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Kelvin–Helmholtz instability in a rotating ideally conducting inhomogeneous plasma

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Abstract. The Kelvin–Helmholtz instability in sheared magnetohydrodynamic flow of an ideally conducting rotating inhomogeneous compressible plasma is investigated. The asymptotic behaviour in x of the Kelvin–Helmholtz eigenfunctions for the case of finite compressibility in the presence of rotation is discussed and instability condition is derived. In the incompressible limit, a dispersion relation is derived which has been solved numerically and discussed in detail. It is found that the inhomogeneous system is unstable in an incompressible plasma.

Keywords. Magnetohydrodynamic; Kelvin–Helmholtz instability; rotation; eigenfunctions.

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1. Introduction

The Kelvin–Helmholtz (K–H) instability occurs when there is a relative motion between the two fluids separated by the interface. In plasma it occurs when plasma flows perpendicular to the magnetic field. The K–H instability is of utmost importance in understanding and investigating a variety of physical situations in space, astrophysical and geophysical plasmas. The problems involving sheared plasma flow such as the stability of interface between the solar wind and the magnetosphere, interaction between adjacent streams of different velocities in the solar wind and the dynamic structure of the comet tails, have been studied by several researchers. Early investigations of Kelvin–Helmholtz instability were concerned with the magnetohydrodynamic (MHD) stability of tangential velocity discontinuity in an incompressible plasma. The linear stability results are given by Chandrashekhar [1]. The effect of compressibility on the stability of the tangential velocity discontinuity has been studied in detail by Sen [2], Fejer [3], Talwar [4], Southwood [5] and Pu and Kivelson [6]. Miura and Pritchett [7] have investigated the stability of compressible magnetized finite width shear layer for a hyperbolic tangent velocity profile. Ray and Erschkovich [8] have studied the homogeneous compressible plasma containing a velocity shear and a magnetic field. They concluded that in the incompressible limit the shear layer is stable to modes making arbitrary angles in the flow provided the total velocity jump across the shear layers is less than the Alfvén speed. Roy Choudhury and Lovelace [9] and Roy Choudhury [10] made the study of the stability of finite thickness compressible shear layer for different linear velocity profiles considering magnetic field in z-direction and in the y–z plane. The transport of energy and momentum on both sides of the magnetopause caused by unstable compressible K–H waves were investigated by Pu and Kivelson [11] using a linear approximation. They showed that the compressible K–H instability has an important effect on the coupling between the solar wind and the magnetosphere. The K–H instability converts the energy of relative bulk plasma flow into other forms, i.e., vortex kinetic energy and magnetic energy. The study of K–H instability was extended to a compressible isotropic pressure plasma with different orientations of magnetic field and flow by Gonzalez and Gratton [12] and Thomas [13]. In the present paper, the linear stability analysis is performed for the K–H instability in sheared magnetohydrodynamic flow of an ideally conducting rotating inhomogeneous compressible plasma. This work aims at presenting the extension of the work of Miura and Pritchett [7] to a rotating inhomogeneous compressible plasma. In the stability analysis, a single second-order differential equation has been obtained. The asymptotic behaviour in x of the K–H eigenfunctions for the case of finite compressibility in the presence of rotation is discussed and instability condition is derived. In the incompressible limit, a dispersion relation is derived which has been solved numerically and discussed in detail. In §2, we derive perturbation equations and obtain eigenvalue equation for the stability. Asymptotic behaviour of K–H instability is discussed in §3. In §4, the dispersion relation in the incompressible limit is derived and discussed in detail.

2. Perturbation equations and eigenvalue equation for the stability

The relevant set of ideal MHD equations governing the rotating ideally conducting inhomogeneous plasma is

$$\frac{\partial \rho}{\partial t} + \nabla .(\rho \mathbf{V}) = 0, \tag{1}$$

$$\rho \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = -\nabla \left(p + \frac{B_0^2}{2\mu_0} \right) + \mu_0^{-1} (\mathbf{B}.\nabla)\mathbf{B} + 2\rho(\mathbf{V} \times \mathbf{\Omega}) + \mathbf{g}\rho, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(p\rho^{-\gamma}) = 0, \tag{4}$$

with $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$, where ρ is the mass density, $\mathbf{V}$ is the velocity of the plasma, $\mathbf{B}$ is the magnetic field, p is the pressure, $\mathbf{\Omega}$ is the vorticity vector and γ is the adiabatic gas constant. In equilibrium, we consider the magnetic field, velocity flow, gravitational field and rotation to have the following form:

$$\mathbf{B_0} = B_{0y}(x)\hat{y} + B_{0z}(x)\hat{z},$$

$$\mathbf{V_0} = V_{0y}\hat{y},$$

$$\mathbf{g} = -g\hat{x},$$

$$\boldsymbol{\Omega} = \Omega_0\hat{z}.$$

The condition for equilibrium is

$$\frac{\mathrm{d}}{\mathrm{d}x}[p_0 + (B_{0y}^2 + B_{0z}^2)/2\mu_0] = -\rho_0 g + 2\rho_0\Omega_0 V_{0y}, \tag{5}$$

where $p_0(x)$ and $\rho_0(x)$ are the pressure and the mass density, respectively. Linearizing the MHD equations (1)–(4) and then taking the first-order perturbation quantities of the form $\delta f(x)\exp[i(k_y\hat{y} + k_z\hat{z} - \omega t)]$, where ω is the growth rate, k_y and k_z are the wave numbers in y- and z-directions respectively, we obtain

