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Sea-boson theory of Landau–Fermi liquids, Luttinger liquids and Wigner crystals

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Abstract. It is shown how Luttinger liquids may be studied using sea-bosons. The main advantage of the sea-boson method is its ability to provide information about short-wavelength physics in addition to the asymptotics and is naturally generalizable to more than one dimension. In this article, we solve the Luttinger model and the Calogero–Sutherland model, the latter in the weak-coupling limit. The anomalous exponent we obtain in the former case is identical to the one obtained by Mattis and Lieb. We also apply this method to solve the two-dimensional analog of the Luttinger model and show that the system is a Landau–Fermi liquid. Then we solve the model of spinless fermions in one dimension with long-range (gauge) interactions and map the Wigner crystal phase of the system.

Keywords. Bosonization; Luttinger liquids.

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1. Introduction

In this article, we show how Luttinger liquids may be obtained from sea-bosons. This rectifies some serious errors of judgement of previous works [1]. In addition, we are able to solve for the one-particle properties of various systems highlighted in the abstract. Specifically, we are able to write down a closed formula for the momentum distribution of the Wigner crystal mentioned in the abstract that does not involve arbitrarily chosen momentum cutoffs. Thus our formalism is able to provide information about short-wavelength physics as well as the asymptotics. The subject of fermions in one dimension is vast and we shall only provide some representative references. Many of the more important references on this subject have already been included in our earlier work [1]. Some others are by Haldane [2], Schulz *et al* [3] and Schulz [4].

2. The sea-boson method

In this article, we lay down the prescription for computing the momentum distribution of Fermi systems using the sea-boson method. We shall be quite cryptic at times since by now the terminology is well-known [1,5]. In our earlier work [5], we made it quite clear that the sea-displacement annihilation operator has to be defined as follows:

$$A_{\mathbf{k}}(\mathbf{q}) = \frac{n_{\rm F}(\mathbf{k}-\mathbf{q}/2)(1-n_{\rm F}(\mathbf{k}+\mathbf{q}/2))}{\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}} c^{\dagger}_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2}. \tag{1}$$

This definition is ambiguous due to the square root of the number operator in the denominator, but leaving that aside (see [5] for more details), it is clear that we may use this to rewrite the number operator [5] as follows (where $n_{\rm F}(\mathbf{k}) = \theta(k_{\rm F} - |\mathbf{k}|)$ is the momentum distribution of the free Fermi gas):

$$n_{\mathbf{k}} = n_{\rm F}(\mathbf{k}) n^{<}_{\mathbf{k}} + (1-n_{\rm F}(\mathbf{k})) n^{>}_{\mathbf{k}}. \tag{2}$$

Here,

$$n^{<}_{\mathbf{k}} = \hat{\mathbf{1}} - \sum_{\mathbf{q}\neq 0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) A_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}), \tag{3}$$

$$n^{>}_{\mathbf{k}} = \sum_{\mathbf{q}\neq 0} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) A_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}), \tag{4}$$

and $\hat{\mathbf{1}} = \widehat{N}/N^0$. Here $N^0 = \sum_{\mathbf{k}} n_{\rm F}(\mathbf{k})$ and $\widehat{N} = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}}$. From eq. (2) it is clear that in general we should expect a discontinuity in the momentum distribution at $|\mathbf{k}| = k_{\rm F}$ since in general, $\langle n^{<}_{\mathbf{k}_{\rm F}} \rangle \neq \langle n^{>}_{\mathbf{k}_{\rm F}} \rangle$. This means that we may regard eq. (2) as a proof of the Luttinger theorem. This is true no matter how exactly or approximately we have solved our equations. It seems therefore that Landau–Fermi liquids are generic. But there is a way in which this theorem can be violated. The only way in which Luttinger liquids can arise for arbitrarily weak repulsion is if the sums over $\mathbf{q}$ in eq. (4) and eq. (3) diverge in such a way as to compensate for the individual terms in the sum being vanishingly small. Indeed, eq. (2) suggests a natural classification of spinless one component Fermi liquids. We may consider the functions $\langle n^{>}_{\mathbf{k}} \rangle$ and $\langle n^{<}_{\mathbf{k}} \rangle$ to be well-defined and non-zero in general (possibly with infinite slope at $|\mathbf{k}| = k_{\rm F}$) for all $\mathbf{k}$ after factoring out the discontinuous $n_{\rm F}(\mathbf{k})$ and $1 - n_{\rm F}(\mathbf{k})$.

