PRAMANA — journal of physics · © Indian Academy of Sciences · Vol. 62, No. 4, April 2004, pp. 943–955

Real solution of monochromatic wave propagation in inhomogeneous media

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MS received 1 May 2003; revised 31 August 2003; accepted 8 September 2003

Abstract. The earlier phenomenological descriptions of the propagation of signals in inhomogeneous media – except the mirror-type, scattering-type descriptions – have an inherent misunderstanding and therefore these methods are wrong except the non-coupled WKB approximation. The cause of the problem is the wrong physical concept about the structure of the signals propagating in inhomogeneous media. Using a better physical concept of the signal structure propagating in inhomogeneous media and the method of inhomogeneous basic modes (MIBM) it was possible to derive correct full-wave solutions for propagating signals. The investigated example is a monochromatic plane wave propagating in an isotropic, inhomogeneous, linear media parallel to the gradient of the one-dimensional inhomogeneity. However, a generalization of the process for more complex inhomogeneities is possible.

Keywords. Wave propagation; electromagnetic theory; radio wave propagation.

PACS Nos 03.50.De; 03.65.Ge; 41.20.Jb; 42.25.Bs; 46.40.Cd; 52.35.Hr; 94.20.Bb

1. Introduction

'Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concept with which the theory operates' [1]. We must take into account this distinction in the case of the electromagnetic (or other) wave propagation in inhomogeneous media even if we use the phenomenological description of the problem. The 'objective reality' in the case of wave propagation in (linear) inhomogeneous media is the fact, that the original (e.g. 'forward' propagating or 'source-') signal will attenuate during the propagation and simultaneously generates another, reflected or scattered signal propagating in other direction(s) in every point of space in which the inhomogeneity exists. The 'physical concept' is the description of this phenomenon by solving the Maxwell's equations.

The whole process of derivation of a real (full wave) solution of Maxwell's equations is a self-consistent field (SCF) process (see figure 1, ref. [2]). During this process we must make two crucial decisions. The first decision is the supposition of the starting form of the solution, e.g. time harmonic [3–7] or an arbitrary shaped signal [8]. The second is the description of the medium – signal interaction for the given case of inhomogeneity and for the supposed form of the solution [5,8–11].

Earlier solution-methods and results for the propagating electromagnetic (e.m.) signals in different inhomogeneous media were demonstrated. The most frequently used methods (excluding the scattering) are the eikonal-equation [2,5], the WKB approximation [2,5], the method of generalized vector of propagation [3,4,6] etc. The signal – medium interaction was clarified in the most important cases, when the structure of the inhomogeneity do not cause a full scattering of the signal, i.e., the description of the phenomena is possible by attenuation, reflection, refraction, diffraction etc. of the signals. It was clarified that the constitutional parameters of the medium (i.e. permittivity, permeability, etc.) in the case of strong inhomogeneities are tensors even in the most simple cases, in which a scalar approximation is valid only in slowly-varying, non-dispersive media [10]. However, applying this knowledge in the most common way, the well-known results have an essential inaccuracy. For example, using these results it is possible in a passive, linear, time-invariant... simple, inhomogeneous medium that the computed propagating source-signal will locally amplify depending on the direction of the gradient of permittivity.

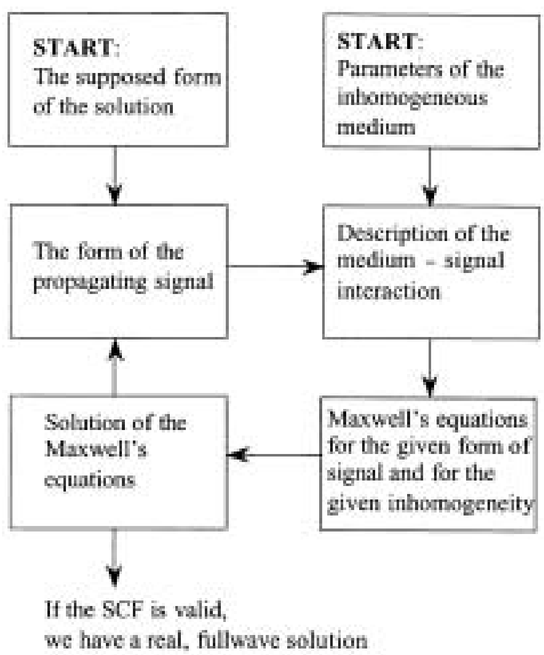

Figure 1. The self-consistent field process for deriving a solution of Maxwell's equations.