$$-i\tilde{\omega}\delta\rho = -(\mathrm{d}\rho_0/\mathrm{d}x)\delta v_x - \rho_0\nabla.(\delta\mathbf{v}), \tag{6}$$

$$-i\rho_0\tilde{\omega}\delta v_x = -\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta p^*\right] + i(f/\mu_0)\delta B_x - g\delta\rho + 2\Omega_0\rho_0\delta v_y, \tag{7}$$

$$\begin{aligned}-i\rho_0\tilde{\omega}\delta v_y = {} & -\rho_0 V_{0y}'\delta v_x - ik_y\delta p^* + i(f/\mu_0)\delta B_y \\ & +\mu_0^{-1}B'_{0y}\delta B_x - 2\Omega_0\rho_0\delta v_x,\end{aligned} \tag{8}$$

$$-i\rho_0\tilde{\omega}\delta v_z = -ik_z\delta p^* + i(f/\mu_0)\delta B_z + \mu_0^{-1}B_{0z}'\delta B_x, \tag{9}$$

$$-i\Omega\delta B_x = if\delta v_x, \tag{10}$$

$$-i\tilde{\omega}\delta B_y = V'_{0y}\delta B_x - B_{0y}\nabla\cdot(\delta\mathbf{v}) + if\delta v_y - B'_{0y}\delta v_x, \tag{11}$$

$$-i\tilde{\omega}\delta B_z = if\delta v_z - B_{0z}\nabla\cdot(\delta\mathbf{v}) - B'_{0z}\delta v_x, \tag{12}$$

$$-i\tilde{\omega}\delta p = -p'_0\delta v_x - \gamma p_0\nabla\cdot(\delta\mathbf{v}). \tag{13}$$

Here $\delta\rho, \delta v_x, \delta v_y, \delta v_z, \delta B_x, \delta B_y, \delta B_z$ and δp are perturbations in the density, velocity, magnetic field and pressure respectively. $\tilde{\omega}(= \omega - k_yV_{0y})$ and $\delta p^* = (\delta p + (\mathbf{B_0}\cdot\delta\mathbf{B})/\mu_0)$ are Doppler shift frequency and total pressure perturbations respectively, and $f = (\mathbf{k}\cdot\mathbf{B_0}) = k_yB_{0y} + k_zB_{0z}$. Dash ($\prime$) on different parameters stands for differentiation with respect to x.

From the above equations, we get

$$\begin{aligned}-i\tilde{\omega}\delta p^* = {} & \left[\rho_0 g g_\mathrm{A} + \frac{2f\Omega_0B_{0y}}{\tilde{\omega}\mu_0} - 2\rho_0\Omega_0 g_\mathrm{A}V_{0y}\right]\delta v_x \\ & -[g_\mathrm{A}\gamma p_0 + (B_{0y}^2 + B_{0z}^2)/\mu_0]\nabla\cdot(\delta\mathbf{v}),\end{aligned} \tag{14}$$

where $g_\mathrm{A} = (1 - f^2/\mu_0\rho_0\tilde{\omega}^2)$.

Using eq. (14), the compression term $\nabla\cdot(\delta v)$ can be written in terms of the total pressure perturbation δp^* as

$$\nabla\cdot(\boldsymbol{\delta}\mathbf{v}) = \frac{i\tilde{\omega}}{\gamma p_0 H_{\mathrm{A}}}\left(H_{\mathrm{B}}\delta p^* - H_{\mathrm{C}}\frac{\mathrm{d}}{\mathrm{d}x}\delta p^*\right), \tag{15}$$

where

$$H_{\mathrm{A}} = \left[m_{\mathrm{A}} + \frac{2gf\Omega_0 B_{0y}}{\tilde{\omega}^3\mu_0 F_{\mathrm{A}}\gamma p_0} - \frac{2g\rho_0\Omega_0 g_{\mathrm{A}} V_{0y}}{\tilde{\omega}^2 F_{\mathrm{A}}\gamma p_0} + \frac{2f\Omega_0 g g_{\mathrm{A}} B_{0y}}{\tilde{\omega}^3\mu_0 F_{\mathrm{A}}\gamma p_0}\right.$$
$$\left. + \frac{4f^2\Omega_0^2 B_{0y}^2}{\rho_0\mu_0^2\tilde{\omega}^4 F_{\mathrm{A}}\gamma p_0} - \frac{4f\rho_0\Omega_0^2 B_{0y} g_{\mathrm{A}} V_{0y}}{\mu_0\rho_0\tilde{\omega}^3 F_{\mathrm{A}}\gamma p_0}\right],$$

$$H_{\mathrm{B}} = \left[1 + \frac{2g\Omega_0 k_y}{\tilde{\omega}^3 F_{\mathrm{A}}} + \frac{4f\Omega_0^2 k_y B_{0y}}{\rho_0\tilde{\omega}^4\mu_0 g_{\mathrm{A}} F_{\mathrm{A}}} - \frac{4\Omega_0^2 k_y V_{0y}}{\tilde{\omega}^3 F_{\mathrm{A}}}\right],$$

$$H_{\mathrm{C}} = \left[\frac{g g_{\mathrm{A}}}{\tilde{\omega}^2 F_{\mathrm{A}}} + \frac{2f\Omega_0 B_{0y}}{\rho_0\tilde{\omega}^3\mu_0 F_{\mathrm{A}}} - \frac{2\Omega_0 g_{\mathrm{A}} V_{0y}}{\tilde{\omega}^2 F_{\mathrm{A}}}\right],$$