Definition

If we have,

(i) $\langle n^{<}_{\mathbf{k}_{\rm F}} \rangle > \langle n^{>}_{\mathbf{k}_{\rm F}} \rangle$: Landau–Fermi liquid.

(ii) $\langle n^{<}_{\mathbf{k}_{\rm F}} \rangle = \langle n^{>}_{\mathbf{k}_{\rm F}} \rangle$: Residueless Fermi system.

(iii) $\langle n^{<}_{\mathbf{k}_{\rm F}} \rangle < \langle n^{>}_{\mathbf{k}_{\rm F}} \rangle$: Inverse Landau–Fermi liquid.

It seems that the third possibility has not been investigated at all, and with good reason. Later in this article, we argue that there is no room for such a possibility, the most exotic ones are Luttinger liquids (or Wigner crystals).

The operator in eq. (1) is not an exact boson but we may treat it as such and impose canonical boson commutation rules (this time we include spin for the sake of generality): $[A_{\mathbf{k}\sigma}(\mathbf{q}\sigma'), A^{\dagger}_{\mathbf{k}\sigma}(\mathbf{q}\sigma')] = n_{\mathrm{F}}(\mathbf{k}-\mathbf{q}/2)(1-n_{\mathrm{F}}(\mathbf{k}+\mathbf{q}/2))$. However, now it appears that we must pay a price for this luxury. The reason is quite subtle. Treating this object as an exact boson and using the formulas for the momentum distribution written down in the main text of our first work leads to logarithmic divergences. In order to tame these divergences we have to adopt the following prescription:

$$\langle n_{\mathbf{k}\sigma}\rangle = n_{\mathrm{F}}(\mathbf{k})\langle n^{<}_{\mathbf{k}\sigma}\rangle + (1-n_{\mathrm{F}}(\mathbf{k}))\langle n^{>}_{\mathbf{k}\sigma}\rangle, \tag{5}$$

$$\langle n^{<}_{\mathbf{k}\sigma}\rangle = \frac{1}{2}\left[1 + \frac{1}{\exp[2S^{0}_{B}(\mathbf{k}\sigma)]}\right], \tag{6}$$

$$\langle n^{>}_{\mathbf{k}\sigma}\rangle = \frac{1}{2}\left[1 - \frac{1}{\exp[2S^{0}_{A}(\mathbf{k}\sigma)]}\right]. \tag{7}$$

By definition, we have,

$$S^{0}_{B}(\mathbf{k}\sigma) = \sum_{\mathbf{q}\sigma_1}\langle G_0|a^{\dagger}_{\mathbf{k}+\mathbf{q}/2\sigma}(\mathbf{q}\sigma_1)a_{\mathbf{k}+\mathbf{q}/2\sigma}(\mathbf{q}\sigma_1)|G_0\rangle(1-n_{\mathrm{F}}(\mathbf{k}+\mathbf{q})), \tag{8}$$

$$S^{0}_{A}(\mathbf{k}\sigma) = \sum_{\mathbf{q}\sigma_1}\langle G_0|a^{\dagger}_{\mathbf{k}-\mathbf{q}/2\sigma_1}(\mathbf{q}\sigma)a_{\mathbf{k}-\mathbf{q}/2\sigma_1}(\mathbf{q}\sigma)|G_0\rangle n_{\mathrm{F}}(\mathbf{k}-\mathbf{q}). \tag{9}$$

We define, $A_{\mathbf{k}\sigma}(\mathbf{q}\sigma') = n_{\mathrm{F}}(\mathbf{k}-\mathbf{q}/2)(1-n_{\mathrm{F}}(\mathbf{k}+\mathbf{q}/2))a_{\mathbf{k}\sigma}(\mathbf{q}\sigma')$. Here $|G_0\rangle$ is the ground state of the full Hamiltonian with the sea-displacement operators being treated as canonical bosons with the commutation rule $[a_{\mathbf{k}\sigma}(\mathbf{q}\sigma'), a^{\dagger}_{\mathbf{k}\sigma}(\mathbf{q}\sigma')] = 1$; and all other commutators involving any two of these operators are zero. Also it involves ignoring the presence of the square root making the Fermi bilinear a simple linear combination. If $\mathbf{q} \neq 0$ then,

