1}{B^{2n}}\ \frac{1}{\cosh 2\lambda} + \frac{2(n-1)}{B^n}\ \frac{B_4}{B}\tanh 2\lambda.\end{aligned} \tag{2.21}$$

From eq. (2.15), we have

$$\frac{B_{44}}{B_4} + (n-1)\frac{B_4}{B} = 0 \tag{2.22}$$

which on integration leads to

$$B = n^{1/n}(\ell t + m)^{1/n}, \tag{2.23}$$

$$A^2 = n^2(\ell t + m)^2, \tag{2.24}$$

and

$$B^2 = n^{2/n}(\ell t + m)^{2/n}, \tag{2.25}$$

where ℓ, m are constants of integration.

Hence, the metric (2.1) reduces to the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + n^2(\ell t + m)^2\mathrm{d}x^2 + [n(\ell t + m)]^{2/n}\mathrm{e}^{2x}(\mathrm{d}y^2 + \mathrm{d}z^2). \tag{2.26}$$

After the suitable transformations of coordinates, the metric (2.6) reduces to the form

$$\mathrm{d}s^2 = -\frac{\mathrm{d}T^2}{\ell^2} + n^2T^2\mathrm{d}X^2 + n^{2/n}T^{2/n}\mathrm{e}^{2X}(\mathrm{d}Y^2 + \mathrm{d}Z^2). \tag{2.27}$$

3. Some physical and geometrical features

The pressure and density for the model (2.27) are given by

$$8\pi p = \frac{(1-\ell^2)}{n^2T^2} \tag{3.1}$$

and

$$8\pi\varepsilon = 3\frac{(\ell^2-1)}{n^2T^2}. \tag{3.2}$$

The tilt angle λ is given by

$$\cosh^2\lambda = \beta^2 \text{ (constant)}, \tag{3.3}$$

where

$$\beta^2 = \frac{(n\ell^2-1)^2}{(\ell^2-1)(2n^2\ell^2 - n\ell^2 - 1) + 2\ell^2(n-1)\sqrt{(n^2-n)(1-\ell^2)}}$$

$$\sinh^2\lambda = \gamma^2, \tag{3.4}$$

where

$$\gamma^2 = \frac{(n-1)(2n\ell^2 - n\ell^4 - \ell^2) - 2\ell^2(n-1)\sqrt{(n^2-n)(1-\ell^2)}}{(\ell^2-1)(2n^2\ell^2 - n\ell^2 - 1) + 2\ell^2(n-1)\sqrt{(n^2-n)(1-\ell^2)}}.$$

The strong energy conditions given by Hawking and Ellis [24] as $\varepsilon - p \geq 0$ and $\varepsilon + p \geq 0$ lead to $\ell^2 \geq 1$. We have $\varepsilon \to 0, p \to 0$ when $T \to \infty$. Thus for large values of T, the pressure and density for the model vanish.

The scalar of expansion (θ) calculated for the flow vector v^i is given by

$$\theta = \frac{2\gamma + (n+2)\beta\ell}{nT}. \tag{3.5}$$

The non-vanishing components of shear tensor (σ_{ij}) are given by

$$\sigma_{11} = \beta n^2 \ell T - \frac{\beta^2}{3}\{2\gamma + \beta\ell(n+2)\}nT + nT\beta\gamma^2 n\ell, \tag{3.6}$$

$$\sigma_{22} = \sigma_{33} = (\gamma + \beta\ell - \beta n\ell)\frac{n^{2/n}T^{2/n}\mathrm{e}^{2X}}{3nT}, \tag{3.7}$$

$$\sigma_{44} = -\frac{\gamma^2}{3nT}\{2\gamma + \beta\ell(n+2)\} + \frac{\beta\gamma^2\ell}{T}, \tag{3.8}$$

$$\sigma_{14} = -\gamma^3 n\ell - n\ell\gamma + \frac{\beta\gamma}{3}\{2\gamma + \beta\ell(n+2)\}. \tag{3.9}$$

The rotation vanishes.

Thus $\sigma_{\mu\nu}v^\nu = 0$ leads to

$$\begin{aligned}\sigma_{11}v^1 + \sigma_{14}v^4 &= \beta\gamma n\ell - \frac{\beta^2\gamma}{3}\{2\gamma + \beta\ell(n+2)\} + \beta\gamma^3 n\ell - \beta\gamma^3 n\ell \\ &\quad -\beta\gamma n\ell + \frac{\beta^2\gamma}{3}\{2\gamma + \beta\ell(n+2)\} \\ &= 0.\end{aligned} \tag{3.10}$$

Similarly $w_{\mu\nu}v^\nu = 0$ leads to $w_{11}v^1 + w_{14}v^4 = 0$.

