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# Near-horizon states of black holes and Calogero models

B BASU-MALLICK, PIJUSH K GHOSH and KUMAR S GUPTA
Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata 700 064, India

**Abstract.** We find self-adjoint extensions of the rational Calogero model in the presence of the harmonic interaction. The corresponding eigenfunctions may describe the near-horizon quantum states of certain types of black holes.

**Keywords.** Calogero model; self-adjoint extension; black holes.

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The dynamics of particles or fields in the near-horizon region of black holes [1,2] is often described in terms of the Calogero model [3]. In particular, it has been shown that the existence of the near-horizon conformal symmetry [4] as well as the logarithmic correction to the black hole entropy [5] can be described in terms of the self-adjoint extension [6] of the Calogero model in the absence of the confining potential [7,8]. On the other hand, it has been argued that in certain string theoretic description of black holes, the near-horizon dynamics is governed by a many-particle Calogero model in the presence of the confining potential [2]. It is therefore of interest to find the quantum states of this model in the presence of the self-adjoint extension. These states would be expected to encode the dynamics for such black holes. A more detailed description of the analysis presented below can be found in ref. [9].

The Hamiltonian of the $N$-particle rational Calogero model [3] is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \left[ \frac{a^2 - \frac{1}{4}}{(x_i - x_j)^2} + \frac{\Omega^2}{16}(x_i - x_j)^2 \right], \tag{1}$$

where $a$, $\Omega$ are constants, $x_i$ is the coordinate of the $i$th particle and units have been chosen such that $2m\hbar^{-2} = 1$. We are interested in finding normalizable solutions of the eigenvalue problem $H\psi = E\psi$ in a situation where the system admits self-adjoint extensions. We consider the above eigenvalue equation in a sector of configuration space corresponding to a definite ordering of particles given by $x_1 \geq x_2 \geq \cdots \geq x_N$. The translation-invariant eigenfunctions of the Hamiltonian $H$ can be written as

$$\psi = \prod_{i<j} (x_i - x_j)^{a+\frac{1}{2}} \ \phi(r) \ P_k(x), \tag{2}$$

where $x \equiv (x_1, x_2, \ldots, x_N)$, $r^2 = \frac{1}{N}\sum_{i<j}(x_i - x_j)^2$ and $P_k(x)$ is a translation-invariant as well as homogeneous polynomial of degree $k(\geq 0)$ [3]. The radial part of the wave function satisfies the equation $\tilde{H}\phi = E\phi$ where

$$\tilde{H} = \left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - (1+2\nu)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + w^2 r^2\right], \tag{3}$$

$w^2 = \frac{1}{8}\Omega^2 N$ and $\nu = k + \frac{1}{2}(N-3) + \frac{1}{2}N(N-1)(a+\frac{1}{2})$.

The Hamiltonian $\tilde{H}$ is a symmetric (Hermitian) operator on the domain $D(\tilde{H}) \equiv \{\phi(0) = \phi'(0) = 0,\ \phi,\ \phi' \text{ absolutely continuous}\}$. However, when $-1 < \nu < 1$, $\tilde{H}$ is not self-adjoint in $D(\tilde{H})$ but admits a one-parameter family of self-adjoint extensions [6,9] labelled by $\mathrm{e}^{iz}$ where $z \in R$ (mod $2\pi$). For $\nu \neq 0$, the spectrum is determined by the equation

$$f(E) \equiv \frac{\Gamma\left(\frac{1-\nu}{2} - \frac{E}{4w}\right)}{\Gamma\left(\frac{1+\nu}{2} - \frac{E}{4w}\right)} = \frac{\xi_2 \cos(\frac{z}{2} - \eta_1)}{\xi_1 \cos(\frac{z}{2} - \eta_2)}, \tag{4}$$