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2\sigma}c_{\mathbf{k}-\mathbf{q}/2\sigma'} = A_{\mathbf{k}\sigma}(-\mathbf{q}\sigma') + A^{\dagger}_{\mathbf{k}\sigma'}(\mathbf{q}\sigma), \tag{10}$$

$$\begin{aligned} c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma'} = n_{\mathrm{F}}(\mathbf{k})\delta_{\sigma,\sigma'} &+ \sum_{\mathbf{q}_1\sigma_1} A^{\dagger}_{\mathbf{k}-\mathbf{q}_1/2\sigma_1}(\mathbf{q}_1\sigma)A_{\mathbf{k}-\mathbf{q}_1/2\sigma_1}(\mathbf{q}_1\sigma') \\ &- \sum_{\mathbf{q}_1\sigma_1} A^{\dagger}_{\mathbf{k}+\mathbf{q}_1/2\sigma'}(\mathbf{q}_1\sigma_1)A_{\mathbf{k}+\mathbf{q}_1/2\sigma}(\mathbf{q}_1\sigma_1). \end{aligned} \tag{11}$$

From eq. (11) it is clear that we must also have, $\langle n^{<}_{\mathbf{k}\sigma}\rangle = 1 - S_B(\mathbf{k}\sigma)$ and $\langle n^{>}_{\mathbf{k}\sigma}\rangle = S_A(\mathbf{k}\sigma)$ where

$$S_{B}(\mathbf{k}\sigma) = \sum_{\mathbf{q}\sigma_1}\langle G|a^{\dagger}_{\mathbf{k}+\mathbf{q}/2\sigma}(\mathbf{q}\sigma_1)a_{\mathbf{k}+\mathbf{q}/2\sigma}(\mathbf{q}\sigma_1)|G\rangle(1-n_{\mathrm{F}}(\mathbf{k}+\mathbf{q})), \tag{12}$$

$$S_{A}(\mathbf{k}\sigma) = \sum_{\mathbf{q}\sigma_1}\langle G|a^{\dagger}_{\mathbf{k}-\mathbf{q}/2\sigma_1}(\mathbf{q}\sigma)a_{\mathbf{k}-\mathbf{q}/2\sigma_1}(\mathbf{q}\sigma)|G\rangle n_{\mathrm{F}}(\mathbf{k}-\mathbf{q}). \tag{13}$$

Here $|G\rangle$ is the ground state of the full Hamiltonian obtained by treating the operators $A_{\mathbf{k}\sigma}(\mathbf{q}\sigma')$ more carefully and also by retaining the square root of the number operator and so on. All this is needed presumably, since Luttinger liquids being non-ideal, their momentum distributions fluctuate and one must solve the system self-consistently. A simple exercise illustrates this point. Take for example the fluctuation in the number operator $N(\mathbf{k},\mathbf{k}) = \langle n^2_{\mathbf{k}}\rangle - \langle n_{\mathbf{k}}\rangle^2$. From idempotence we may write, $N(\mathbf{k},\mathbf{k}) = \langle n_{\mathbf{k}}\rangle(1-\langle n_{\mathbf{k}}\rangle)$. For a non-interacting system at zero temperature this quantity is zero. For a Landau–Fermi liquid with quasiparticle residue close to unity, this quantity is small. However, for a Luttinger liquid this quantity is large and is equal to the (square of the) mean itself, since in the vicinity of $|\mathbf{k}| = k_{\rm F}$ for a Luttinger liquid, $\langle n_{\mathbf{k}_{\rm F}}\rangle \approx 1/2$. This points to the importance of including fluctuations in the momentum distribution. In our work this is done implicitly by resumming the infrared divergences. However, we have found that the off-diagonal counterpart namely, $N(\mathbf{k},\mathbf{k}') = \langle n_{\mathbf{k}} n_{\mathbf{k}'}\rangle - \langle n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}'}\rangle$ is vanishingly small in the thermodynamic limit. A systematic mathematically rigorous approach is beyond the scope of this article, but the claim is that the prescriptions in eqs (6) and (7) capture all these essential features. In order to make this plausible, we first note the following feature. Since $0 \le S^0_{A,B} \le \infty$, we have $0 \le n^{\langle,\rangle} \le 1$, as required for fermions. Also in the case of Landau–Fermi liquid for weak coupling, $0 \le S^0_{A,B}(\mathbf{k}\sigma) \ll 1$, this leads to the simple formulas presented above with $|G\rangle = |G_0\rangle$ and $S^0_{A,B} = S_{A,B}$. In the case of a Luttinger liquid, the quantities $S^0_{A,B}(\mathbf{k}\sigma)$ diverge logarithmically in the vicinity of $|\mathbf{k}| = k_{\rm F}$. The use of eq. (6) gives us the following momentum distribution of a Luttinger liquid in the case of the Luttinger Hamiltonian and also in the case of the Calogero–Sutherland Hamiltonian (here Λ is a momentum cut-off, see below).