The cause of the problem is a non-discussed assumption which is part of the assumptions about the signal form, that the different signal-parts – appearing during the propagation in the inhomogeneous medium as a direct consequence of the propagation of the source-signal – will propagate independently of each other in such a manner that the coupling between them is possible to determine by additional calculations, if necessary. This assumption is an extremely important part of the 'physical concept' and it is wrong. In the following let us demonstrate first the problem itself in well-known example, concentrating during this presentation to the non-scattering propagation with monochromatic excitation. In the next step let us present the solution of the problem together with a method that can be used for the derivation of the real full-wave solutions.

2. The problem itself

Let us investigate the propagation of a strictly monochromatic signal in a linear, isotropic, time-invarying, lossless, non-dispersive simple medium. If this e.m. signal propagates through an inhomogeneity, a part of the propagating energy reflects (or scatters) from the inhomogeneity and therefore the amplitude of the original signal must attenuate. The task is to find the solution of the Maxwell's equations knowing that the amplitude of the solution must change.

Let the model of the medium be as simple as possible, i.e., we can define permittivity and this permittivity $\varepsilon(\bar{r})$ is scalar and real quantity, the permeability is μ_0, and ε_0 is the permittivity and μ_0 is the permeability of vacuum. Let the form of the solution be strictly monochromatic and let us use the common complex formalism,

$$\bar{G}(\bar{r},t) \mathrel{\hat{=}} \bar{G}_0(\bar{r})e^{j[\omega t-\varphi(\bar{r})]}, \tag{1}$$

where $\bar{G}$ means $\bar{E},\bar{B},\bar{D},\bar{H}$; $\bar{G}_0$ is the amplitude, φ is the phase function, $\bar{r}$ is the space vector, t is the time and ω is the angular frequency. In this case the Maxwell's equations are

$$\begin{aligned}
&\bar{\nabla}\times\bar{H} = j\omega\varepsilon\bar{E},\\
&\bar{\nabla}\times\bar{E} = -j\omega\mu_0\bar{H},\\
&\bar{\nabla}\cdot\bar{H} = 0,\\
&\bar{\nabla}\cdot(\varepsilon\bar{E}) = 0.
\end{aligned} \tag{2}$$

After simple and known manipulations

$$\begin{aligned}
&(\bar{\nabla}\times\bar{H}_0) - j\bar{\nabla}\varphi\times\bar{H}_0 = j\omega\varepsilon\bar{E}_0,\\
&(\bar{\nabla}\times\bar{E}_0) - j\bar{\nabla}\varphi\times\bar{E}_0 = -j\omega\mu_0\bar{H}_0.
\end{aligned} \tag{3}$$

Let us use the notations

$$\bar{k} \mathrel{\hat{=}} \bar{\nabla}\varphi \quad \text{and} \quad \bar{k}\times\bar{u} \mathrel{\hat{=}} \bar{\bar{k}}\cdot\bar{u}, \tag{4}$$

where $\bar{u}$ is an arbitrary vector.

In the most simple cases it is usual to suppose that $\bar{G}_0$ and φ are real quantities (see e.g. the eikonal function, the WKB approximation or some other methods) and therefore eq. (3) could be separated as the real part and the imaginary part.

The real part

$$\bar{\nabla} \times \bar{H}_0 = 0,$$
$$\bar{\nabla} \times \bar{E}_0 = 0. \tag{5}$$

The imaginary part

$$\bar{k} \times \bar{H}_0 = -\omega\varepsilon\bar{E}_0,$$
$$\bar{k} \times \bar{E}_0 = \omega\mu_0\bar{H}_0. \tag{6}$$

From (6) we get the dispersion equation, i.e.,

$$|\,\overline{\overline{k}}\,\overline{\overline{k}} + \omega^2\varepsilon\mu_0\, \overline{\overline{1}}\,| = 0. \tag{7}$$

In this very simple case $\bar{k}$ is a real quantity and

$$k^2 = \omega^2\varepsilon\mu_0 \quad \text{or} \quad k = \pm\omega\sqrt{\varepsilon\mu_0}, \tag{8}$$

suggesting a forward and a backward propagating signal, or

$$\bar{k} = k \cdot \bar{e}_k, \tag{9}$$

where $\bar{e}_k$ is a unit vector with arbitrary direction. As is well-known, the signal amplitude will change deriving it from (6). However, by deriving the solution of (5) it is shown that in every case

$$\bar{H}_0 = \text{constant} \quad \text{and} \quad \bar{E}_0 = \text{constant}. \tag{10}$$

We must realize that using this form of solution – relation (1) – the amplitude of the signal could not change even if $\varepsilon = \varepsilon(\bar{r})$. Therefore the supposed form of the solution could exist only in homogeneous, lossless, time-invarying media.

