PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 62, No. 3
March 2004
pp. 699–702

Logarithmic corrections to entropy and AdS/CFT

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Abstract. We calculate the correction to the Bekenstein–Hawking entropy formula for five-dimensional AdS–Schwarzschild black holes due to thermodynamic fluctuations. The result is then compared with the boundary gauge theory entropy corrections via AdS/CFT correspondence.

Keywords. AdS/CFT; blackholes.

Recently, holography has shed many interesting insights on the relation between the space-time physics and the physics without gravity through AdS/CFT correspondence. In particular, the correspondence tells us that type-IIB string theory in a five-dimensional AdS space times a five-sphere is dual to the large N limit of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in four dimensions [1,4]. More interestingly, in [5], it was further argued that the gauge theory in question, at high temperature, is dual to AdS–Schwarzschild black hole at the same temperature. It further follows from [5] that the thermodynamic variables of CFT can be read off from the boundary of AdS–Schwarzschild black hole. In particular, at high temperature, CFT calculation gives the correct entropy of the black hole up to a constant numerical multiplicative factor.

Recently, there are several works in [6–13] suggesting that for a large class of black holes (AdS–Schwarzschild being one), the Bekenstein–Hawking area-entropy law receives additive logarithmic corrections due to thermal fluctuations. Typically, the corrected formula has the form

$$\mathcal{S} = S_0 - c\,\log S_0, \tag{1}$$

where S_0 is the standard Bekenstein–Hawking term and c is a number. In this talk [13a], we shall analyse this correction of the entropy of AdS–Schwarzschild black hole in the light of AdS/CFT correspondence. In particular, we present the correction to the entropy of CFT due to thermal fluctuations and then compare it with the entropy correction of the black hole (1) following AdS/CFT dictionary. The logarithmic correction to the entropy due to thermal fluctuation comes out to be of the form given in (1) with $c = 1/2$. We show that this matches with what is expected from AdS/CFT correspondence. However, this number differs from what was found in the existing literature, see for example [11]. The partition function in the grand canonical ensemble is given by [13b].

$$Z(\alpha,\beta)=\int_0^\infty\int_0^\infty \rho(n,E)\mathrm{e}^{(\alpha n-\beta E)}\mathrm{d}n\mathrm{d}E, \tag{2}$$

where $\alpha=\beta\mu$, μ is the chemical potential and $\beta=1/T$ is the inverse temperature. Here we have set $k_B=1$ so that the temperature is having the dimension of energy. The density of states, $\rho(n,E)$, can be obtained from the above equation by inverse Laplace transformation [14], and the integral will be performed by the saddle-point approximation and the main contribution to this integral will come around the equilibrium point, where the integrand is stationary. See [15] for detailed calculations.

We get the following from of the density of states:

$$\rho(n,E)=\frac{\mathrm{e}^{S(\alpha_0,\beta_0)}}{2\pi\sqrt{\frac{\partial^2 \ln Z}{\partial\alpha^2}|_{(\alpha_0,\beta_0)}\times\frac{\partial^2 \ln Z}{\partial\beta^2}|_{(\alpha_0,\beta_0)}}}, \tag{3}$$

where α_0 and β_0 are the stationary points. Hence, the entropy is

$$\mathcal{S}\equiv\ln(\mathcal{E}\rho)=S(\alpha_0,\beta_0)+\ln\frac{\mathcal{E}}{\sqrt{\frac{\partial^2 \ln Z}{\partial\beta^2}|_{(\alpha_0,\beta_0)}}}+\text{higher order terms.} \tag{4}$$

As can be seen, ρ in our calculation has a dimension of inverse temperature. To get entropy as a log of dimensionless quantity, we have multiplied the density of state with $\mathcal{E}$ which has the dimension of energy. This scale $\mathcal{E}$ is set by the particular system in question. Since $\frac{\partial^2 \ln Z}{\partial\beta^2}|_{(\alpha_0,\beta_0)}=C_{\mathrm{v}}T^2$, where C_{v} is the specific heat at constant volume, the correction to the entropy becomes

$$\mathcal{S}=S(\beta_0)+\ln\frac{\mathcal{E}}{\sqrt{C_{\mathrm{v}}T^2}}+\cdots. \tag{5}$$