$$m_{\mathrm{A}} = \left[g_{\mathrm{A}} + V_{\mathrm{A}}^2/c_{\mathrm{s}}^2 + g^2 g_{\mathrm{A}}/\tilde{\omega}^2 F_{\mathrm{A}} c_{\mathrm{s}}^2\right],$$

$$F_{\mathrm{A}} = \left[f_{\mathrm{A}} - \frac{2\Omega_0 {V'}_{0y}}{\tilde{\omega}^2} - \frac{4\Omega_0^2}{\tilde{\omega}^2 g_{\mathrm{A}}}\right],$$

$$f_{\mathrm{A}} = \left[g_{\mathrm{A}} + \frac{g}{\rho_0\tilde{\omega}^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right], \quad g_{\mathrm{A}} = \left[1 - f^2/\mu_0\rho_0\tilde{\omega}^2\right],$$

and V_{A}, c_{s} are Alfvén and sound speeds respectively.

Substituting for $\delta B_x, \delta B_y, \delta B_z$ in terms of δp^* into the condition $\nabla\cdot(\delta\mathbf{B}) = \mathbf{0}$, we get,

$$\frac{1}{\rho_0\tilde{\omega}^2 F_{\mathrm{A}}}\left[1 - \frac{gH_{\mathrm{C}}}{H_{\mathrm{A}} c_{\mathrm{s}}^2} - \frac{2f\Omega_0 B_{0y} H_{\mathrm{C}}}{\rho_0\mu_0 H_{\mathrm{A}}\tilde{\omega} c_{\mathrm{s}}^2}\right]\frac{\mathrm{d}^2\delta p^*}{\mathrm{d}x^2}$$
$$+ \left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\rho_0\tilde{\omega}^2 F_{\mathrm{A}}}\left[1 - \frac{gH_{\mathrm{C}}}{H_{\mathrm{A}} c_{\mathrm{s}}^2} - \frac{2f\Omega_0 B_{0y} H_{\mathrm{C}}}{\rho_0 H_{\mathrm{A}}\mu_0\tilde{\omega} c_{\mathrm{s}}^2}\right]\right]\right.$$
$$+ \frac{1}{\rho_0\tilde{\omega}^2 F_{\mathrm{A}}}\left[\frac{gH_{\mathrm{B}}}{H_{\mathrm{A}} c_{\mathrm{s}}^2} + \frac{2f\Omega_0 B_{0y} H_{\mathrm{B}}}{\rho_0\mu_0 H_{\mathrm{A}}\tilde{\omega} c_{\mathrm{s}}^2}\right] - \frac{H_{\mathrm{C}}}{\gamma p_0 H_{\mathrm{A}} g_{\mathrm{A}}}$$
$$\left. - \frac{2g\Omega_0 k_y H_{\mathrm{C}}}{\rho_0\tilde{\omega}^3 g_{\mathrm{A}} F_{\mathrm{A}} H_{\mathrm{A}} c_{\mathrm{s}}^2} - \frac{4f\Omega_0^2 k_y B_{0y} H_{\mathrm{C}}}{\rho_0^2\tilde{\omega}^4\mu_0 g_{\mathrm{A}} F_{\mathrm{A}} H_{\mathrm{A}} c_{\mathrm{s}}^2}\right\}\frac{\mathrm{d}\delta p^*}{\mathrm{d}x}$$
$$+ \left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\rho_0\tilde{\omega}^2 F_{\mathrm{A}}}\left[\frac{gH_{\mathrm{B}}}{H_{\mathrm{A}} c_{\mathrm{s}}^2}\frac{2f\Omega_0 B_{0y} H_{\mathrm{B}}}{\rho_0\mu_0 H_{\mathrm{A}}\tilde{\omega} c_{\mathrm{s}}^2} - \frac{2\Omega_0 k_y}{\tilde{\omega} g_{\mathrm{A}}}\right]\right]\right.$$
$$+ \frac{2g\Omega_0 k_y H_{\mathrm{B}}}{\rho_0\tilde{\omega}^3 g_{\mathrm{A}} F_{\mathrm{A}} H_{\mathrm{A}} c_{\mathrm{s}}^2} + \frac{4f\Omega_0^2 H_{\mathrm{B}} k_y B_{0y}}{\rho_0^2\mu_0\tilde{\omega}^4 g_{\mathrm{A}} F_{\mathrm{A}} H_{\mathrm{A}} c_{\mathrm{s}}^2} + \frac{H_{\mathrm{B}}}{\gamma p_0 H_{\mathrm{A}} g_{\mathrm{A}}}$$
$$\left. - \frac{(k_y^2 + k_z^2)}{\rho_0\tilde{\omega}^2 g_{\mathrm{A}}} - \frac{4\Omega_0^2 k_y^2}{\rho_0\tilde{\omega}^4 g_{\mathrm{A}}^2 F_{\mathrm{A}}}\right\}\delta p^* = 0. \tag{16}$$

This is the eigenvalue equation for the stability of the general MHD slab equilibrium in the presence of uniform rotation. In the absence of rotation, the above equation reduces to the eigenvalue equation obtained by Miura and Pritchett [7].

