PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 62, No. 4
April 2004
pp. 803–814

On the Bohmian quantum friction

FATIMAH SHOJAI[1] and ALI SHOJAI[1,2]
[1]Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5531, Tehran, Iran
[2]Physics Department, Tehran University, Tehran, Iran
E-mail: fatimah@ipm.ir; shojai@ipm.ir

MS received 7 May 2003; accepted 18 November 2003

Abstract. The problem of quantum friction in the framework of Bohmian quantum mechanics is studied. The appropriate equations for such a system is written and solved exactly for some cases. Also two approximate solutions are found which represent the transition of a system from an upper state to the ground state caused by the friction. The physical nature of these solutions are examined.

Keywords. de Broglie–Bohm theory; causal interpretation of quantum mechanics; quantum friction.

PACS Nos 03.65.Ta; 03.65.Ca; 03.65.Yz

1. Introduction

Bohmian mechanics [1,2] is an attempt to bring the concept of trajectory to the domain of quantum phenomena. Although it is built in such a way that all the results of the standard quantum are reproduced, it solves some problems of the standard quantum mechanics. In particular, in Bohmian mechanics the measurement process has a causal description. The projection principle of the standard quantum mechanics is emerged [1,2].

The introduction of friction in the classical Lagrangian mechanics needs care. We concentrate our attention on the case in which the friction force is proportional to velocity. The equation of motion of a particle under the influence of this kind of friction is given by

$$m\frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2} + 2m\gamma\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} + \left(\vec{\nabla}V(\vec{x};t)\right)_{\vec{x}=\vec{r}(t)} = 0, \tag{1}$$

where γ is the friction coefficient and V represents the potential due to any other force. This equation of motion can be derived from different distinct Lagrangians [3].

One particular Lagrangian and perhaps the most simple one, which we shall concentrate on, is

$$\mathcal{L}(\vec{r},\dot{\vec{r}};t) = \mathrm{e}^{2\gamma t}\left[\frac{1}{2}m|\dot{\vec{r}}|^2 - V(\vec{r})\right]. \tag{2}$$

The canonical momentum is

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m\mathrm{e}^{2\gamma t}\dot{\vec{r}} \tag{3}$$

and thus the Hamiltonian is given by

$$\mathcal{H}(\vec{r},\vec{p};t) = \mathrm{e}^{-2\gamma t}\frac{|\vec{p}|^2}{2m} + \mathrm{e}^{2\gamma t}V(\vec{r}). \tag{4}$$

This Hamiltonian leads to an apparently important result when one looks for a damped harmonic oscillator, i.e., when one takes $V = \frac{1}{2}m\omega_0^2 r^2$. In this case, it is seen that a damped harmonic oscillator is identical to an ordinary one with increasing mass $m\mathrm{e}^{2\gamma t}$. Note that this resemblance is not actual. Because we factor by hand, an m in V, and this enables one to state the above equivalence. This factorization can be done for any potential V. But it is not true to say that any damped system is equivalent to the corresponding undamped system with increasing mass. In classical mechanics this correspondence makes no error, but as we know, in the quantum domain, using this correspondence or not would lead to different physics [4,5].

To study a damped system in the standard quantum mechanics, it is sufficient to use the Dirac ansatz $\vec{p} \to -i\hbar\vec{\nabla}$ in the Hamiltonian (4) and write out the damped Schrödinger equation as

$$-\mathrm{e}^{-2\gamma t}\frac{\hbar^2}{2m}\nabla^2\psi + \mathrm{e}^{2\gamma t}V\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{5}$$

This wave equation is investigated in [6].

Vandyck [5] has investigated the Bohmian trajectories for a damped harmonic oscillator with one degree of freedom using this Schrödinger equation.

In this paper we shall try to tackle the problem of quantum friction in the framework of Bohmian mechanics in a different way. We shall write out the equations of motion for a quantum damped system. And then, for a damped harmonic oscillator we obtain exact solutions, as well as two approximate ones (which we call transitive solutions) that demonstrate how a quantum system can loose energy and goes from an upper level to a lower one.