In (4) and (5), we have set the chemical potential to zero. Furthermore, the scale $\mathcal{E}$ has been set to be the temperature T of the system. This is because we have temperature as the only available scale in canonical ensemble. Hence, the above equation becomes

$$\mathcal{S}=S(\beta_0)-\frac{1}{2}\ln C_{\mathrm{v}}+\cdots. \tag{6}$$

Having obtained the general form of correction to entropy, we would like to apply it for black holes. Note that, for (6) to make sense, C_{v} has to be positive. In the context of black hole, it means that the variation of mass with respect to temperature is positive. Five-dimensional AdS–Schwarzschild black hole does indeed satisfy this property in certain choice of parameters. The metric of such a black hole is given by

$$\mathrm{d}s^2=-\left(1-\frac{16\pi MG_5}{3\Omega_3 r^2}+\frac{r^2}{l^2}\right)\mathrm{d}t^2 \\ +\left(1-\frac{16\pi MG_5}{3\Omega_3 r^2}+\frac{r^2}{l^2}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega_3^2, \tag{7}$$

where $\mathrm{d}\Omega_3^2$ is the metric on unit sphere S^3 and Ω_3 is the area of that unit sphere. G_5, M and $\Lambda = -6/l^2$ are five-dimensional Newton's constant, mass and the cosmological constant. Evaluating the mass, temperature and specific heat of black hole, for $r_+^2 > (l^2/2)$ where r_+ is the horizon of black hole and substituting the above values of mass, temperature and specific heat into eq. (6), we get the corrected entropy as

$$\mathcal{S}_{\mathrm{BH}} = S_{\mathrm{BH}} - \frac{1}{2}\ln S_{\mathrm{BH}} + \dots \tag{8}$$

According to AdS/CFT correspondence, on the boundary of AdS, we have conformally invariant $\mathcal{N} = 4$ $U(N)$ SYMs theory. The particle contents of this theory are: N^2 gauge fields, $6N^2$ massless scalars and $4N^2$ Weyl fermions, that is, we have $8N^2$ bosonic and $8N^2$ fermionic degrees of freedom. The free energy and hence the entropy can be calculated in the free field limit $g_{\mathrm{YM}}^2 N \to 0$. The free energy of a gas of bosons and fermions in a box of volume $V^{(n)}$, the superscript denotes the number of spatial dimensions, calculated in the grand canonical ensemble is

$$F = -T\,\ln Z = T\sum_{s_i,p} s_i \ln(1 - s_i \mathrm{e}^{-\beta p}), \tag{9}$$

where s_i is $+1$ for bosons and -1 for fermions. Now taking the volume of the box to be very large implies $\sum_p = V^{(n)} \int \mathrm{d}^n p/(2\pi)^n$. The free energy, then, becomes [16], for $\mathcal{N} = 4$, $U(N)$ SYM theory in four dimension

$$F = -\frac{\pi^2 N^2 T^4 V^{(3)}}{6}. \tag{10}$$

Evaluating the specific heat from the free energy and substituting in eq. (6), we get

$$\mathcal{S}_{\mathrm{CFT}} = S_{\mathrm{CFT}} - \frac{1}{2}\ln(S_{\mathrm{CFT}}) + \text{ constant}, \tag{11}$$

where

$$S_{\mathrm{CFT}} = \frac{2\pi^2 N^2 T^3 V^{(3)}}{3}. \tag{12}$$

Here, $T = T_{\mathrm{CFT}}$ is clearly associated to the temperature of the gauge theory that resides on the boundary.

We would now like to compare the corrected entropy formula for the bulk (8) and for the boundary CFT (11) following AdS/CFT prescription. In order to do so, let us concentrate at an asymptotic distance $r \equiv L \gg r_+$ of the AdS–Schwarzschild black hole where lives the gauge theory. In this region, the geometry of the above space-time becomes $\mathrm{d}s^2 \sim L^2[-\frac{\mathrm{d}t^2}{l^2} + \mathrm{d}\Omega_3^2]$. Due to red-shift, the temperature at this boundary therefore is

$$T_{\mathrm{CFT}} = \frac{T_{\mathrm{BH}}}{\sqrt{-g_{00}}} = \frac{l}{L} T_{\mathrm{BH}}, \tag{13}$$

where $T_{\rm BH}$ is the Hawking temperature associated with the black hole. Substituting (13) in (12) and using N^2 from the AdS/CFT dictionary [1], $N^2 = (\pi l^3/2G_5)$, we get the standard $S_{\rm CFT} = 2\pi^2 r_+^3/3G_5 = \frac{4}{3}S_{\rm BH}$. This, in turn, implies that (11) can be written as

$$\mathcal{S}_{\rm CFT} = \frac{4}{3}S_{\rm BH} - \frac{1}{2}\ln\left(\frac{4}{3}S_{\rm BH}\right) + \text{ constant}. \tag{14}$$

To conclude, we have analysed the correction to entropy due to thermal fluctuations. We then applied our result to AdS black holes and compared it with what is expected from AdS/CFT correspondence. We found that the corrected entropy formula is of the form (1) with $c = 1/2$.

Acknowledgements

I wish to thank Sudipta Mukherji for collaboration, discussion and careful reading of the manuscript.

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