It would be possible to present other well-known examples (e.g. propagation in a lossy, homogeneous medium; propagation through boundary planes) to investigate the nature of variation of the signal amplitudes. However, the result will be the same. If the pre-supposed form of the solution has the general structure given in (1), then the amplitude of the solution produced by the Maxwell's equations could not change, only the complex phase of the computed signal will change.

Therefore we must change our 'physical concept' [1] about the (mathematical) structure of the solution knowing that the source-part (generated by the original excitation) and the part reflected by the inhomogeneity of the whole solution form a single, non-separable propagating mode.

3. Derivation of the correct full-wave solution in an inhomogeneous medium – the simplest example

Let the source-signal be a strictly monochromatic plane wave propagating in the $\bar{e}_k$ direction (i.e. parallel to axis x). Let the inhomogeneity be one-dimensional and the gradient of the permittivity be parallel to axis x. In this case the inhomogeneous permittivity could be isotropic (scalar) if the medium itself is isotropic in homogeneous cases. Therefore,

$$\varepsilon = \varepsilon(x) \quad \text{and} \quad \mu = \mu_0. \tag{11}$$

(In the cases of 'oblique' propagation etc. the permittivity will be tensorial $\bar{\bar{\varepsilon}}_{inh}(x)$ – even for the media which are isotropic in homogeneous cases – [2,10].) The goal in this investigation is only to find a correct mode of derivation of a real full-wave solution.

The excitation is monochromatic as was seen in (1). Therefore,

$$\bar{G} = \bar{G}_0 \mathrm{e}^{j(\omega t - \varphi)} \quad \text{and} \quad \varphi = \int \bar{k}(\bar{r}) \cdot \mathrm{d}\bar{r}, \quad \text{i.e.} \quad \bar{k}(\bar{r}) = \bar{\nabla}\varphi(\bar{r}).$$

However, let the structure of the whole signal be

$$\bar{G} = \sum_i \bar{G}_{0i}(\bar{r}) \cdot \mathrm{e}^{j(\omega t - \int \bar{k}_i \mathrm{d}\bar{r})} \tag{12}$$

and therefore let us use the method of inhomogeneous basic modes (MIBM) [2,7,8] and let the index i – $i = 1, \ldots, n_{\max}$ – be the index of the basic modes, which are not solutions of the Maxwell's equations alone, however, the sum of these modes, i.e. (12) is a solution. The Maxwell's equations after simplification by $\mathrm{e}^{j\omega t}$ are

$$\begin{aligned}
&\sum_i [\bar{\nabla} \times \bar{H}_{0i}(\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} - \sum_i j\bar{k}_i(\bar{r}) \times \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} \\
&= \sum_i j\omega\varepsilon(x)\bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}}, \\
&\sum_i [\bar{\nabla} \times \bar{E}_{0i}(\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} - \sum_i j\bar{k}_i(\bar{r}) \times \bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} \\
&= -\sum_i j\omega\mu_0 \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} \\
&\sum_i \mu_0[\bar{\nabla} \cdot \bar{H}_{0i}(\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} - \sum_i j\mu_0\bar{k}_i(\bar{r}) \cdot \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0, \\
&\sum_i [\bar{\nabla}\varepsilon(x)]\bar{E}_{0i}(x)\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} + \sum_i \varepsilon(x)[\bar{\nabla} \cdot \bar{E}_{0i}\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} \\
&\quad - \sum_i j\bar{k}_i(\bar{r}) \cdot \varepsilon(x)\bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0.
\end{aligned} \tag{13}$$

Let us separate eq. (13) as it was defined by MIBM (see [8]) and then it follows: The equation system defining the inhomogeneous basic modes are

$$\begin{aligned}
&\sum_i \bar{k}_i(\bar{r}) \times \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = -\omega\varepsilon(x) \sum_i \bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}},\\
&\sum_i \bar{k}_i(\bar{r}) \times \bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = \omega\mu_0 \sum_i \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}},\\
&\sum_i \bar{k}_i(\bar{r}) \cdot \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0,\\
&\varepsilon(x) \sum_i \bar{k}_i(\bar{r}) \cdot \bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0.
\end{aligned} \tag{14}$$