2. Bohmian quantum friction

In Bohmian mechanics a particle is always accompanied by a field – the ψ-field – which satisfies the Schrödinger equation. This objectively real field exerts a quantum force on the particle given by

$$\vec{F}_Q = -\vec{\nabla} Q(\vec{x};t); \quad Q = -\frac{\hbar^2}{2m}\frac{\nabla^2|\psi|}{|\psi|}. \tag{6}$$

Thus the complete state of the system is determined by giving $(\vec{r}(t);\psi(\vec{x},t))$ where $\vec{r}(t)$ is the particle trajectory. The equations of motion are

$$m\frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2} = -\left[\vec{\nabla}V(\vec{x}) + \vec{\nabla}Q(\vec{x},t)\right]_{\vec{x}=\vec{r}(t)}, \tag{7}$$

$$\psi(\vec{x},t) \ \text{ satisfies the Schrödinger equation.} \tag{8}$$

An essential property of the Bohmian mechanics is that if one writes the equation of motion (7) in terms of the Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{|\vec{\nabla}S|^2}{2m} + V + Q = 0, \tag{9}$$

then the Hamilton–Jacobi function S coincides exactly with $\hbar$ times the phase of the wave function. So eq. (8) is only a relation for the amplitude of the wave function which is

$$\frac{\partial R^2}{\partial t} + \vec{\nabla}\cdot\left(R^2\frac{\vec{\nabla}S}{m}\right) = 0. \tag{10}$$

In order to guarantee that the results of Bohmian mechanics coincide with those of quantum mechanics, Bohm made the following ansatz, that for an ensemble the distribution function is given by

$$\rho(\vec{x},t) = |\psi(\vec{x},t)|^2 = R^2. \tag{11}$$

Accordingly, one can express Bohmian mechanics in two identical [7] ways:

1. - $\psi = R\mathrm{e}^{iS/\hbar}$ satisfies the Schrödinger equation.
   - S is the Hamilton–Jacobi function of the particle, so $m(\mathrm{d}\vec{r}(t)/\mathrm{d}t) = \vec{\nabla}S|_{\vec{x}=\vec{r}(t)}$.
2. - The particle is acted on by a quantum potential $Q = -(\hbar^2/2m)\nabla^2\sqrt{\rho}/\sqrt{\rho}$.
   - ρ is the ensemble density satisfying the continuity equation $(\partial\rho/\partial t) + \vec{\nabla}\cdot(\rho\vec{v}) = 0$.

A straightforward application of Bohmian mechanics to systems with friction can be achieved [5] by adopting the first method and accepting eq. (5) as the Schrödinger equation for ψ. Another way to construct a Bohmian theory for quantum friction is to use the second method. In this method the equation of motion of the particle is nothing but eq. (1) except that the quantum potential is presented:

$$m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} + 2m\gamma\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} + \left(\vec{\nabla}V + \vec{\nabla}Q\right)_{\vec{x}=\vec{r}(t)} = 0, \tag{12}$$

where

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{13}$$

and

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right) = 0. \tag{14}$$

A Lagrangian which leads to this equation of motion is

$$\mathcal{L}(\vec{r},\dot{\vec{r}};t) = \mathrm{e}^{2\gamma t}\left[\frac{1}{2}m|\dot{\vec{r}}|^2 - V - Q\right]. \tag{15}$$

The canonical momentum is again given by (3), so the Hamiltonian is

$$\mathcal{H}(\vec{r},\vec{p};t) = \mathrm{e}^{-2\gamma t}\frac{|\vec{p}|^2}{2m} + \mathrm{e}^{2\gamma t}(V+Q). \tag{16}$$

The Hamilton–Jacobi equation, which is identical to (12) is as follows:

$$\frac{\partial S}{\partial t} + \mathrm{e}^{-2\gamma t}\frac{|\vec{\nabla}S|^2}{2m} + \mathrm{e}^{2\gamma t}(V+Q) = 0 \tag{17}$$

which is certainly different from that of Vandyck [5].

It is a simple task to use relations (17) and (14) to show that the corresponding Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\mathrm{e}^{-2\gamma t}\nabla^2\psi + \mathrm{e}^{2\gamma t}V\psi + 2\sinh(2\gamma t)Q\psi. \tag{18}$$

Again this is different from Schrödinger equation (5). Note that the above equation has the correct limit $\gamma \to 0$. In this limit the ordinary Schrödinger equation emerges.