where $\Gamma\left(\frac{1+\nu}{2} + \frac{i}{4w}\right) \equiv \xi_1 \mathrm{e}^{i\eta_1}$ and $\Gamma\left(\frac{1-\nu}{2} + \frac{i}{4w}\right) \equiv \xi_2 \mathrm{e}^{i\eta_2}$. This equation is plotted in figure 1. The corresponding eigenfunctions are given by

$$\phi(r) = B\mathrm{e}^{-\frac{wr^2}{2}} U(d, c, wr^2), \tag{5}$$

where $c = 1+\nu$, $d = \frac{1+\nu}{2} - \frac{E}{4w}$, $B$ is a constant and $U$ denotes the confluent hypergeometric function of the second kind. For given values of the parameters $\nu$ and $w$, the bound state energy $E$ is obtained from eq. (4) as a function of $z$. Different choices of $z$ thus leads to inequivalent quantizations of the many-body Calogero model. It may be mentioned that the self-adjoint extensions described above exist for all values of $N$ and for higher 'angular momentum' ($k \neq 0$) sectors of the theory as well.

The following features about the spectrum may be noted:

(1) We have obtained the spectrum analytically when the r.h.s. of eq. (4) is either 0 or $\infty$. When the r.h.s. of eq. (4) is 0, we get $E_n = 2w(2n + \nu + 1)$ where $n$ is a positive integer, corresponding to the choice of $z = z_1 = \pi + 2\eta_1$. Similarly, when the r.h.s. of eq. (4) is $\infty$, we get $E_n = 2w(2n - \nu + 1)$ corresponding to the value of $z$ given by $z = z_2 = \pi + 2\eta_2$. For choices of $z$ other than $z_1$ or $z_2$, the nature of the spectrum can be understood from figure 1, which is a plot of eq. (4) for specific values of $\nu, z$ and $w$. In that plot, the horizontal straight line corresponds to the r.h.s of eq. (4). The energy eigenvalues are obtained from the intersection of $f(E)$ with the horizontal straight line. Note that the spectrum generically consists of infinite number of positive energy solutions and at most one negative energy solution.

(2) Contrary to the spectrum of the rational Calogero model, the energy spectrum obtained from eq. (4) is not equispaced for finite values of $E$ and for generic values of $z$. This may seem surprising with the presence of $SU(1,1)$ as the spectrum

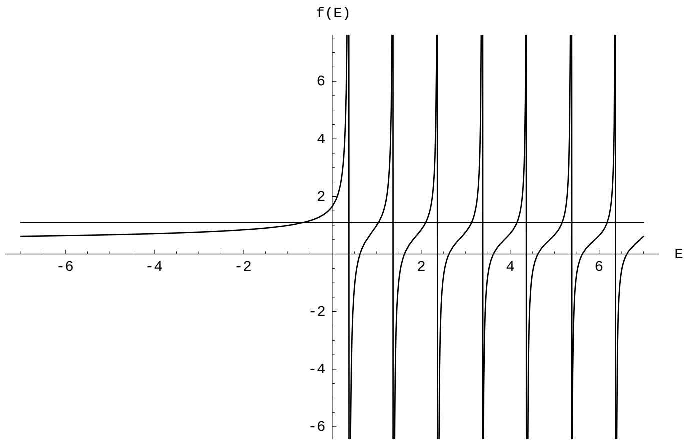


**Figure 1.** A plot of eq. (4) using *Mathematica* with $w = 0.25$, $\nu = 0.25$ and $z = -1.5$. The horizontal straight line corresponds the value of the r.h.s of eq. (4).

generating algebra in this system [10], which demands that the eigenvalues be evenly spaced. However, when $z \neq z_1$, $z_2$, the generator of dilatations does not in general leave the domain of the Hamiltonian invariant [11,8]. Consequently, $SU(1,1)$ cannot be implemented as the spectrum generating algebra except for $z = z_1$, $z_2$.

It is plausible that the eigenfunctions described above would describe the near-horizon quantum states of certain types of black holes as described in ref. [2]. Calculation for the corresponding density of states and entropy would be of future interest.

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