The coupling equations are

$$\begin{aligned}
&\sum_i [\bar{\nabla} \times \bar{H}_{0i}(\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0,\\
&\sum_i [\bar{\nabla} \times \bar{E}_{0i}(\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0,\\
&\sum_i [\bar{\nabla} \cdot \bar{H}_{0i}(\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0,\\
&[\bar{\nabla}\varepsilon(x)] \cdot \sum_i \bar{E}_{0i}(x)\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} + \varepsilon(x) \sum_i [\bar{\nabla} \cdot \bar{E}_{0i}(\bar{r})]\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0.
\end{aligned} \tag{15}$$

The physical concept behind this separation is the practical fact that the inhomogeneities have finite (geometrical) size, and out of the inhomogeneity the medium is homogeneous. Therefore let the basic modes be chosen in such a manner that every mode, when propagating out of the inhomogeneity, is supposed to be equal to one of the signals propagating independently of each other in the homogeneous volumes in this linear case. Equation (14) can guarantee this.

If we use the MIBM philosophy, then we shall solve eq. (14) for each value of i independently of each other. After this process – using the results of this derivation, i.e., the values of the derived inhomogeneous basic modes – we must solve eq. (14) without any further separation.

3.1 *Derivation of the basic modes*

Equation (13) for a single i value is

$$\begin{aligned}
&\bar{k}_i(\bar{r}) \times \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = -\omega\varepsilon(x)\bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}},\\
&\bar{k}_i(\bar{r}) \times \bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = \omega\mu_0\bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}},\\
&\bar{k}_i(\bar{r}) \cdot \bar{H}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0,\\
&\varepsilon(x)\bar{k}_i(\bar{r}) \cdot \bar{E}_{0i}(\bar{r})\mathrm{e}^{-j\int \bar{k}_i \mathrm{d}\bar{r}} = 0.
\end{aligned} \tag{16}$$

After common simplifications we can solve the equations and the solution of (16) is well-known from homogenous investigations as follows:

$$\begin{aligned}
&k_i^2(x) = \omega^2\varepsilon(x)\mu_0 \mathrel{\hat{=}} k^2(x),\\
&\bar{k}_i(\bar{r}) = k_i(x)\bar{e}_{ki},
\end{aligned} \tag{16a}$$

where $\bar{e}_{ki}$ are unit vectors. Without any additional restrictions

$$\bar{e}_{ki} = \cos\alpha \cdot \bar{e}_1 + \sin\alpha \cdot \bar{e}_3 \quad \text{and} \quad \bar{k}_i = k_i(x)\cos\alpha \cdot \bar{e}_1 + k_i(x)\sin\alpha \cdot \bar{e}_3. \tag{16b}$$

In our special case when the direction of propagation is the x-axis,

$$\bar{k}_i = k_i(x) \cdot \bar{e}_1 \quad \text{and} \quad \sin\alpha = 0. \tag{16c}$$

Applying (16b) in (16)

$$\bar{e}_{ki} \times \bar{H}_{0i}(\bar{r}) = -\frac{\bar{E}_{0i}(\bar{r})}{Z_0(x)} \quad \text{and} \quad \bar{e}_{ki} \times \bar{E}_{0i}(\bar{r}) = Z_0(x)\bar{H}_{0i}(\bar{r}). \tag{17}$$

3.2 *Derivation of the 'coupling' equations*

Let us use the basic mode solution (16) and (17) in (15). Then

$$\begin{aligned}
&\sum_i [\bar{\nabla} \times \bar{H}_{0i}(\bar{r})] \mathrm{e}^{-j\int[k_i(x)\cos\alpha \mathrm{d}x + k_i(x)\sin\alpha \mathrm{d}z]} = 0,\\
&\sum_i [\bar{\nabla} \times \bar{E}_{0i}(\bar{r})] \mathrm{e}^{-j\int[k_i(x)\cos\alpha \mathrm{d}x + k_i(x)\sin\alpha \mathrm{d}z]} = 0,\\
&\sum_i [\bar{\nabla} \cdot \bar{H}_{0i}(\bar{r})] \mathrm{e}^{-j\int[k_i(x)\cos\alpha \mathrm{d}x + k_i(x)\sin\alpha \mathrm{d}z]} = 0,\\
&[\bar{\nabla}\varepsilon(x)] \cdot \sum_i \bar{E}_{0i}(x) \mathrm{e}^{-j\int[k_i(x)\cos\alpha \mathrm{d}x + k_i(x)\sin\alpha \mathrm{d}z]}\\
&\qquad + \varepsilon(x) \sum_i [\bar{\nabla} \cdot \bar{E}_{0i}(\bar{r})] \mathrm{e}^{-j\int[k_i(x)\cos\alpha \mathrm{d}x + k_i(x)\sin\alpha \mathrm{d}z]} = 0.
\end{aligned} \tag{18}$$