$$\bar{n}_k = \left[\frac{1}{2} + \frac{1}{2}\mathrm{sgn}(k_{\rm F} - |k|)\left(\frac{||k| - k_{\rm F}|}{\Lambda}\right)^{\gamma}\right]\theta(\Lambda - ||k| - k_{\rm F}|)$$
$$+\theta(||k| - k_{\rm F}| - \Lambda)\theta(k_{\rm F} - |k|). \quad (14)$$

The anomalous exponent γ is characteristic of the model used. Thus we may now proceed to compute the exponent in either case.

The other question worth answering is the one posed in the beginning, namely, is there such a thing as an 'Inverse Landau–Fermi liquid'? An examination of eqs (6) and (7) tells us that the answer is 'No'. This is because we find there that $\langle n^<_{\mathbf{k}_{\rm F}\sigma}\rangle \ge \langle n^>_{\mathbf{k}_{\rm F}\sigma}\rangle$. Thus we have accounted for Landau–Fermi liquids, Luttinger liquids (and Wigner crystals) and there is no such thing as an Inverse Landau–Fermi liquid.

3. The Luttinger model in the sea-boson language

Here we write down the spinless Luttinger model in the sea-displacement language. This is then solved by a straightforward diagonalization technique and the momentum distribution is calculated. The Hamiltonian is as follows:

$$H = \sum_{kq} \left(\frac{k \cdot q}{m} \right) A^{\dagger}_{k}(q) A_{k}(q) + \frac{\lambda}{L} \sum_{q} v(q) \left[S(q)S(-q) + S^{\dagger}(-q)S^{\dagger}(q) \right]. \tag{15}$$

We have tried to retain a close similarity with the notation in Mattis and Lieb [6]. Here, the interactions couple only the off-diagonal terms, since we anticipate that these are responsible for breaking Fermi liquid behavior. Here $S(q) = \sum_{k} A_{k}(q)C(k,q)$ and $C(k,q) = \theta(\Lambda - |k_{\mathrm{F}} - |k+q/2||)\theta(\Lambda - |k_{\mathrm{F}} - |k-q/2||)$. The cut-off Λ is needed in order to ensure that only electrons in a shell of $[\pm k_{\mathrm{F}} - \Lambda, \pm k_{\mathrm{F}} + \Lambda]$ interact and the rest are free. We also postulate that $v(q) = v(0)\theta(2\Lambda - |q|)$. In other words, the delta-function repulsion, but with $|q|$ restricted to be small enough. If $q > 0$ then we find that the sum over k in the definition of $S(q)$ is in the region $k \in [k_{\mathrm{F}} - \Lambda, k_{\mathrm{F}} + \Lambda]$. Thus this corresponds to right movers or ρ_1 in the notation of Lieb and Mattis. Similarly, $S(-q)$ involves the sum peaked at $k = -k_{\mathrm{F}}$ and corresponds to left movers. Thus this term couples the right movers to the left movers and is responsible for breaking Fermi liquid behavior. In the absence of $\theta(2\Lambda - |q|)$, we find, due to the identity below and the repulsion–attraction duality [5] (more commonly called back scattering) the interaction term vanishes identically ($\sum_{q} S(q)S(-q) \equiv 0$). Thus we must really make sure that $v(k-k')$ can be ignored in comparison with $v(q)$. This is possible if we include $\theta(2\Lambda - |q|)$ because, then $\theta(2\Lambda - |k-k'|) \approx \theta(2\Lambda - |2k_{\mathrm{F}}|) = 0$. We shall not provide the details of the diagonalization which is by now quite well-known to the readers. The dispersion of the collective mode is $\omega_{\mathrm{c}}(q) = v_{\mathrm{eff}}|q| = v_{\mathrm{F}}(1 - [\lambda v(0)/\pi v_{\mathrm{F}}]^2)^{1/2}|q|$. Thus we arrive at the following formula for the sea-boson number operator.

