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Exact periodic wave solutions to the coupled Schrödinger–KdV equation and DS equations

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Abstract. The exact solutions for the coupled non-linear partial differential equations are studied by means of the mapping method proposed recently by the author. Taking the coupled Schrödinger–KdV equation and DS equations as examples, abundant periodic wave solutions in terms of Jacobi elliptic functions are obtained. Under the limit conditions, soliton wave solutions are given.

Keywords. Periodic wave solution; non-linear partial differential equation; the mapping method.

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1. Introduction

It has long been known that the exact solutions to some non-linear partial differential equations (PDEs) can be explicitly expressed by the Jacobi elliptic functions. Recently, searching for the Jacobi elliptic periodic wave solutions to the non-linear PDEs attracts considerable interest [1–6]. The Jacobi elliptic functions $sn\xi = sn(\xi|m)$, $cn\xi = cn(\xi|m)$, and $dn\xi = dn(\xi|m)$, where m $(0 < m < 1)$ is the modulus of the elliptic function, are double periodic and possess the following properties:

$$\begin{aligned} &sn^2\xi + cn^2\xi = 1, \quad dn^2\xi + m^2 sn^2\xi = 1, \\ &(sn\xi)' = cn\xi dn\xi, \quad (cn\xi)' = -sn\xi dn\xi, \\ &(dn\xi)' = -m^2 sn\xi cn\xi. \end{aligned} \tag{1}$$

When $m \to 0$, the Jacobi elliptic functions degenerate to the triangular functions, i.e.,

$$sn\xi \to \sin\xi, \quad cn\xi \to \cos\xi, \quad dn\xi \to 1. \tag{2}$$

When $m \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, i.e.,

$$sn\xi \to \tanh\xi, \quad cn\xi \to \operatorname{sech}\xi, \quad dn\xi \to \operatorname{sech}\xi. \tag{3}$$

Detailed explanations about Jacobi elliptic functions can be found in [7–9]. On the other hand, there are many methods for finding special solutions of a non-linear PDE. Some of the most important methods are the inverse scattering transformation [10], the bilinear method [11], symmetry reductions [12], Bäcklund and Darboux transformations [13] and so on. Very recenlty, we proposed the mapping method [4–6] to obtain abundant travelling wave solutions to some non-linear PDEs. In limit conditions, more solitary wave solutions and shock wave (kink wave) solutions may be obtained. The basic idea of the mehtod is as follows: For a given non-linear evolution equation, say in two independent variables,

$$N(u, u_t, u_x, \ldots) = 0. \tag{4}$$

We search for its travelling wave solution of the form

$$u(x,t) \equiv u(\xi), \quad \xi = kx - \omega t, \tag{5}$$

where k, ω are constants to be determined. Without loss of generality, we define $k > 0$. Substituting eq. (5) into eq. (4) yields an ordinary differential equation, the solution of which is searched for in the form

$$u(\xi) = \sum_{i=0}^{n} A_i f^i, \tag{6}$$

where n is a positive integer that can be determined by balancing the linear term of highest order with non-linear term in eq. (4), A_i are the constants to be determined, and f satisfies the elliptic equation of first kind

$$f'' = pf + qf^3, \quad f'^2 = pf^2 + \frac{1}{2}qf^4 + r. \tag{7}$$

Here the prime denotes the derivative with respect to ξ, and p, q and r are the three parameters to be determined. After substituting eq. (6) into the ordinary differential equation and using eq. (7), the constants A_i, k, ω, p, q and r may be determined. In general, if any of the parameters is left unspecified, then it is to be regarded as arbitrary for the solution of eq. (4). The algebra mapping relation is thus established through eq. (6) between the solution to eq. (7) and that of eq. (4). If we assume $f = \tanh\xi$ in the expansion (6), the method is called tanh-function method [14–16], $f = \operatorname{sech}\xi$, sech-function method [17,18], $f = sn\xi$, $cn\xi$, $dn\xi$, Jacobi elliptic function method [1–3]. We choose eq. (7) because $\tanh\xi$, $\operatorname{sech}\xi$, $sn\xi$, $cn\xi$, and $dn\xi$ are all the solutions of it for the appropriate parameters p, q and r, which will be seen in the study of the exact solutions to the systems in §2. Thus we may obtain multiple exact solutions in the unified way by introducing eq. (7) and much tedious and repetitive calculation can be avoided. In this paper, the mapping method will be used to study the exact solutions for the coupled non-linear PDEs. The outline of the paper is as follows: In §2 we obtain abundant Jacobi elliptic periodic wave solutions to the coupled Schrödinger–KdV equation and DS equations and study their limit cases. In §3 we discuss our results and a simple review of the mapping method is given.