In the absence of gravity, eq. (16) simplifies considerably:

$$\begin{aligned}&\frac{g_{\mathrm{A}}}{F_{\mathrm{A}}}\left[1-\frac{2f\Omega_0 B_{0y}H_{\mathrm{C}}}{\rho_0\mu_0 H_{\mathrm{A}}\tilde{\omega}c_{\mathrm{s}}^2}\right]\frac{\mathrm{d}^2\delta p^*}{\mathrm{d}x^2}\\&\quad+\left\{\rho_0\tilde{\omega}^2 g_{\mathrm{A}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\rho_0\tilde{\omega}^2F_{\mathrm{A}}}\left[1-\frac{2f\Omega_0B_{0y}H_{\mathrm{C}}}{\rho_0H_{\mathrm{A}}\mu_0\tilde{\omega}c_{\mathrm{s}}^2}\right]\right]\right.\\&\quad\left.+\frac{2g_{\mathrm{A}}f\Omega_0B_{0y}H_{\mathrm{B}}}{F_{\mathrm{A}}\rho_0\mu_0\tilde{\omega}H_{\mathrm{A}}c_{\mathrm{s}}^2}-\frac{\tilde{\omega}^2H_{\mathrm{C}}}{c_{\mathrm{s}}^2H_{\mathrm{A}}}-\frac{4f\Omega_0^2k_yB_{0y}H_{\mathrm{C}}}{\rho_0\tilde{\omega}^2\mu_0F_{\mathrm{A}}H_{\mathrm{A}}c_{\mathrm{s}}^2}\right\}\frac{\mathrm{d}\delta p^*}{\mathrm{d}x}\\&\quad+\left\{\rho_0\tilde{\omega}^2g_{\mathrm{A}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\rho_0\tilde{\omega}^2F_{\mathrm{A}}}\left[\frac{2f\Omega_0B_{0y}H_{\mathrm{B}}}{\rho_0\mu_0H_{\mathrm{A}}\tilde{\omega}c_{\mathrm{s}}^2}-\frac{2\Omega_0k_y}{\tilde{\omega}g_{\mathrm{A}}}\right]\right]\right.\\&\quad\left.+\frac{4f\Omega_0^2H_{\mathrm{B}}k_yB_{0y}}{\rho_0\mu_0\tilde{\omega}^2F_{\mathrm{A}}H_{\mathrm{A}}c_s^2}\frac{\tilde{\omega}^2H_{\mathrm{B}}}{c_{\mathrm{s}}^2H_{\mathrm{A}}}-(k_y^2+k_z^2)-\frac{4\Omega_0^2k_y^2}{\tilde{\omega}^2g_{\mathrm{A}}F_{\mathrm{A}}}\right\}\delta p^*=0.\end{aligned}\tag{17}$$

If the transverse case (i.e. $B_{0y}=0$ with $k_z=0$) is considered in the absence of rotation, the above equation takes the form as the eigenvalue equation in fluid dynamics except that the magnetosonic speed appears in place of sound speed, i.e., c_{s} is replaced by $(c_{\mathrm{s}}^2+V_{\mathrm{A}}^2)^{1/2}$.

3. Asymptotic behaviour of K–H eigenfunctions

We now discuss the asymptotic behaviour in x of the K–H eigenfunctions for the case of finite compressibility in the presence of rotation. If we assume all the plasma parameters to be uniform as $x\to\pm\infty$, as has been considered by Miura and Pritchett [7], eq. (17) reduces to

$$\frac{\mathrm{d}^2\delta p^*}{\mathrm{d}x^2}+P\frac{\mathrm{d}\delta p^*}{\mathrm{d}x}+Q\delta p^*=0,\tag{18}$$

where $P=A_2/A_1$, $Q=A_3/A_1$,

$$A_1=\frac{g_{\mathrm{A}}}{F_{\mathrm{A}}}\left[1-\frac{2f\Omega_0B_{0y}H_{\mathrm{C}}}{\rho_0\mu_0\tilde{\omega}c_{\mathrm{s}}^2H_{\mathrm{A}}}\right],$$

$$A_2=\left[\frac{2g_{\mathrm{A}}f\Omega_0B_{0y}H_{\mathrm{B}}}{F_{\mathrm{A}}\rho_0\mu_0\tilde{\omega}H_{\mathrm{A}}c_{\mathrm{s}}^2}-\frac{\tilde{\omega}^2H_{\mathrm{C}}}{c_{\mathrm{s}}^2H_{\mathrm{A}}}-\frac{4f\Omega_0^2k_yB_{0y}H_{\mathrm{C}}}{\rho_0\tilde{\omega}^2\mu_0F_{\mathrm{A}}H_{\mathrm{A}}c_{\mathrm{s}}^2}\right],$$

$$A_3=\left[\frac{4f\Omega_0^2H_{\mathrm{B}}k_yB_{0y}}{\rho_0\mu_0\tilde{\omega}^2F_{\mathrm{A}}H_{\mathrm{A}}c_{\mathrm{s}}^2}+\frac{\tilde{\omega}^2H_{\mathrm{B}}}{c_{\mathrm{s}}^2H_{\mathrm{A}}}-(k_y^2+k_z^2)-\frac{4\Omega_0^2k_y^2}{\tilde{\omega}^2g_{\mathrm{A}}F_{\mathrm{A}}}\right].$$