After a detailed analysis of the structure of (18) we can say that the equation system (18) could be valid for every value of z only if

$$k_i(x)\sin\alpha \mathrel{\hat{=}} \zeta = \text{constant}. \tag{19a}$$

Therefore the possible values existing in this case are

$$k_i(x)\cos\alpha = \pm\sqrt{k^2(x) - \zeta^2} \quad \text{and} \quad i = 1 \text{ and } 2, \tag{19b}$$

depending on the sign of $k_i \cos\alpha$. In our special case

$$\bar{k}_i(\bar{r}) \equiv \bar{k}_i(x) = \pm k(x)\bar{e}_1 \mathrel{\hat{=}} k_{ix}(x)\bar{e}_1. \tag{19c}$$

Applying (19c) in (18) it is possible to simplify by $\mathrm{e}^{-j\zeta z}$ and therefore the new form of (18) to find the simplest full-wave solution of this equation system is

$$\sum_{i=1}^{2}[\bar{\nabla} \times \bar{H}_{0i}(\bar{r})]\mathrm{e}^{-j\int k_{ix}(x)\mathrm{d}x} = 0,$$
$$\sum_{i=1}^{2}[\bar{\nabla} \times \bar{E}_{0i}(\bar{r})]\mathrm{e}^{-j\int k_{ix}(x)\mathrm{d}x} = 0,$$
$$\sum_{i=1}^{2}[\bar{\nabla} \cdot \bar{H}_{0i}(\bar{r})]\mathrm{e}^{-j\int k_{ix}(x)\mathrm{d}x} = 0,$$
$$[\bar{\nabla}\varepsilon(x)] \cdot \sum_{i=1}^{2} \bar{E}_{0i}(\bar{r})]\mathrm{e}^{-j\int k_{ix}(x)\mathrm{d}x} + \varepsilon(x)\sum_{i=1}^{2}[\bar{\nabla} \cdot \bar{E}_{0i}(\bar{r})]\mathrm{e}^{-j\int k_{ix}(x)\mathrm{d}x} = 0. \tag{20}$$

In the following, let us use the notations,

$$k_{1x}(x) = k(x) \quad \text{and} \quad k_{2x}(x) = -k(x). \tag{21}$$

Applying (19c) in (17) we can derive the detailed form of the basic modes. The new and detailed form of (17) is

For EM (so-called 'electric mode') signals

$$\begin{aligned} &\mp e_{k1}H_{0iz}(\bar{r}) + e_{k3}H_{0ix}(\bar{r}) = -\frac{1}{Z_0(x)}E_{0iy}(\bar{r}),\\ &-e_{k3}E_{0iy}(\bar{r}) = Z_0(x)H_{0ix}(\bar{r}),\\ &\pm e_{k1}E_{0iy}(\bar{r}) = Z_0(x)H_{0iz}(\bar{r}), \end{aligned} \tag{22a}$$

and for HM (so-called 'magnetic mode') signals

$$\begin{aligned} &-e_{k3}H_{0iy}(\bar{r}) = -\frac{1}{Z_0(x)}E_{0ix}(\bar{r}),\\ &\pm e_{k1}H_{0iy}(\bar{r}) = -\frac{1}{Z_0(x)}E_{0iz}(\bar{r}),\\ &\mp e_{k1}E_{0iz}(\bar{r}) + e_{k3}E_{0ix}(\bar{r}) = Z_0(x)H_{0iy}(\bar{r}). \end{aligned} \tag{22b}$$

From (22a) the EM signals are

$$H_{0ix}(\bar{r}) = 0 \quad \text{and} \quad H_{0iz}(\bar{r}) = \pm\frac{1}{Z_0(x)}E_{0iy}(\bar{r}), \tag{23}$$

and from (22b) the HM signals are

$$E_{0ix}(\bar{r}) = 0 \quad \text{and} \quad E_{0iz}(\bar{r}) = \mp Z_0(x)H_{0iy}(\bar{r}). \tag{24}$$

Applying (23) and (24) in (20) it is possible to derive a new form of the 'coupling' equations. However, during this process and in the final phase of derivation we must define and apply the boundary conditions of the problem.