2. Application

2.1 *The coupled Schrödinger–KdV equation*

The coupled Schrödinger–KdV equation

$$\begin{aligned} &iu_t = u_{xx} + uv, \\ &v_t + 6vv_x + v_{xxx} = (|u|^2)_x, \end{aligned} \tag{8}$$

describes various processes in dusty plasma, such as Langmuir, dust-acoustic wave and electromagnetic waves [19–21]. A kind of soliton solution was obtained by Hase and Satsuma [22]. Here the abundant Jacobi elliptic periodic wave solutions to eq. (8) will be reported and the limit cases are studied. For eq. (8), we seek the following solution:

$$u = \phi(\xi)\mathrm{e}^{i(Kx-\Omega t)}, \quad v = \psi(\xi), \tag{9}$$

where $\xi = kx - \omega t$ and $\phi(\xi)$ and $\psi(\xi)$ are real functions. Substituting eq. (9) into eq. (8), setting $\omega = -2Kk$ and integrating the equation obtained above once, we have

$$\begin{aligned} &k^2\phi'' - (K^2 + \Omega)\phi + \phi\psi = 0, \\ &k^2\psi'' + 3\psi^2 + 2K\psi - \phi^2 = C, \end{aligned} \tag{10}$$

where C is the integration constant. According to the mapping method, we assume that eq. (10) has the solution of the form

$$\begin{aligned} &\phi = A_0 + A_1 f + A_2 f^2, \\ &\psi = B_0 + B_1 f + B_2 f^2, \end{aligned} \tag{11}$$

where A_i and B_i are constants to be determined, and f satisfies eq. (7). The substitution of eq. (11) into eq. (10) and use of eq. (7) yields (equating the coefficients of like powers of f) two kinds of solutions

$$\begin{aligned} &A_1 = B_1 = 0, \quad A_2 = \pm 3\sqrt{2}qk^2, \\ &B_2 = -3qk^2, \quad A_0 = \pm\frac{\sqrt{2}}{5}(10pk^2 - 3K^2 - 3\Omega - K), \\ &B_0 = -\frac{1}{5}(10pk^2 - 2K^2 - 2\Omega + K), \\ &\Omega = -K^2 - \frac{1}{3}K \pm \frac{5}{3}k^2\sqrt{4p^2 - 6qr}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &A_0 = A_2 = B_1 = 0, \quad B_0 = -pk^2 + K^2 + \Omega, \\ &B_2 = -qk^2, \quad A_1 = \pm\sqrt{q(2pk^2 - 6K^2 - 2K - 6\Omega)}\,k. \end{aligned} \tag{13}$$

Thus we obtain two kinds of exact solutions to eq. (8).

$$u_1 = \left[\pm\frac{\sqrt{2}}{5}(10pk^2 - 3K^2 - 3\Omega - K) \pm 3\sqrt{2}qk^2 f^2(\xi)\right] \mathrm{e}^{i(Kx-\Omega t)},$$
$$v_1 = -\frac{1}{5}(10pk^2 - 2K^2 - 2\Omega + K) - 3qk^2 f^2(\xi), \tag{14}$$

with

$$\Omega = -K^2 - \frac{1}{3}K \pm \frac{5}{3}k^2\sqrt{4p^2 - 6qr}, \tag{15}$$

where $\xi = kx + 2Kkt$, f satisfies eq. (7) and K, k, p, q and r are arbitary constants, and

$$u_2 = \pm\sqrt{q(2pk^2 - 6K^2 - 2K - 6\Omega)}\, kf(\xi)\mathrm{e}^{i(Kx-\Omega t)},$$
$$v_2 = -pk^2 + K^2 + \Omega - qk^2 f^2(\xi), \tag{16}$$

where $\xi = kx + 2Kkt$, f satisfies eq. (7) and K, Ω, k, p, q and r are arbitrary constants. We notice that in all the results of this paper, arbitrary constants should make the expressions non-negative in the square root or denominator not being zero. In what follows, we discuss the specific form of f according to eq. (7).