In the absence of rotation, eq. (18) reduces to the equation, asymptotic behaviour of which has been discussed by Miura and Pritchett [7]. After removing the first derivative of eq. (18), we require the following condition for eigenmode to be evanescent as $x \to \pm\infty$, for the case of finite compressibility in the presence of rotation,

$$\lambda = \left[\frac{(k_y^2 + k_z^2) g_{\mathrm{A}}}{F_{\mathrm{A}}} - \frac{\tilde{\omega}^2 g_{\mathrm{A}} M_{\mathrm{R}}}{(g_{\mathrm{A}} c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2) F_{\mathrm{A}}} \right] > 0, \tag{19}$$

where

$$\begin{aligned} M_{\mathrm{R}} = \Bigg[& \frac{H_{\mathrm{B}}}{H_{\mathrm{A}}} + \frac{4 f \Omega_0^2 k_y B_{0y} H_{\mathrm{B}}}{\rho_0 \mu_0 \tilde{\omega}^4 F_{\mathrm{A}} H_{\mathrm{A}}} - \frac{4 \Omega_0^2 k_y^2 (g_{\mathrm{A}} c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2)}{\tilde{\omega}^4 F_{\mathrm{A}} g_{\mathrm{A}}} \\ & - \frac{4 f^2 \Omega_0^3 k_y B_{0y}^2 H_{\mathrm{B}} H_{\mathrm{C}}}{\rho_0^2 \mu_0^2 \tilde{\omega}^5 H_{\mathrm{A}} (g_{\mathrm{A}} c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2)} - \frac{f \Omega_0 B_{0y} H_{\mathrm{B}} H_{\mathrm{C}}}{\rho_0 \mu_0 \tilde{\omega} H_{\mathrm{A}}^2 (g_{\mathrm{A}} c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2)} \\ & + \frac{2 f \Omega_0 B_{0y} H_{\mathrm{C}} (k_y^2 + k_z^2)}{\rho_0 \mu_0 \tilde{\omega}^3 H_{\mathrm{A}}} + \frac{8 f \Omega_0^3 B_{0y} k_y^2 H_{\mathrm{C}}}{\rho_0 \mu_0 \tilde{\omega}^5 F_{\mathrm{A}} H_{\mathrm{A}} g_{\mathrm{A}}} \\ & - \frac{f^2 \Omega_0^2 B_{0y}^2 H_{\mathrm{B}}^2 g_{\mathrm{A}}}{\rho_0^2 \mu_0^2 \tilde{\omega}^4 H_{\mathrm{A}} F_{\mathrm{A}} (g_{\mathrm{A}} c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2)} - \frac{\tilde{\omega}^2 H_{\mathrm{C}}^2 F_{\mathrm{A}}}{4 (g_{\mathrm{A}} c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2) H_{\mathrm{A}}^2 g_{\mathrm{A}}} \\ & - \frac{4 \Omega_0^4 f^2 k_y^2 B_{0y}^2 H_{\mathrm{C}}^2}{\rho_0^2 \mu_0^2 \tilde{\omega}^6 F_{\mathrm{A}} g_{\mathrm{A}} H_{\mathrm{A}}^2 (g_{\mathrm{A}} c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2)} - \frac{2 f \Omega_0^2 k_y B_{0y} H_{\mathrm{C}}^2}{\rho_0 \mu_0 g_{\mathrm{A}} H_{\mathrm{A}}^2 \tilde{\omega}^2 (g_{\mathrm{A}}^2 c_{\mathrm{s}}^2 + V_{\mathrm{A}}^2)} \Bigg]. \end{aligned}$$

For the validity of the medium and short wavelength modes, we use the approximation $|\gamma| \ll (k_y V_0/2)$, where V_0 is the total velocity jump. From eq. (19), we now arrive at the condition

$$\lambda = \left[(k_y^2 + k_z^2)\alpha - \frac{k_y^2 M_S^2 M_{\mathrm{A}}^2 M_{\mathrm{R}}}{4} \right] \Big/ \alpha\beta, \tag{20}$$

where

$$\begin{aligned} M_{\mathrm{R}} = \Bigg[& \frac{m_3}{m_2} + \frac{64 (\mathbf{k} \cdot \mathbf{B_0}) B_{0y} d^2 m_3}{k_y B_0^2 \beta m_2 M_{\mathrm{A}}^4} - \frac{64 \alpha d^2}{M_{\mathrm{A}}^4 M_{\mathrm{s}}^2 m_1^2 \beta} \\ & - \frac{64 (\mathbf{k} \cdot \mathbf{B_0})^2 B_{0y}^2 d^3 M_S^2 m_3 m_4}{k_y^2 B_0^4 M_{\mathrm{A}}^4 m_2^2} - \frac{2 (\mathbf{k} \cdot \mathbf{B_0}) B_{0y} d m_3 m_4 M_S^2}{k_y B_0^2 M_{\mathrm{A}} m_2^2} \\ & + \frac{16 (\mathbf{k} \cdot \mathbf{B_0}) B_{0y} d m_4 (k_y^2 + k_z^2)}{k_y^3 B_0^2 M_{\mathrm{A}}^3 m_2} + \frac{256 (\mathbf{k} \cdot \mathbf{B_0}) B_{0y} d^3 m_4}{k_y B_0^2 M_{\mathrm{A}}^5 m_1^2 m_2 \beta} \\ & - \frac{16 (\mathbf{k} \cdot \mathbf{B_0})^2 B_{0y}^2 d^2 m_3^2 M_S^2}{k_y^2 B_0^4 M_{\mathrm{A}}^4 m_2^2} - \frac{M_S^2 M_{\mathrm{A}}^2 \beta m_4^2}{16 \alpha m_2^2} \\ & - \frac{256 (\mathbf{k} \cdot \mathbf{B_0})^2 B_{0y}^2 d^4 m_3^2 m_4^2}{k_y^2 B_0^2 M_{\mathrm{A}}^6 m_1^2 \beta \alpha m_2^2} - \frac{8 (\mathbf{k} \cdot \mathbf{B_0}) B_{0y} d^2 m_4^2 M_S^2}{k_y B_0^4 M_{\mathrm{A}}^2 m_1 m_2^2} \Bigg], \end{aligned}$$

