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Muon $g-2$ measurements and non-commutative geometry of quantum beams

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Abstract. We discuss a completely quantum mechanical treatment of the measurement of the anomalous magnetic moment of the muon. A beam of muons move in a strong uniform magnetic field and a weak focusing electrostatic field. Errors in the classical beam analysis are exposed. In the Dirac quantum beam analysis, an important role is played by non-commutative muon beam coordinates leading to a discrepancy between the classical and quantum theories. We obtain a quantum limit to the accuracy achievable in BNL type experiments. Some implications of the quantum corrected data analysis for supersymmetry are briefly mentioned.

Keywords. Muon anomalous magnetic moment; electric field corrections; quantum beams; non-Euclidean geometry; supersymmetry.

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1. Introduction

A muon's magnetic moment anomaly $\kappa = (g-2)/2$ is measured by having a beam of muons circulate in a uniform magnetic field $\mathbf{B}$. Precision measurements are possible because the anomaly frequency $\omega_\kappa = \kappa|e\mathbf{B}/Mc|$ depends upon the muon mass unlike the cyclotron frequency $\omega_L = |ec\mathbf{B}/\mathcal{E}|$, which depends upon the less precisely measured muon energy $\mathcal{E} = \gamma Mc^2$.

To focus the muons one also applies an electrostatic field $\mathbf{E}$. Quantum mechanics then makes both the cyclotron orbit and the center of the orbit non-commutative and uncertain [1–3]. There are errors in the classical analysis of the BNL [4] experiment. Even for a free particle, the Dirac quantum theory conserves only the sum of the spin and orbital angular momentum while the classical theory conserves each of the orbital angular momentum types separately. For the muon interacting with an electromagnetic field, the classical analysis generates incorrect notions of focusing electrostatic field corrections to κ. The quantum Dirac analysis is simple and elegant. The proper role of previously redundant [4] electric field corrections has been clarified [2]. The transverse electric field produce no change in the measured frequency ω_κ. This lessens the need for non-standard model contributions to the measured κ. The quantum analysis produces two sets of dynamical non-commutative coordinates. The non-commutative geometry of cyclotron beams gives an absolute quantum limit to the precision achievable in future κ experiments.

2. Quantum mechanical muon beams

In the classical theory, the position $\mathbf{r}$ of the muon may be decomposed into a center position of a cyclotron orbit $\mathbf{R}$ and a radius of curvature ρ; i.e.

$$\mathbf{r} = \rho + \mathbf{R}, \text{ where } \rho = \left(\frac{Mc\gamma(\mathbf{B}\times\mathbf{v})}{eB^2}\right). \tag{1}$$

The classical equations of motion include

$$M\frac{\mathrm{d}(\gamma\mathbf{v})}{\mathrm{d}t} = e\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right), \tag{2}$$

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = c\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right). \tag{3}$$

Quantum corrections are scaled by the Landau length $L = \sqrt{\hbar c/e|\mathbf{B}|}$. One employs Dirac Hamiltonian for the muon in the presence of a uniform magnetic field $\mathbf{B}$, and a focusing electric field $\mathbf{E} = -\mathrm{grad}\ \Phi$; i.e.

$$H = c\alpha\cdot(\mathbf{p} - (e/c)\mathbf{A}) + e\Phi + \beta Mc^2 + H_\kappa, \tag{4}$$

$$H_\kappa = \left(\frac{\hbar e\kappa}{2Mc}\right)\beta\mathbf{\Sigma}\cdot\{\mathbf{B} - i\gamma_5\mathbf{E}\}. \tag{5}$$

After the classical to quantum replacement $M\gamma\mathbf{v} \to (\mathbf{p} - (e/c)\mathbf{A})$, the components of ρ and $\mathbf{R}$ (in the plane transverse to the magnetic field) obey the commutation relations

$$[\rho_x, \rho_y] = -[R_x, R_y] = iL^2. \tag{6}$$

For two non-commuting spatial coordinates (as above) we have the quantum Pythagorean theorem, wherein the hypotenuse squared (say $|\rho|^2$ or $|\mathbf{R}|^2$) is quantized in units of L^2.