Case 1. $p = -(1 + m^2)$, $q = 2m^2$, $r = 1$.

The solution of eq. (7) is $f(\xi) = sn\xi$. Thus we get two kinds of Jacobi elliptic periodic wave solutions to eq. (8).

$$u_1 = \Big\{ \pm\frac{\sqrt{2}}{5}[-10(1 + m^2)k^2 - 3K^2 - 3\Omega - K)]$$
$$\pm 6\sqrt{2}m^2k^2 sn^2(kx + 2Kkt)\Big\}\mathrm{e}^{i(Kx-\Omega t)},$$
$$v_1 = -\frac{1}{5}[-10(1 + m^2)k^2 - 2K^2 - 2\Omega + K]$$
$$-6m^2k^2 sn^2(kx + 2Kkt), \tag{17}$$

with

$$\Omega = -K^2 - \frac{1}{3}K \pm \frac{10}{3}k^2\sqrt{1 - m^2 + m^4}, \tag{18}$$

and

$$u_2 = \pm\sqrt{2[-2(1 + m^2)k^2 - 6K^2 - 2K - 6\Omega]}mksn(kx + 2Kkt)\mathrm{e}^{i(Kx-\Omega t)},$$
$$v_2 = (1 + m^2)k^2 + K^2 + \Omega - 2m^2k^2 sn^2(kx + 2Kkt). \tag{19}$$

As $m \to 1$, eqs (17) and (19) degenerate to

$$u_1 = \left[\pm\frac{\sqrt{2}}{5}(-20k^2 - 3K^2 - 3\Omega - K) \pm 6\sqrt{2}k^2\tanh^2(kx + 2Kkt)\right]\mathrm{e}^{i(kx-\Omega t)},$$
$$v_1 = -\frac{1}{5}(-20k^2 - 2K^2 - 2\Omega + K) - 6k^2\tanh^2(kx + 2Kkt), \tag{20}$$

with

$$\Omega = -K^2 - \frac{1}{3}K \pm \frac{10}{3}k^2, \tag{21}$$

and

$$\begin{aligned} u_2 &= \pm\sqrt{2(-4k^2 - 6K^2 - 2K - 6\Omega)}\, k \tanh(kx + 2Kkt)\mathrm{e}^{i(Kx-\Omega t)}, \\ v_2 &= 2k^2 + K^2 + \Omega - 2k^2 \tanh^2(kx + 2Kkt), \end{aligned} \tag{22}$$

respectively.

Case 2. $p = 2m^2 - 1$, $q = -2m^2$, $r = m'^2 \equiv 1 - m^2$.

It follows from eq. (7) that $f(\xi) = cn\xi$ and eq. (8) has periodic wave solutions

$$\begin{aligned} u_1 &= \left\{ \pm \frac{\sqrt{2}}{5}[10(2m^2 - 1)k^2 - 3K^2 - 3\Omega - K] \right. \\ &\qquad \left. \pm(-6\sqrt{2})m^2k^2cn^2(kx + 2Kkt) \right\} \mathrm{e}^{i(Kx-\Omega t)}, \\ v_1 &= -\frac{1}{5}[10(2m^2 - 1)k^2 - 2K^2 - 2\Omega + K] + 6m^2k^2cn^2(kx + 2Kkt), \end{aligned} \tag{23}$$

with Ω given by eq. (18), and

$$\begin{aligned} u_2 &= \pm\sqrt{2[-2(2m^2 - 1)k^2 + 6K^2 + 2K + 6\Omega]}\, mkcn(kx + 2Kkt)\mathrm{e}^{i(Kx-\Omega t)}, \\ v_2 &= -(2m^2 - 1)k^2 + K^2 + \Omega + 2m^2k^2cn^2(kx + 2Kkt). \end{aligned} \tag{24}$$