In what follows we accept that eqs (14) and (17) (or identically the above equation) describe the correct physics of quantum friction. Then we try to solve them.

3. Exact solutions

First of all we look for exact solutions of eqs (14) and (17) which are rewritten as

$$\frac{\partial S}{\partial t} + \mathrm{e}^{-2\gamma t}\frac{|\vec{\nabla}S|^2}{2m} + \mathrm{e}^{2\gamma t}(V+Q) = 0, \tag{19}$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\frac{\vec{\nabla}S}{m}\mathrm{e}^{-2\gamma t}\right) = 0, \tag{20}$$

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}. \tag{21}$$

We make the following ansatz for exact solutions of these equations:

$$\sqrt{\rho_n} = u_n^{(0)}(x), \tag{22}$$

and

$$S_n = S_n^{(0)} + \Sigma, \tag{23}$$

where $S_n^{(0)} = -E_n^{(0)}t$, in which $E_n^{(0)}$ is the eigenvalue of the ordinary Schrödinger equation (i.e. with $\gamma = 0$) and $\psi_n^{(0)} = u_n^{(0)}\mathrm{e}^{iS_n^{(0)}/\hbar}$ are the eigenfunctions. We also assume that $u_n^{(0)}$ are real functions.

Inserting (22) and (23) in eqs (14) and (17) one gets

$$\vec{\nabla}\Sigma = 0 \Rightarrow \Sigma = \Sigma(t), \tag{24}$$

$$\frac{\partial\Sigma}{\partial t} = E_n^{(0)}(1 - \mathrm{e}^{2\gamma t}) \tag{25}$$

with the solution

$$\Sigma = E_n^{(0)}t - \frac{E_n^{(0)}}{2\gamma}(\mathrm{e}^{2\gamma t} - 1). \tag{26}$$

Note that the constant of integration is chosen such that Σ goes to zero as γ goes to zero, in order to have the correct limit for S_n.

In brief, the exact solutions of our equations of quantum friction are

$$\sqrt{\rho_n} = u_n^{(0)}(x), \tag{27}$$

$$S_n = -\frac{E_n^{(0)}}{2\gamma}(\mathrm{e}^{2\gamma t} - 1) \tag{28}$$

or

$$\psi_n^{(0)}(x,t) = u_n^{(0)}(x)\exp\left[-i\frac{E_n^{(0)}}{2\gamma\hbar}(\mathrm{e}^{2\gamma t} - 1)\right]. \tag{29}$$

The motion of the particle for such a state is given by

$$\vec{p} = m\mathrm{e}^{2\gamma t}\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{\nabla}S|_{\vec{x}=\vec{r}(t)} = 0. \tag{30}$$

So $\mathrm{d}\vec{r}/\mathrm{d}t = 0$ and the particle is at rest. The energy of the particle is given by

$$\mathcal{E} = -\frac{\partial S}{\partial t} = E_n^{(0)}\mathrm{e}^{2\gamma t}. \tag{31}$$

Since the particle is at rest for these states, but energy is not conserved we call such states quasi-stationary states. Note that in a quasi-stationary state the particle is at rest and since the friction force is proportional to the velocity there is no friction force for such a state and so the paricle remains in this state forever. We shall also see this result in the next section.

4. Transitive solutions

Intuitively, one expects that when quantum systems with friction are considered, there must exist transitive solutions. A transitive solution is a state that is in an upper state and in the course of time looses energy and goes to a lower state. An important question now is that, if our model of quantum friction predicts such a solution?