The expressions for fluid velocity components v^1, v^4 and heat conduction vector q^1, q^4 are given by

$$v^1 = \frac{\gamma}{nT}, \tag{3.11}$$

$$v^4 = \beta, \tag{3.12}$$

$$q^1 = -\frac{\beta[(\ell^2-1)\beta\gamma + (n-1)\ell]}{4\pi n^3 T^3(\beta^2+\gamma^2)}, \tag{3.13}$$

$$q^4 = -\frac{\gamma[(\ell^2-1)\beta\gamma + (n-1)\ell]}{4\pi n^2 T^2(\beta^2+\gamma^2)}. \tag{3.14}$$

The rate of expansion H_i in the direction of X, Y, Z-axes are given by

$$H_1 = \frac{\ell}{T} \tag{3.15}$$

$$H_2 = H_3 = \frac{\ell}{nT}. \tag{3.16}$$

4. Discussions

The model starts expanding with a big-bang at $T = 0$ and the expansion in the model stops at $T = \infty$ and $\ell = -2\gamma/(n+2)\beta$. The model in general represents shearing, non-rotating and tilted universe. The expansion in the model decreases as time increases. For $\ell = 1, n = 1$, we have heat conduction vector $q^1 = 0, q^4 = 0$. When $T \to \infty, v^1 = 0, v^4 =$ constant, $q^1 = 0, q^4 = 0$.

Since $\lim_{T\to\infty}(\sigma/\theta) \neq 0$, the model does not approach isotropy for large values of T. There is a real physical singularity in the model at $T = 0$.

5. Special case

For $n = 2$, the metric (2.27) reduces to the form

$$\mathrm{d}s^2 = -\frac{\mathrm{d}T^2}{\ell^2} + 4T^2\mathrm{d}X^2 + 2T\mathrm{e}^{2X}(\mathrm{d}Y^2 + \mathrm{d}Z^2). \tag{5.1}$$

The pressure and the density for model (5.1) are given by

$$8\pi p = \frac{(1-\ell^2)}{4T^2} \tag{5.2}$$

and

$$8\pi\varepsilon = \frac{3}{4}\,\frac{(\ell^2-1)}{T^2}. \tag{5.3}$$

The strong energy conditions $\varepsilon + p \geq 0$ and $\varepsilon - p \geq 0$ lead to

$$\ell^2 \geq 1.$$

The tilt angle λ is given by

$$\cosh^2\lambda = \beta^2 \text{ (constant)}, \tag{5.4}$$

where

$$\beta^2 = \frac{(2\ell^2-1)^2}{[(6\ell^4 - 7\ell^2 + 1) + 2\ell^2\sqrt{2(1-\ell^2)}]}$$

and

$$\sinh^2\lambda = \gamma^2, \tag{5.5}$$

where

$$\gamma^2 = \frac{\ell^2[3 - 2\ell^2 - 2\sqrt{2(1-\ell^2)}]}{[(6\ell^4 - 7\ell^2 + 1) + 2\ell^2\sqrt{2(1-\ell^2)}]}.$$

The scalar of expansion (θ) calculated for flow vector v^i is given by

$$\theta = \frac{\gamma + 2\beta\ell}{T}. \tag{5.6}$$

The expansion for non-vanishing components of fluid velocity vectors v^1, v^4, heat conduction vectors q^1, q^4, and shear tensors σ_{ij} are given by

$$v^1 = \frac{\gamma}{2T}, \tag{5.7}$$

$$v^4 = \beta, \tag{5.8}$$

$$q^1 = \frac{-\beta[\ell + \beta\gamma(\ell^2 - 1)]}{32\pi(\beta^2 + \gamma^2)T^3}, \tag{5.9}$$

$$q^4 = \frac{-\gamma[\ell + \beta\gamma(\ell^2 - 1)]}{16\pi(\beta^2 + \gamma^2)T^2}, \tag{5.10}$$

$$\sigma_{11} = 4\beta\ell T - \frac{4\beta^2}{3}\{\gamma + 2\beta\ell\}T + 4T\beta\gamma^2\ell, \tag{5.11}$$

$$\sigma_{22} = \sigma_{33} = \frac{(\gamma - \beta\ell)T\mathrm{e}^{2X}}{3T}, \tag{5.12}$$

$$\sigma_{44} = -\frac{\gamma^2}{3T}\{\gamma + 2\beta\ell\} + \frac{\beta\gamma^2\ell}{T}, \tag{5.13}$$

$$\sigma_{14} = -2\gamma^3\ell - 2\ell\gamma + \frac{2\beta\gamma}{3}(\gamma + 2\beta\ell), \tag{5.14}$$

and rotation vanishes.

The rate of expansion (H_i) in the direction of X, Y, Z-axes is given by

$$H_1 = \frac{\ell}{T} \tag{5.15}$$

$$H_2 = H_3 = \frac{\ell}{2T}. \tag{5.16}$$

6. Conclusions

The model starts expanding with a big-bang at $T = 0$ and the expansion in the model stops at $T = \infty$ and $\ell = -\gamma/2\beta$.

The model in general represents non-rotating and shearing universe. For expansion in the model, we have $\ell \neq 0, \gamma \neq 0$ when $T \to \infty$ then heat conduction vectors $q^1 = 0$ and $q^4 = 0, v^1 = 0, v^4 = \beta$.

Since $\lim_{T\to\infty}(\sigma/\theta) \neq 0$, the model does not approach isotropy for large values of T. There is a real physical singularity in the model at $T = 0$. When $\beta = 1$ then $\lambda = 0$. Thus the tilted cosmological model leads to non-tilted case for $\beta = 1$.

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