$$m_1 = \left[1 - \frac{4 (\mathbf{k} \cdot \mathbf{B_0})^2}{k_y^2 B_0^2 M_{\mathrm{A}}^2} \right],$$

$$m_2 = \left[1 + \frac{64(\mathbf{k}\cdot\mathbf{B_0})^2 B_{0y}^2 d^2 M_S^2}{k_y^2 B_0^4 M_\mathrm{A}^4 \alpha} - \frac{16(\mathbf{k}\cdot\mathbf{B_0}) B_{0y} d^2 M_S^2}{k_y B_0^2 M_\mathrm{A}^2 \alpha\beta}\right],$$

$$m_3 = \left[1 + \frac{64(\mathbf{k}\cdot\mathbf{B_0}) B_{0y} d^2}{k_y B_0^2 M_\mathrm{A}^4 m_1^2 \beta} - \frac{16 d^2}{M_\mathrm{A}^2 m_1 \beta}\right],$$

$$m_4 = \left[\frac{16(\mathbf{k}\cdot\mathbf{B_0}) B_{0y} d}{k_y B_0^2 M_\mathrm{A}^3 m_1 \beta} - \frac{4d}{M_\mathrm{A}\beta}\right]$$

$$\alpha = \left[M_\mathrm{A}^2 + M_S^2 - \frac{4(\mathbf{k}\cdot\mathbf{B_0})^{\mathbf{2}}}{k_y^2 B_0^2}\right], \quad \beta = \left[1 - \frac{16 d^2}{M_\mathrm{A}^2\left(1 - \frac{4(\mathbf{k}\cdot\mathbf{B_0})^2}{k_y^2 B_0^2 M_\mathrm{A}^2}\right)}\right],$$

$$M_\mathrm{A} = \frac{V_0}{V_\mathrm{A}}, \quad M_S = \frac{V_0}{c_\mathrm{s}}, \quad d = \frac{\Omega_0}{k_y v_\mathrm{A}}, \quad v_\mathrm{A} = \frac{B_0}{(\mu_0\rho_0)^{1/2}}, \quad c_\mathrm{s}^2 = \frac{\gamma p_0}{\rho_0}.$$

Equations (19) and (20) resemble with that of Miura and Pritchett [7] in the absence of rotation.

For the instability, $\alpha > 0$ implies that β must be positive. β occurs because of rotation. For $\alpha > 0, \beta > 0$, the necessary condition (20) for the existence of an evanescent eigenmode at $x = \pm\infty$ becomes

$$(k_y^2 + k_z^2)\alpha - \frac{k_y^2 M_S^2 M_\mathrm{A}^2 M_\mathrm{R}}{4} > 0. \tag{21}$$

Putting for α, the above inequality takes the form as

$$M_f^2 = \frac{M_\mathrm{A}^2 M_S^2}{M_\mathrm{A}^2 + M_S^2} < 4\frac{(k_y^2 + k_z^2)}{k_y^2 M_\mathrm{R}}\left[1 - \frac{4(\mathbf{k}\cdot\mathbf{B_0})^2}{k_y^2 B_0^2 (M_\mathrm{A}^2 + M_S^2)}\right]. \tag{22}$$

This is the general condition for the eigenfunction to be evanescent as $x \to \pm\infty$ and it shows the asymptotic behaviour of K–H eigenfunction for the case of compressibility in the presence of rotation for modes satisfying $|\gamma| \ll \frac{1}{2}k_y V_0$. In the absence of rotation ($d = 0$) (22) reduces to the condition obtained by Miura and Pritchett [7].

We discuss the instability in two cases:

(i) Transverse case: ($\hat{B}_0 = \hat{z}$) with $k_z = 0$ from (22), we get

$$M_f^2 < 4/M_\mathrm{R}. \tag{23}$$

(ii) Parallel case: ($\hat{B}_0 = \hat{y}$) with $k_z = 0$ the inequality from (22) reduces to

$$M_f^2 = \frac{M_\mathrm{A}^2 M_S^2}{M_\mathrm{A}^2 + M_S^2} < \frac{4}{M_\mathrm{R}}\left[1 - \frac{4}{(M_\mathrm{A}^2 + M_S^2)}\right]. \tag{24}$$

In the absence of rotation, the above inequalities in both the cases reduce to the inequalities obtained by Miura and Pritchett [7]. Furthermore, in the presence of rotation, if we allow for the conditions $M_\mathrm{A}^2 > 4$ given by Miura and Pritchett [7] and $\beta > 0$ the inequalities (23) and (24) are satisfied for instability only when $d^2 < (M_\mathrm{A}^2/16)$ for the transverse case and $d^2 < ((M_\mathrm{A}^2 - 4)/16)$ for the parallel case.

It is concluded that for both the cases $d^2 < ((M_\mathrm{A}^2 - 4)/16)$ for the instability.