As $m \to 1$, eqs (23) and (24) degenerate to

$$\begin{aligned} u_1 &= \left[\pm\frac{\sqrt{2}}{5}(10k^2 - 3K^2 - 3\Omega - K) \pm (-6\sqrt{2})k^2\mathrm{sech}^2(kx + 2Kkt) \right] \mathrm{e}^{i(Kx-\Omega t)}, \\ v_1 &= -\frac{1}{5}(10k^2 - 2K^2 - 2\Omega + K) + 6k^2\mathrm{sech}^2(kx + 2Kkt), \end{aligned} \tag{25}$$

with Ω given by eq. (21), and

$$\begin{aligned} u_2 &= \pm\sqrt{2(-2k^2 + 6K^2 + 2K + 6\Omega)}\, k\mathrm{sech}\,(kx + 2Kkt)\mathrm{e}^{i(Kx-\Omega t)}, \\ v_2 &= -k^2 + K^2 + \Omega + 2k^2\mathrm{sech}^2(kx + 2Kkt), \end{aligned} \tag{26}$$

respectively.

Case 3. $p = 2 - m^2$, $q = -2$, $r = -m'^2$.

The solution of eq. (7) reads $f(\xi) = dn\xi$ and we obtain periodic wave solutions to eq. (8).

$$u_1 = \left\{ \pm \frac{\sqrt{2}}{5}[10(2-m^2)k^2 - 3K^2 - 3\Omega - K) \right.$$
$$\left. \pm(-6\sqrt{2})k^2 dn^2(kx + 2Kkt) \right\} \mathrm{e}^{i(Kx-\Omega t)},$$
$$v_1 = -\frac{1}{5}[10(2-m^2)k^2 - 2K^2 - 2\Omega + K) + 6k^2 dn^2(kx + 2Kkt), \tag{27}$$

with Ω given by eq. (18), and

$$u_2 = \pm\sqrt{2[-2(2-m^2)k^2 + 6K^2 + 2K + 6\Omega]}\, kdn(kx + 2Kkt)\mathrm{e}^{i(Kx-\Omega t)},$$
$$v_2 = -(2-m^2)k^2 + K^2 + \Omega + 2k^2 dn^2(kx + 2Kkt). \tag{28}$$

As $m \to 1$, from eqs (27) and (28) we obtain again eqs (25) and (26) respectively. By choosing different values of p, q and r in eq. (7), we can obtain other Jacobi elliptic function wave solutions to eq. (8), such as $nc\xi$, $nd\xi$, $sc\xi$, $sd\xi$ and $ds\xi$. However, these solutions have singularity and we do not discuss them.

2.2 *The DS equations*

The Davey–Stewartson (DS) equation [23]

$$iu_t + \alpha u_{xx} + u_{yy} + \beta|u|^2 u - 2uv = 0,$$
$$\alpha v_{xx} - v_{yy} - \alpha\beta(|u|^2)_{xx} = 0, \tag{29}$$

where $\alpha = \pm 1$, β is a constant. Equation (29) with $\alpha = 1$ and $\alpha = -1$ are called the DSI and DSII equations, respectively. These equations were introduced in order to discuss the modulational instability of uniform train of weakly non-linear water waves in the two-dimensional space. Here we will obtain a series of Jacobi elliptic periodic wave solutions to eq. (29) for arbitrary constants α and β. By means of the mapping method and using the same procedure as the first example, we obtain the exact solution to eq. (29) as follows:

$$u = \pm\sqrt{\frac{q(\alpha k^2 - l^2)}{\beta}} f(\xi)\mathrm{e}^{i(Kx+Ly-\Omega t)},$$
$$v = \frac{C_0}{\alpha k^2 - l^2} + q\alpha k^2 f^2(\xi), \tag{30}$$

with

$$\Omega = -p(\alpha k^2 + l^2) + \alpha K^2 + L^2 + \frac{2C_0}{\alpha k^2 - l^2}, \tag{31}$$

where $\xi = kx + ly - 2(\alpha Kk + Ll)t$, f satisfies eq. (7) and K, L, k, l, p, q and r are arbitrary constants. In the following, we discuss the specific form of eq. (30).

Case 1. $p = -(1+m^2)$, $q = 2m^2$, $r = 1$.