$$\begin{aligned} \langle G_0 | a^{\dagger}_{k}(q) a_{k}(q) | G_0 \rangle &= \left(\frac{2\pi}{|q|} \right) \left(\frac{\lambda v(0)}{\pi} \right)^2 \frac{1}{L} \frac{C(k,q)}{2 v_{\mathrm{eff}} (v_{\mathrm{F}} + v_{\mathrm{eff}})} \\ &= \left(\frac{2\pi}{|q|} \right) \frac{1}{2L} \, C(k,q) \\ &\quad \times \left[\left(1 - \left(\frac{\lambda v(0)}{\pi v_{\mathrm{F}}} \right)^2 \right)^{-1/2} - 1 \right]. \end{aligned} \tag{16}$$

Thus we see the emergence of the sort of exponents found by Lieb and Mattis [6]. Now we calculate $S^{0}_{A}(k) \approx S^{0}_{B}(k) = S_0(k)$ as follows:

$$\begin{aligned} S_0(k) &= \sum_{q} \langle G_0 | a^{\dagger}_{k-q/2}(q) a_{k-q/2}(q) | G_0 \rangle \theta(k_{\mathrm{F}} - |k-q|) \\ &= \frac{L}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}q \, \left(\frac{2\pi}{|q|} \right) \frac{1}{2L} \, \theta(\Lambda - |k_{\mathrm{F}} - |k||)\theta(\Lambda - |k_{\mathrm{F}} - |k-q||)) \\ &\quad \times \theta(k_{\mathrm{F}} - |k-q|) \left[\left(1 - \left(\frac{\lambda v(0)}{\pi v_{\mathrm{F}}} \right)^2 \right)^{-1/2} - 1 \right] \\ &= \frac{1}{2} \theta(\Lambda - |k_{\mathrm{F}} - |k||) \left[\left(1 - \left(\frac{\lambda v(0)}{\pi v_{\mathrm{F}}} \right)^2 \right)^{-1/2} - 1 \right] \mathrm{Ln} \left(\frac{\Lambda}{|k_{\mathrm{F}} - |k||} \right). \end{aligned} \tag{17}$$

The momentum distribution is given by (using eqs (6), (7) when $||k| - k_{\rm F}| < \Lambda$),

$$\bar{n}_k = \frac{1}{2}\left[1 + {\rm sgn}(k_{\rm F} - |k|)\left(\frac{|k_{\rm F} - |k||}{\Lambda}\right)^{\gamma}\right] \tag{18}$$

with the exact exponent completely identical to the one in Mattis and Lieb,

$$\gamma = \left[\left(1 - \left(\frac{\lambda v(0)}{\pi v_{\rm F}}\right)^2\right)^{-1/2} - 1\right]. \tag{19}$$

They [6] set $v_{\rm F} = 1$ which we have displayed explicitly here.