4.1 *Undamped eigenstate as initial condition*

In order to have a transitive solution, we first write our Schrödinger equation for $\gamma t \ll 1$. Note that as γ is a small number this is not a serious limitation on time. Expanding in terms of γt we have

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi + \gamma V_p\psi, \tag{32}$$

where

$$V_p = \frac{\hbar^2 t}{m}\frac{\nabla^2\psi}{\psi} + 2tV + 4tQ. \tag{33}$$

Since γ is a small parameter one can look at solving the above equation as a time-dependent perturbation problem. For simplicity we assume a two-state system and write the wave function as

$$\psi = \psi_0 + \gamma\psi_1 = \psi_0 + \gamma\sum_{n=1}^{2}\mathrm{e}^{-iE_n^{(0)}t/\hbar}X_n(t)u_n^{(0)}(x). \tag{34}$$

Then one can simply get the following relation for the wave function coefficients:

$$X_n = -\frac{i}{\hbar}\int_0^t \mathrm{d}t'\mathrm{e}^{iE_n^{(0)}t'/\hbar}\langle u_n^{(0)}|V_p|\psi_0\rangle. \tag{35}$$

We assume that the unperturbed wave function is a combination of the two states of the system

$$\psi_0 = \varepsilon\mathrm{e}^{-iE_1^{(0)}t/\hbar}u_1^{(0)} + \sqrt{1-\varepsilon^2}\mathrm{e}^{-iE_2^{(0)}t/\hbar}u_2^{(0)} \tag{36}$$

with ε small which implies that the system is almost in the first excited state, initially. One can show that V_p simplifies to (up to first order in ε)

$$V_p = 2E_2^{(0)}t - 2\varepsilon t\mathrm{e}^{-i\omega t}\hbar\omega\frac{u_1^{(0)}}{u_2^{(0)}} \tag{37}$$

for the above initial condition, in which $\omega = (E_2^{(0)} - E_1^{(0)})/\hbar$, and the solution to the wave function coefficients is

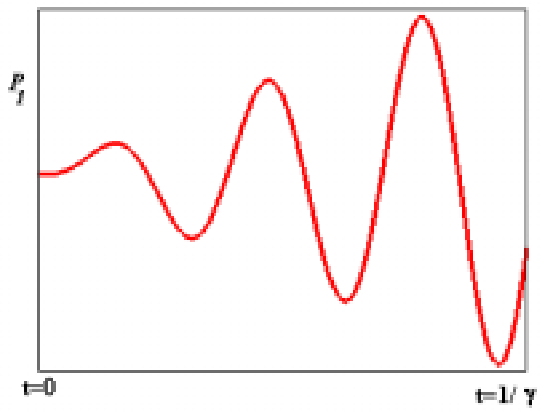

Figure 1. Ground state probability as a function of time.

$$X_1 = -\frac{i\varepsilon}{\hbar}\left[E_2^{(0)}t^2 + \frac{\hbar}{2\omega}\left(1 - \mathrm{e}^{-2i\omega t} - 2i\omega t\mathrm{e}^{-2i\omega t}\right)\right] \tag{38}$$

and

$$X_2 = -\frac{i}{\hbar}E_2^{(0)}t^2. \tag{39}$$

Now we can calculate the probability that the system is in the ground state (P_1) and in the first excited state (P_2), as functions of time:

$$P_1 = \varepsilon^2\left(1 + \frac{\gamma}{\omega}\sin 2\omega t - 2\gamma t\cos 2\omega t\right), \tag{40}$$

$$P_2 = 1 - P_1. \tag{41}$$

It is important to note that P_1 is proportional to ε^2. Thus setting $\varepsilon = 0$, i.e., starting with a particle completely in the upper state, it remains in that state. This is because in such a state the particle is at rest and thus there is no friction force, so the particle would remain in that state.

P_1 is plotted in figure 1. As seen in this figure, the probability of being in the ground state has an oscillatory nature. This can be explained in this way: Initially almost all of the systems are in the first excited state. They would come in the ground state and radiate because of friction. But some of the systems in the ground state can absorb this radiation and jump to the first excited state. As a result one has an oscillating probability.