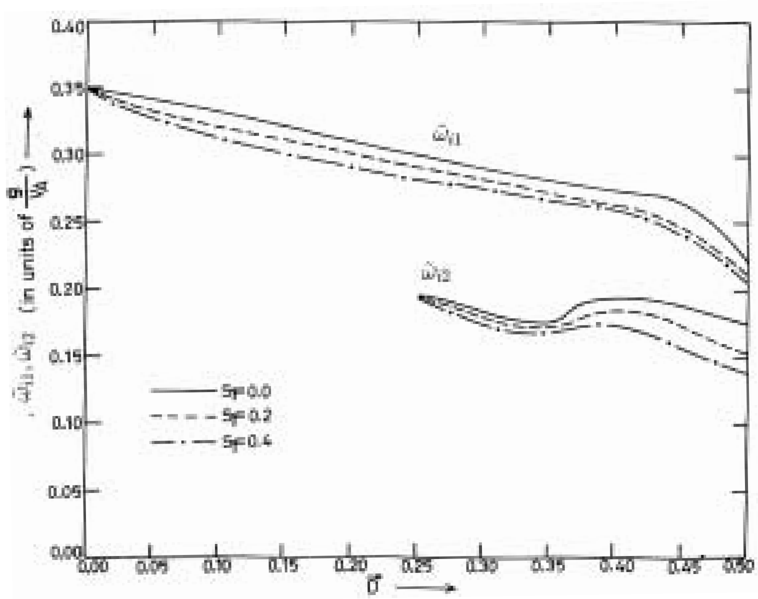

Figure 1. Variation of $\tilde{\omega}_{i1}$ and $\tilde{\omega}_{i2}$ (in units of $g/V_{\rm A}$) with the parameter D for $S_1 = 0.0, 0.2, 0.4, l_1 = l_2 = l_3 = 0.01$ and $\theta = 60°$.

4. Dispersion relation and discussion

In the incompressible limit, i.e., $g_{\rm A}^2 c_S^2 + V_{\rm A}^2 \to \infty$ the eq. (16) is given by

$$\frac{g_{\rm A}}{F_{\rm A}}\frac{{\rm d}^2\delta p^*}{{\rm d}x^2} + \left\{\rho_0\tilde{\omega}^2 g_{\rm A}\frac{\rm d}{{\rm d}x}(\rho_0\tilde{\omega}^2 F_{\rm A})\right\}\frac{{\rm d}\delta p^*}{{\rm d}x} - \left\{\rho_0\tilde{\omega}^2 g_{\rm A}\frac{\rm d}{{\rm d}x}\left[\frac{2\Omega_0 k_y}{\rho_0\tilde{\omega}^3 g_{\rm A} F_{\rm A}}\right] - (k_y^2+k_z^2) + \frac{4\Omega_0^2 k_y^2}{\tilde{\omega}^2 g_{\rm A} F_{\rm A}}\right\}\delta p^* = 0. \tag{25}$$

In analyzing the behaviour of the solutions of eq. (25) for the case $k_z = 0, B_0||V_0$, it is convenient to subtract out a spatial decay (due to density stratification) through the transformation

$$\delta p^*(x) = \exp[-x/2H]\bar{\delta p}(x), \tag{26}$$

where $H(= 1/k_n)$ is the scale length of inhomogeneity in equilibrium quantities. The purified variable $\bar{\delta p}(x)$ then satisfies an associated wave equation. This wave equation admits a plane-wave solution varying as

$$\exp(-ik_x x), \tag{27}$$

provided the following dispersion relation holds:

$$m_1^2 m_2\left[\frac{1}{H^2}+\frac{ik_x}{H}-k_x^2\right]+\left\{l_2 m_1^2 m_2-\frac{2k_y S_1 m_1^2 m_2}{X}+m_1 m_2'\right\}$$
$$\times\left[-\frac{1}{2H}-ik_x\right]-\left\{\frac{2l_2 k_y D m_1 m_2}{X}-\frac{6k_y^2 S_1 D m_1 m_2}{X^2}\right.$$
$$\left.+\frac{2k_y D m_1' m_2}{X}+\frac{2k_y D m_2'}{X}+k_y^2 m_2^2+\frac{4\Omega_0^2 k_y m_2}{X^2}\right\}=0, \tag{28}$$

where

$$m_1=\left[1-\frac{l_1^2}{X^2}\right],\quad m_1'=\frac{-2l_1^2 S_1}{X^3},$$
$$m_2=\left[m_1^2+\frac{m_1 l_2}{X^2}-\frac{2DS_1 m_1}{X^2}-\frac{4D_2}{X^2}\right],$$
$$m_2'=\left[m_1^2 m_1'+\frac{2k_y l_2 S_1 m_1^2}{X^3}-\frac{4k_y D S_1^2 m_1^2}{X^3}+\frac{4D^2 m_1'}{X^2}-\frac{8D^2 k_y S_1 m_1}{X^3}\right].$$