3.3 *The boundary conditions*

We know that in this example the source-signal is generated out of the inhomogeneity and it is a plane wave which propagates in $\bar{e}_k$ direction, and in the final version of this case it propagates parallel to the axis x. The gradient of permittivity inside the inhomogeneity is parallel to the axis x and the inhomogeneity is finite in space along the axis x. (The nature of the medium was defined earlier.) The boundary conditions are defined out of inhomogeneity and they are the following:

– the medium is inhomogeneous from $x = 0$ to $x = x_M$ and it is homogeneous in the $x < 0$ and the $x > x_M$ half-spaces;
– the source of the signal (the excitation) exists in the $x < 0$ half-space and defines the $E_{01y}(x < 0) = \text{constant}$ and $H_{01y}(x < 0) = \text{constant}$ amplitudes;
– the amplitudes of the reflected (backward propagating) part of the signal are in half-space $x > x_M$ $E_{02y}(x > x_M) \equiv 0$ and $H_{02y}(x > x_M) \equiv 0$.

Knowing and applying the boundary conditions we can continue the solution of the 'coupling' equations.

3.4 *Solution of the 'coupling' equations*

Applying (23) and (24) in (20) and the consequence of the boundary conditions, i.e., the amplitudes can change only as a function of x, we can derive a new form of (20) which is

$$\begin{aligned}
&-\frac{\mathrm{d}H_{01z}(x)}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x} - \frac{\mathrm{d}H_{02z}(x)}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x} = 0,\\
&\frac{\mathrm{d}H_{01y}(x)}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x} - \frac{\mathrm{d}H_{02y}(x)}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x} = 0,\\
&-\frac{\mathrm{d}E_{01z}(x)}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x} - \frac{\mathrm{d}E_{02z}(x)}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x} = 0,\\
&\frac{\mathrm{d}E_{01y}(x)}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x} + \frac{\mathrm{d}E_{02y}(x)}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x} = 0,\\
&\frac{\mathrm{d}H_{01x}(x)}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x} + \frac{\mathrm{d}H_{02x}(x)}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x} = 0,\\
&\frac{\mathrm{d}\varepsilon}{\mathrm{d}x}E_{01x}(x)\mathrm{e}^{-j\int k\mathrm{d}x} + \frac{\mathrm{d}\varepsilon}{\mathrm{d}x}E_{02x}(x)\mathrm{e}^{j\int k\mathrm{d}x}\\
&\qquad + \varepsilon\frac{\mathrm{d}E_{01x}(x)}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x} + \varepsilon\frac{\mathrm{d}E_{02x}(x)}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x} = 0. \qquad (25)
\end{aligned}$$

It is clear from (25) that in general the EM and HM parts of the signal propagate independently. The rewritten form of (25) is

$$\begin{aligned}
&\frac{\mathrm{d}E_{01y}}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x} + \frac{\mathrm{d}E_{02y}}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x} = 0,\\
&-\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{E_{01y}}{Z_0(x)}\right\}\mathrm{e}^{-j\int k\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{E_{02y}}{Z_0(x)}\right\}\mathrm{e}^{j\int k\mathrm{d}x} = 0; \qquad (26\text{a})
\end{aligned}$$

for EM signals and

$$\frac{\mathrm{d}H_{01y}}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x}+\frac{\mathrm{d}H_{02y}}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x}=0,$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\{Z_0(x)H_{01y}\}\mathrm{e}^{-j\int k\mathrm{d}x}-\frac{\mathrm{d}}{\mathrm{d}x}\{Z_0(x)H_{02y}\}\mathrm{e}^{j\int k\mathrm{d}x}=0, \tag{26b}$$

for HM signals.

The propagation of the EM and HM parts of a given signal in inhomogeneous medium, even if the medium is isotropic, differs from each other. This difference disappears if the direction of propagation is parallel to the gradient of medium parameter, i.e., of permittivity in this case.