Now one can calculate the Hamilton–Jacobi function

$$S = (1+\gamma t)\left(-E_2^{(0)}t + \varepsilon\hbar\sin\omega t\frac{u_1^{(0)}}{u_2^{(0)}}\right) \tag{42}$$

leading to the velocity field

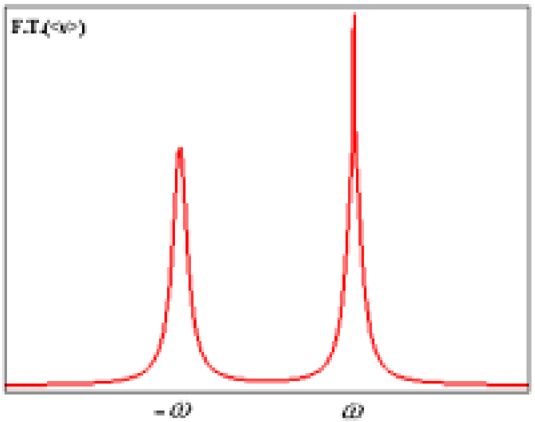

Figure 2. Fourier transform of average velocity.

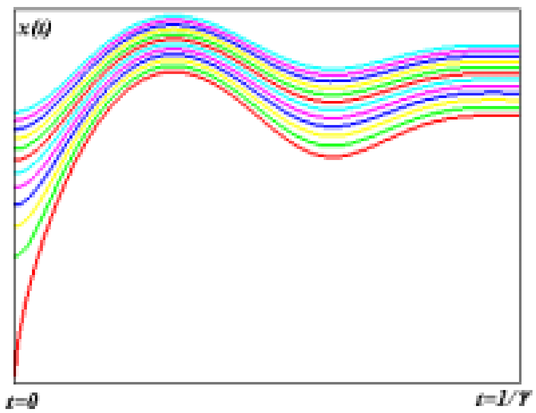

Figure 3. Bohmian trajectories with different initial positions.

$$v(x,t) = \frac{\varepsilon\hbar}{m}(1-\gamma t)\sin\omega t \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_1^{(0)}}{u_2^{(0)}}\right). \tag{43}$$

The Fourier transform of the average velocity is plotted in figure 2. Two peaks are seen at ω (corresponding to the systems that move down) and at $-\omega$ (corresponding to the systems that move up) and that most of the systems are coming down from the upper state (the peak at ω is sharper).

One can integrate the above relation for velocity to get the system trajectory. If we assume that the classical potential is that of a harmonic oscillator, the result is as in figure 3, which shows particle trajectories for different initial conditions. As can be seen in the figure, the particle would be approximately at rest at final time. This is because of the fact that the system would be approximately in the ground state in which the particle is at rest according to Bohm's theory. In figure 4 the corresponding total potential $(V+Q)$ is plotted. This shows again the nature of trajectories. The particle can start from a small value of x and goes to rest at a lager value of x in an oscillatory path.

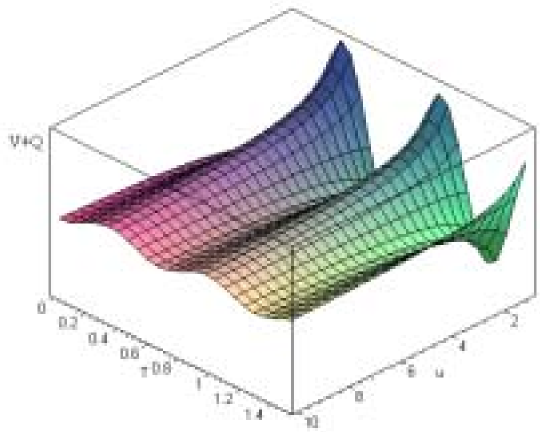

Figure 4. Total potential $(V+Q)$ where we used normalized time $T=\gamma t$ and coordinate $u=x(2m/\hbar^2)^{1/3}$.

4.2 *Quasi-stationary state as initial condition*

We can see the existence of transitive solutions in a more general way. Let us assume the wave function in the form

$$\psi = A(t)\psi_i + B(t)\psi_f, \tag{44}$$

where ψ_i and ψ_f are two exact solutions as in the previous section, and $A(t)$ and $B(t)$ are some functions of time. We are seeking for a solution in which $|A(\infty)|^2$ and $|B(-\infty)|^2$ are small. In the limit $t\to-\infty$ the wave equation leads to

$$i\hbar\dot{A} = -\mathrm{e}^{-2\gamma t}qB \tag{45}$$

and

$$i\hbar\dot{B} = \mathrm{e}^{-2\gamma t}B\Delta Q, \tag{46}$$

where

$$q = \left.\frac{\partial Q}{\partial(B/A)}\right|_{B/A=0} \tag{47}$$

and

$$\Delta Q = Q_f - Q_i. \tag{48}$$

Q_f and Q_i are the quantum potentials calculated for ψ_f and ψ_i and, in the limit $t\to\infty$ one has