If we introduce non-dimensional variables in eq. (28) and simplify, we get

$$\left\{R^2E_1+\frac{RE_2 l_2}{l_1}-1\right\}X^8+\left\{-2S_1RE_2-\frac{2l_2D}{l_1}\right\}X^7$$
$$+\{R^2E_1[-4l_1^2+l_2-2DS_1-4D^2]+\frac{RE_2l_2}{l_1}[-4l_1^2-2DS_1-4D^2]$$
$$+10DS_1+4l_1^2-2l_2+8D^2-4\}X^6$$
$$+\left\{-2S_1RE_2[-4l_1^2+l_2-2DS_1-4D^2]\right.$$
$$\left.-\frac{2l_2D}{l_1}[-3l_1^2-2DS_1-4D^2]\right\}X^5$$
$$+\{R^2E_1[6l_1^4-3l_1^2l_2+6l_1^2DS_1+8l_1^2D^2]$$
$$+RE_2l_2[6l_1^3+6l_1DS_1+8l_1D^2]-20l_1^2DS_1+6l_2DS_1$$
$$-12D^2S_1^2-24S_1D^3-6l_1^4-16D^4+5l_1^2l_2-16l_1^2D^2+8l_2D^2$$
$$+8l_1^2-4l_2+8DS+16D^2\}X^4+\{RE_2[-6l_1^4S_1$$
$$-16l_1^2DS_1^2-8l_1^2D^2S_1]-2l_2D[3l_1^3+4l_1DS_1+4l_1D^2]\}X^3$$
$$+\{R^2E_1[-4l_1^6+3l_1^4l_2-6l_1^4DS_1-4l_1^4D^2]$$
$$+RE_2l_2[-4l_1^5-6l_1^3DS_1-4l_1^3D^2]+4l_1^6-6l_1^4l_2+14l_1^4DS_1$$
$$+8l_1^4D^2-8l_1^2l_2DS_1-8l_1^2l_2D^2+40l_1^2D^2S_1^2+24l_1^2S_1D^3-4l_1^4$$
$$+4l_1^2l_2-8l_1^2DS_1\}X^2+\{2R_1E_2[l_1^6S_1+8l_1^4DS_1^2+4l_1^2D^2S_1]$$
$$-2l_1D[-l_1^5+l_1^3l_2-2DS_1l_1^3]\}X+\{R^2E_1[l_1^8-l_1^6l_2+2DS_1l_1^6]$$
$$+RE_2l_2[l_1^7+2DS_1l_1^5]-l_1^8+2l_1^6l_2-2l_1^6DS_1+2l_1^4l_2DS_1\}=0, \tag{29}$$

where

$$X=\tilde{\omega}V_{\rm A}/g,\quad l_1=k_yV_{\rm A}^2/g,\quad S_1=V_{oy}'V_{\rm A}/g,\quad l_2=\frac{\rho_o'}{\rho_0}V_{\rm A}^2/g,$$

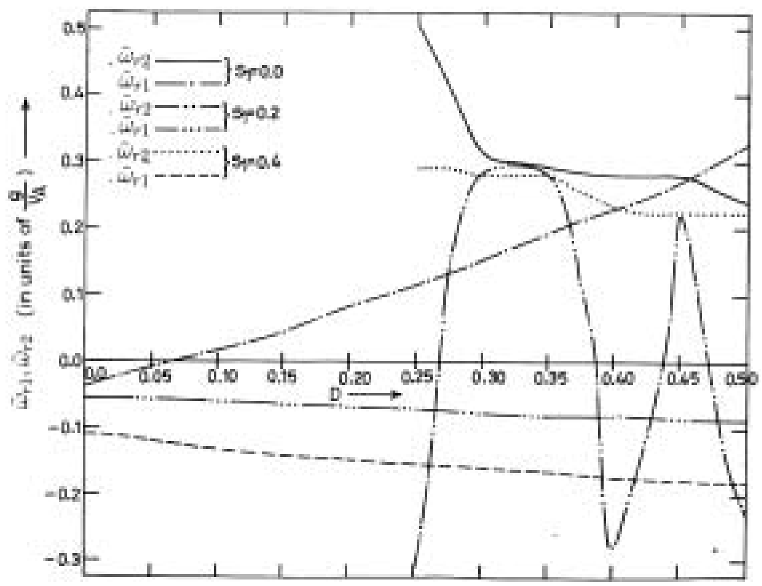

Figure 2. Variation of $\tilde{\omega}_{r1}$ and $\tilde{\omega}_{r2}$ (in units of $g/V_{\rm A}$) with the parameter D for $S_1 = 0.0, 0.2, 0.4, l_1 = l_2 = l_3 = 0.01$ and $\theta = 60°$.

$$D = \Omega_0 V_{\rm A}/g, \quad R = \frac{k_x}{k_y} = \cot\theta, \quad l_3 = Hk_x, \quad E_1 = \left(\frac{1}{l_3^2} + \frac{1}{l_3} + 1\right),$$

$$E_2 = (-i - 1/l_3).$$

Here θ is the angle between the wave vector $\mathbf{k}$ and the z-axis, S_1 is the shear velocity parameter and D measures the effect of rotation.

We computed numerically all the roots of the polynomial of eighth degree in X for different values of D and S_1, taking $l_1 = 0.01, l_2 = 0.01, l_3 = 0.01$, and $\theta = 60°$. A pair of roots appears with non-zero real component $\tilde{\omega}_r$ and imaginary component $\tilde{\omega}_i > 0$ of the frequency $\tilde{\omega}$ which describes the instability of the inhomogeneous system. The largest values of the growth rate $\tilde{\omega}_i > 0$ and the corresponding values of $\tilde{\omega}_r$ in units of $g/v_{\rm A}$ are plotted against D for $s = 0.0, 0.2$ and 0.4 in figures 1 and 2 respectively. The parameter D measures the effect of rotation. From figure 1 it is seen that the growth rate of the first unstable mode has its maximum when there is no rotation but as the parameter D increases it decreases slightly. As D approaches the value 0.25 the second unstable mode appears whose growth rate has decreasing pattern except in the range 0.35 to 0.40 of D in which an increase in the growth rate is observed. Thus, it is concluded that rotation has a destabilizing effect on the inhomogeneous incompressible system. The increase in rotation suppresses the instability up to some extent but is unable to quench the instability completely. The results might be applicable to rotating plasma experiments and rotating astrophysical objects.

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