In this basic investigation let us derive the solution in this special case in which $\bar{e}_k||\bar{e}_1$ and therefore $\sin\alpha=0$. Then the propagation of the EM and HM parts are identical. For example, the propagation of the EM signal is described by the following equations derived from (26a)

$$\frac{\mathrm{d}E_{01y}}{\mathrm{d}x}\mathrm{e}^{-j\int k\mathrm{d}x}+\frac{\mathrm{d}E_{02y}}{\mathrm{d}x}\mathrm{e}^{j\int k\mathrm{d}x}=0,$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{E_{01y}}{Z_0}\right)\mathrm{e}^{-j\int k\mathrm{d}x}+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{E_{02y}}{Z_0}\right)\mathrm{e}^{j\int k\mathrm{d}x}=0. \tag{27}$$

During the following derivation of the solution, we must know that in the homogeneous half-spaces and in every place where Z_0 has local extremum the derivatives $\mathrm{d}Z_0/\mathrm{d}x=0$. From (27)

$$\frac{\mathrm{d}E_{01y}}{\mathrm{d}x}=\frac{1}{2Z_0}\frac{\mathrm{d}Z_0}{\mathrm{d}x}(E_{01y}-E_{02y})\mathrm{e}^{j2\int k\mathrm{d}x}, \tag{28}$$

and

$$\frac{\mathrm{d}E_{02y}}{\mathrm{d}x}=\frac{1}{2Z_0}\frac{\mathrm{d}Z_0}{\mathrm{d}x}(E_{02y}-E_{01y})\mathrm{e}^{-j2\int k\mathrm{d}x}, \tag{29}$$

which differ from the ‘coupled WKB’ formalism or from the earlier description experiments. Let

$$E_{01y}\equiv E_1 \quad \text{and} \quad E_{02y}\equiv E_2, \tag{30}$$

and let us use the successive approximation during the derivation of the solution from eqs (28) and (29). In the first step let $E_2\cong 0$ and therefore from (28)

$$\frac{\mathrm{d}E_1}{\mathrm{d}x}=\frac{1}{2Z_0}\frac{\mathrm{d}Z_0}{\mathrm{d}x}E_1,$$

or

$$E_1(x)=C\sqrt{Z_0(x)}, \tag{31}$$

where $C=$ constant.

This zero-order approximation of the exact solution appearing in (31) is precisely the non-coupled WKB form of amplitudes, deriving here directly from Maxwell's equations without using the WKB philosophy regarding the constant energy density. If we use this approximation as a final solution, then $E_2 \equiv 0$. From the other point of view, (31) verifies that the non-coupled and only the non-coupled WKB approximation is a possible (quasi-homogeneous) description of the problem in which the reflected part of the signal could not exist.

Let us continue the successive approximation by applying (31) in (29), then

$$\frac{\mathrm{d}E_2}{\mathrm{d}x} = \frac{1}{2Z_0}\frac{\mathrm{d}Z_0}{\mathrm{d}x}\left(E_2 - C\sqrt{Z_0}\mathrm{e}^{-j2\int_0^x k(v)\mathrm{d}v}\right). \tag{32}$$

Equation (32) is a known type of differential equations (see e.g. in [12]), i.e.

$$\frac{\mathrm{d}E_2}{\mathrm{d}x} - \frac{1}{2}\frac{\mathrm{d}(\ln Z_0)}{\mathrm{d}x}E_2 = -C\frac{\mathrm{d}\sqrt{Z_0}}{\mathrm{d}x}\mathrm{e}^{-j2\int_0^x k(v)\mathrm{d}v}. \tag{33}$$

Using the notations

$$\begin{aligned}
&E_2 \to y,\\
&-\frac{1}{2}\frac{\mathrm{d}(\ln Z_0)}{\mathrm{d}x} \to f(x),\\
&-\frac{\mathrm{d}\sqrt{Z_0}}{\mathrm{d}x}C\mathrm{e}^{-j2\int_0^x k(v)\mathrm{d}v} \to g(x),
\end{aligned} \tag{34}$$

then the mathematised form of (33) is

$$y' + f(x)y = g(x). \tag{35}$$

If we know a point (ξ, η) on the (x, y) plane, i.e., a point of the solution, then the whole solution of (35) is

$$y = \mathrm{e}^{-F}\left(\eta + \int_\xi^x g(x)\mathrm{e}^F\mathrm{d}x\right),$$

where

$$F = \int_\xi^x f(u)\mathrm{d}u. \tag{36}$$

However, we know a (ξ, η) point of the solution because

$$\begin{aligned}
&E_2 \equiv 0, \quad \text{if} \quad x \geq x_M, \quad \text{i.e.}\\
&y = 0 = \eta, \quad \text{if} \quad x = x_M = \xi.
\end{aligned} \tag{37}$$