$$\begin{aligned} |\rho|^2 &= \rho_x^2 + \rho_y^2 = L^2(2n+1) \quad (n = 0,1,2,\ldots), \\ |\mathbf{R}|^2 &= R_x^2 + R_y^2 = L^2(2n+1) \quad (n = 0,1,2,\ldots). \end{aligned} \tag{7}$$

The fluctuation $(\Delta\rho)$ in ρ is determined by the Landau length L, which when compared to the mean cyclotron radius $\bar{\rho}$ is indeed quite small; $(\Delta\rho/\bar{\rho}) \sim 10^{-9}$. Thus, unless an accuracy to this level is required, we may safely consider ρ as a 'classical' variable. On the other hand, the quantum uncertainties in the orbit center position $\mathbf{R}$ do lead to an interesting quantum error limit to possible κ measurements. The inherent quantum error limit is determined by comparing the beam width W with the Landau length L. The lower bound to the error ε involved in a measurement of κ is given by

$$\varepsilon \geq \frac{L}{W\sqrt{2}} \times 10^{-7} \tag{8}$$

which may be appreciable in an ultra precise measurement.

3. The experimental role of non-commutative geometry

We mention in passing that all charged particles necessarily possess non-commutative coordinates in a plane perpendicular to the ever present (and varying) magnetic fields near the earth or the cosmos. Also, that non-commutative coordinates can be present even in the absence of magnetic fields, e.g., for a massless particle.

According to Schwinger [7], any massless particle with momentum $\mathbf{p}$ and with helicity $\Lambda = (\mathbf{p}\cdot\mathbf{S}/|\mathbf{p}|)$ such as a photon, gluon, graviton or neutrino would have its coordinates (transverse to $\mathbf{p}$) non-commuting

$$[X, Y] = -i\Lambda\hbar^2/|\mathbf{p}|^2. \tag{9}$$

The experimental consequences of the non-commutative Pythagorean theorem,

$$X^2 + Y^2 = (\Lambda\hbar/|\mathbf{p}|)^2(2n+1) \quad (n = 0, 1, 2, \ldots), \tag{10}$$

have yet to be explored.

4. Developments for quantum beams and massive neutrinos

The reaction $\pi^+ \rightarrow \mu^+ + \nu_\mu$ (and its charge conjugate) with fast pions were used in the BNL experiment to measure κ. Let us consider the frame wherein the pions are at rest. If neutrinos have three different possible mixing masses (m_1, m_2, m_3), the recoiling muon momentum would have three different possible four-momenta (p_1, p_2, p_3). Calling the three corresponding neutrino four-momenta (q_1, q_2, q_3), we have from four-momentum conservation

$$P_\pi^2 = (M_\pi c)^2 = (p_1 + q_1)^2 = (p_2 + q_2)^2 = (p_3 + q_3)^2, \tag{11}$$

giving us a different muon momentum, for each decay channel

$$\begin{aligned} p_1 \cdot q_1 - (m_1 c)^2 &= p_2 \cdot q_2 - (m_2 c)^2 = p_3 \cdot q_3 - (m_3 c)^2 \\ &= [M_\pi^2 - M_\mu^2]c^2. \end{aligned} \tag{12}$$

In a fixed magnetic field $\mathbf{B}$, the curvature radius ρ of the muon orbit would thereby be both different and non-commutative geometrically quantized. The possible measured three momentum $|\mathbf{p}_\mu|$ values would be further quantized in virtue of the quantized values of $|\rho|^2$.

It would be an experimental challenge [5] to detect these quantized orbits and muon three-momenta given by

$$\begin{aligned} c|\mathbf{p}_\mu| &= (eB\rho_n) = (eBL)\sqrt{(2n+1)} \\ &= \sqrt{ec\hbar B(2n+1)} \quad (n = 0, 1, 2, \ldots). \end{aligned} \tag{13}$$

While there is little theoretical doubt about the quantization rule in eq. (13), the experimental proof is far from trivial.

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