$$i\hbar\dot{A} = \mathrm{e}^{2\gamma t}A\Delta Q \tag{49}$$

and

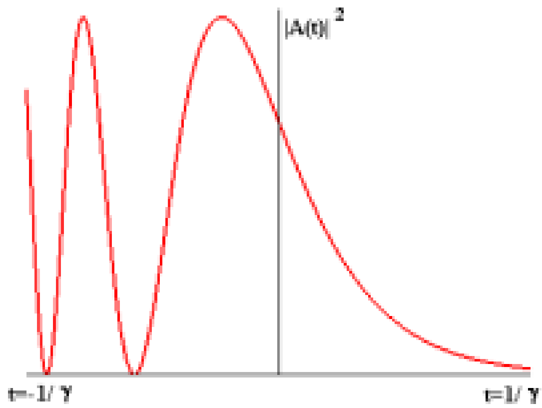

Figure 5. The probability of being in the ground state as a function of time.

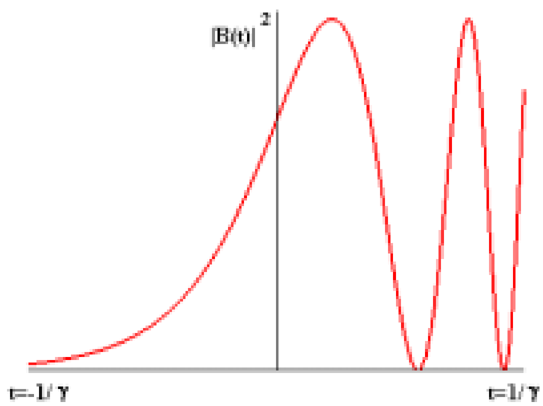

Figure 6. The probability of being in the final state as a function of time.

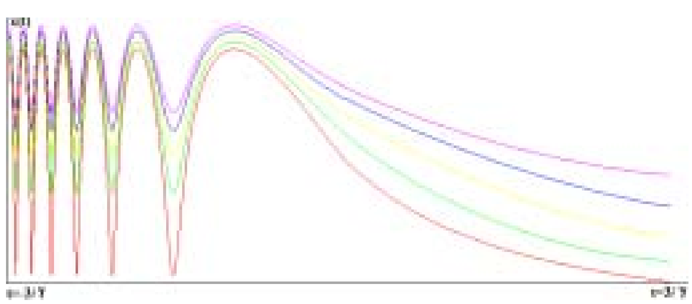

Figure 7. Bohmian trajectories with different initial positions.

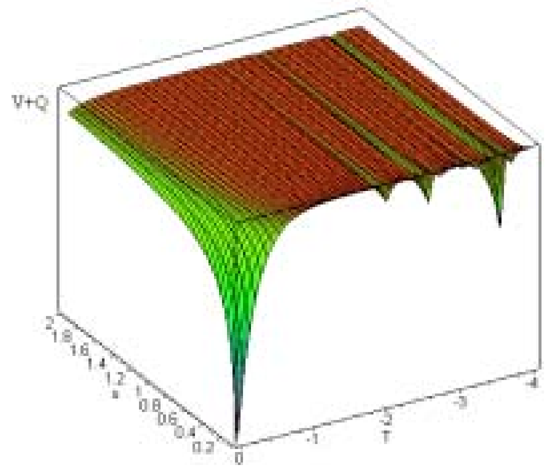

Figure 8. Total potential $(V+Q)$ for negative times where we used normalized time $T=\gamma t$ and coordinate $u=x(2m/\hbar^2)^{1/3}$.