The periodic wave solution of eq. (29) reads

$$u = \pm\sqrt{\frac{2(\alpha k^2 - l^2)}{\beta}}\, msn[kx + ly - 2(\alpha Kk + Ll)t]\mathrm{e}^{i(Kx+Ly-\Omega t)},$$
$$v = \frac{C_0}{\alpha k^2 - l^2} + 2\alpha k^2 m^2 sn^2[kx + ly - 2(\alpha Kk + Ll)t], \tag{32}$$

with

$$\Omega = (1 + m^2)(\alpha k^2 + l^2) + \alpha K^2 + L^2 + \frac{2C_0}{\alpha k^2 - l^2}. \tag{33}$$

As $m \to 1$, eq. (32) degenerates to

$$u = \pm\sqrt{\frac{2(\alpha k^2 - l^2)}{\beta}}\, \tanh[kx + ly - 2(\alpha Kk + Ll)t]\mathrm{e}^{i(Kx+Ly-\Omega t)},$$
$$v = \frac{C_0}{\alpha k^2 - l^2} + 2\alpha k^2 \tanh^2[kx + ly - 2(\alpha Kk + Ll)t], \tag{34}$$

with

$$\Omega = 2(\alpha k^2 + l^2) + \alpha K^2 + L^2 + \frac{2C_0}{\alpha k^2 - l^2}. \tag{35}$$

Case 2. $p = 2m^2 - 1$, $q = -2m^2$, $r = m'^2$.

We have the exact solution to eq. (29)

$$u = \pm\sqrt{\frac{-2(\alpha k^2 - l^2)}{\beta}}\, mcn[kx + ly - 2(\alpha Kk + Ll)t]\mathrm{e}^{i(Kx+Ly-\Omega t)},$$
$$v = \frac{C_0}{\alpha k^2 - l^2} - 2\alpha k^2 m^2 cn^2[kx + ly - 2(\alpha Kk + Ll)t], \tag{36}$$

with

$$\Omega = -(2m^2 - 1)(\alpha k^2 + l^2) + \alpha K^2 + L^2 + \frac{2C_0}{\alpha k^2 - l^2}. \tag{37}$$

As $m \to 1$, eq. (36) degenerates to

$$u = \pm\sqrt{\frac{-2(\alpha k^2 - l^2)}{\beta}}\, \mathrm{sech}[kx + ly - 2(\alpha Kk + Ll)t]\mathrm{e}^{i(Kx+Ly-\Omega t)},$$
$$v = \frac{C_0}{\alpha k^2 - l^2} - 2\alpha k^2 \mathrm{sech}^2[kx + ly - 2(\alpha Kk + Ll)t], \tag{38}$$

with

$$\Omega = -(\alpha k^2 + l^2) + \alpha K^2 + L^2 + \frac{2C_0}{\alpha k^2 - l^2}. \tag{39}$$

Case 3. $p = 2 - m^2$, $q = -2$, $r = -m'^2$.

The exact periodic wave solution to eq. (29) is

$$u = \pm\sqrt{\frac{-2(\alpha k^2 - l^2)}{\beta}}\, dn[kx + ly - 2(\alpha Kk + Ll)t]\mathrm{e}^{i(Kx+Ly-\Omega t)},$$
$$v = \frac{C_0}{\alpha k^2 - l^2} - 2\alpha k^2 dn^2[kx + ly - 2(\alpha Kk + Ll)t], \tag{40}$$

with

$$\Omega = -(2 - m^2)(\alpha k^2 + l^2) + \alpha K^2 + L^2 + \frac{2C_0}{\alpha k^2 - l^2}. \tag{41}$$

As $m \to 1$, we obtain eq. (38) again from eq. (40).

3. Conclusion

Abundant Jacobi elliptic periodic wave solutions to the coupled Schrödinger–KdV equation and DS equations are obtained by means of the mapping method. The limit cases are studied and soliton solutions are obtained. It has been shown that only minimal algebra is needed to find multiple travelling wave solutions by the mapping method. Moreover, this method is readily applicable to a large variety of non-linear PDEs as long as odd- and even-order derivative terms do not coexist in the equation under consideration.

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