Further,

$$\begin{aligned}
&F = \int_\xi^x f(u)\mathrm{d}u = -\frac{1}{2}\int_\xi^x \frac{\mathrm{d}(\ln Z_0)}{\mathrm{d}u}\mathrm{d}u = -\frac{1}{2}\ln Z_0(x) + \frac{1}{2}\ln Z_0(\xi).\\
&Z_0(\xi) = Z_0(x_M) = \text{constant} = Z_{0M}
\end{aligned}$$

and

$$e^{-F} = \sqrt{Z_0(x)/Z_{0M}} \quad \text{and} \quad e^{F} = \sqrt{Z_{0M}/Z_0(x)}.$$

From the excitation we know the original (source) value of the E_1 amplitude, i.e.,

$$C \mathrel{\hat{=}} E_{10} = \text{constant}. \tag{38}$$

Then from (36) we get the first-order approximation of the reflected part of the signal which is

$$E_2 = \frac{E_{10}}{2}\sqrt{Z_0(x)} \int_x^{x_M} \frac{\mathrm{d}(\ln Z_0)}{\mathrm{d}u} e^{-j2\int_0^u k(v)\mathrm{d}v} \mathrm{d}u. \tag{39}$$

Applying (39) in (28) the first approximation of the forward propagating part of the signal is

$$E_1 = E_{10}\sqrt{Z_0(x)} \left\{ 1 - \frac{1}{4}\int_0^x \frac{\mathrm{d}(\ln Z_0)}{\mathrm{d}u} e^{j2\int_0^u k\mathrm{d}v} \cdot \left[\int_u^{x_M} \frac{\mathrm{d}(\ln Z_0)}{\mathrm{d}w} e^{-j2\int_0^u k\mathrm{d}v} \mathrm{d}w \right] \mathrm{d}u \right\}. \tag{40}$$

It is possible to continue this successive approximation process. However, in some practical and space research applications the accuracy of (39) and (40) is enough. These forward and backward propagating parts together form a single propagating mode.

The full form of the solution is

$$E_{1y}(x,t) \cong E_1(x) \cdot e^{j[\omega t - \int_0^x k(v)\mathrm{d}v]}$$

and

$$E_{2y}(x,t) \cong E_2(x) \cdot e^{j[\omega t + \int_0^x k(v)\mathrm{d}v]}, \tag{41}$$

where $E_1(x)$ and $E_2(x)$ are defined in (39) and (40) respectively or by the next steps of the successive approximation presented above.

If we investigate the high-frequency asymptotic values of solution presented in (39) and (40), we get back the known and correct result. If $\omega \to \infty$ then the wavelength $\lambda \to 0$. It can be seen that in this case the high-frequency asymptotic value of (40) is $E_1 = E_{10}\sqrt{Z_0(x)}$ without any reflection, i.e., the asymptotic value $E_2 = 0$. The high-frequency asymptote of the solution (39)–(41) is correct.

4. Conclusions

It was presented that the earlier phenomenological descriptions of the propagation in inhomogeneous media – except the mirror-type, scattering-type descriptions – have an inherent misunderstanding and therefore these methods are wrong. The non-coupled WKB approximation remained usable after this revising of the problem. The cause of the misunderstanding is the wrong 'physical concept' about the structure of the propagating signals which was applied in earlier models.

Using a better 'physical concept' of the structure of the propagating signals and the MIBM it was possible to find correct full-wave solutions for propagation of monochromatic e.m. signals in inhomogeneous media. The presented example was the propagation of a strictly monochromatic wave parallel to a one-dimensional inhomogeneity. However, no problem was found against the generalization of this solution method for non-monochromatic, for oblique propagating [8] and e.g. for moving inhomogeneities. It is important that the final form of the differential equations, which describe the problem, is non-Riccatian (see (35) and (36)).

The main cause of the problem found in the earlier models is the fact that the source- (forward propagating) signal and the reflected (backward propagating, reflected, refracted, scattered) part of the same signal are not signal-parts propagating independently of each other, however, they form a single mode of this propagating energy. We must accept this structure of the signal just at the start of the solution and in this case our 'physical theory' will be correct, the 'objective reality' and the 'physical concept' will be in an acceptable correlation.

Acknowledgements

This work was carried out by the assistance of the Hungarian Space Office and OTKA T037611 funds in cooperation with OTKA F037603 fund.

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