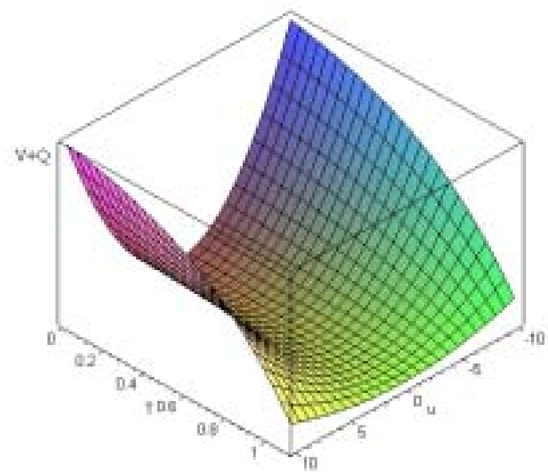

Figure 9. Total potential $(V+Q)$ for positive times where we used normalized time $T=\gamma t$ and coordinate $u=x(2m/\hbar^2)^{1/3}$.

$$i\hbar\dot{B}=\mathrm{e}^{-2\gamma t}q'A, \tag{50}$$

where

$$q'=\left.\frac{\partial Q}{\partial(A/B)}\right|_{A/B=0}. \tag{51}$$

The solutions to these equations are

$$A(t) = \begin{cases} 1 - \exp(i\Delta Q \mathrm{e}^{-2\gamma t}/2\gamma\hbar) & t \to -\infty \\ -\frac{\Delta Q}{q'} \exp(-i\Delta Q \mathrm{e}^{2\gamma t}/2\gamma\hbar) & t \to \infty \end{cases} \tag{52}$$

and

$$B(t) = \begin{cases} \frac{\Delta Q}{q} \exp(i\Delta Q \mathrm{e}^{-2\gamma t}/2\gamma\hbar) & t \to -\infty \\ 1 - \exp(-i\Delta Q \mathrm{e}^{2\gamma t}/2\gamma\hbar) & t \to \infty \end{cases} . \tag{53}$$

The probability of being in ψ_i state and ψ_f state are given by $|A(t)|^2$ and $|B(t)|^2$:

$$|A(t)|^2 = \begin{cases} 2 - 2\cos\left(\frac{\Delta Q}{2\gamma\hbar}\mathrm{e}^{-2\gamma t}\right) & t \to -\infty \\ \left|\frac{\Delta Q}{q'}\right|^2 & t \to \infty \end{cases} . \tag{54}$$

$$|B(t)|^2 = \begin{cases} \left|\frac{\Delta Q}{q}\right|^2 & t \to -\infty \\ 2 - 2\cos\left(\frac{\Delta Q}{2\gamma\hbar}\mathrm{e}^{2\gamma t}\right) & t \to \infty \end{cases} . \tag{55}$$

These two functions are plotted in figures 5 and 6. This shows that the system which is initially in an upper state would decay into a system in a lower state. Again some oscillation is seen in probabilities with the same description as in the previous subsection.

If we assume that the classical potential is that of a harmonic oscillator, the Bohmian trajectories can be obtained. They are plotted in figure 7. Also the coresponding total potential is plotted in figure 8 for negative times and in figure 9 for positive times.

References

[1] D Bohm, *Phys. Rev.* **85**, 166 (1952)
D Bohm, *Phys. Rev.* **85**, 180 (1952)

[2] K Berndl, M Daumer, D Durr, S Goldstein and N Zanghi, *Nuovo Cimento* **B110**, 737 (1995)

[3] I Edwards, *Am. J. Phys.* **47**, 153 (1979)
P Havas, *Nuovo Cim. Suppl.* **5**, 363 (1995)
H Bateman, *Phys. Rev.* **38**, 815 (1931)

[4] D Grenberger, *J. Math. Phys.* **20**, 762 (1979)

[5] M A Vandyck, *J. Phys.* **A27**, 1743 (1994)

[6] D Grenberger, *J. Math. Phys.* **20**, 771 (1979)
P Caldirola, *Nuovo Cimento* **18**, 393 (1941)
E Kanai, *Prog. Theor. Phys.* **3**, 440 (1948)
J Messer, *Acta Phys. Austriaca* **50**, 75 (1979)

[7] These two ways are almost always identical, e.g. in dealing with non-relativistic conservative systems they are identical, but for a relativistic particle they are different [8]. We shall see that when friction is present they are also different

[8] A Shojai and F Shojai, *Phys. Scr.* **64**, 413 (2001)