4. The Calogero–Sutherland model

Now we would like to see if other systems can also be Luttinger liquids, like the Calogero–Sutherland model (CSM). If one uses the form $V_q = -\pi\beta(\beta-1)|q|/(2m)$ (which is the Fourier transform of the inverse square interaction) in the formulas in our earlier work, we find that the integrals are finite at $|k| = k_{\rm F}$ and this means we have a Landau–Fermi liquid for small $\beta(\beta-1)$. This result contradicts the exact solution via Jack polynomials [7] which shows unequivocally that the CSM is a Luttinger liquid. The reason for this fallacy is the repulsion–attraction duality explained in our earlier work [5]. The full Hamiltonian consists of a kinetic plus exchange part and a correlation part. The correlation part in terms of the sea-bosons that takes into account the repulsion–attraction duality has already been written down [5]. We try and solve the model when $0 < \beta(\beta-1) \ll 1$. In general we would have to diagonalize the non-separable Hamiltonian given below.

$$H = \sum_{kq}\left(\frac{k\cdot q}{m}\right)A_k^\dagger(q)A_k(q) + \sum_{k\neq k'}\sum_q \frac{V_q - V_{k-k'}}{2L}[A_k(-q) + A_k^\dagger(q)][A_{k'}(q) + A_{k'}^\dagger(-q)]. \tag{20}$$

In the weak coupling case we may write, retaining only the terms that break Fermi liquid behavior,

$$H = \sum_{kq}\left(\frac{k\cdot q}{m}\right)A_k^\dagger(q)A_k(q) - \sum_{k\neq k'}\sum_q \frac{V_{2k_{\rm F}}}{2L}[A_k(-q)A_{k'}(q) + A_k^\dagger(q)A_{k'}^\dagger(-q)]C(k,q). \tag{21}$$

Using ideas given earlier, we may read off the anomalous exponent $\gamma \approx \lambda^2 v^2(0)/2\pi^2 v_{\rm F}^2 = (\beta(\beta-1))^2/8$. Clearly, for $0 < \beta(\beta-1) \ll 1$ we have $\gamma \ll 1$. But for $\beta = 3$ the anomalous exponent deduced using the Mattis–Lieb formula is imaginary. This means we have to include forward scattering terms (that is V_q in addition to $V_{k-k'}$) that are known to make the exponent real no matter how strong the coupling is. The solution of the CSM using Jack polynomials [7] tells us that $\gamma < 1$ for $\beta = 3$ [7a]. This means we really have to be more careful when dealing with systems that have $|\lambda v(0)| > \pi v_{\rm F}$. Unfortunately, we have not succeeded in reproducing the results obtained via Jack polynomials even qualitatively, apart from being reasonably sure that the system is in fact a Luttinger liquid. This is due to the non-separable nature of the Hamiltonian (eq. (20)) in the sea-boson language brought about by the need to invoke repulsion–attraction duality [5]. We also note that recently Liu [8] has shown how to compute the asymptotics of the one-particle correlation functions in various dimensions for various systems using a powerful method known as eigenfunctional theory.

5. The Luttinger model in two dimensions

We solve the analog of the Luttinger model in two space dimensions and show that ground state of the system is a Landau–Fermi liquid. Many authors have in the past addressed this issue. Notable among those are works by Anderson [9], Baskaran [10] and Bares and Wen [11]. In the present case, the Hamiltonian in the sea-boson language is given by

$$H = \sum_{\mathbf{kq}} \left(\frac{\mathbf{k}\cdot\mathbf{q}}{m}\right) A^{\dagger}_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q}) + \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{kk'}} [A_{\mathbf{k}}(-\mathbf{q}) + A^{\dagger}_{\mathbf{k}}(\mathbf{q})][A_{\mathbf{k'}}(\mathbf{q}) + A^{\dagger}_{\mathbf{k'}}(-\mathbf{q})]. \tag{22}$$

This Hamiltonian describes a self-interacting Fermi gas provided we assume that $v_{\mathbf{q}} = (v_0/m)\theta(\Lambda - |\mathbf{q}|)$ where $\Lambda \ll k_{\rm F}$. Here $0 < v_0 \ll 1$ is a dimensionless parameter. We may solve for the boson occupation number as follows:

$$\langle A^{\dagger}_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle = \frac{1}{V}\sum_{i} \frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{(\omega_i + (\mathbf{k}\cdot\mathbf{q}/m))^2} g_i^2(-\mathbf{q}), \tag{23}$$

where $\Lambda_{\mathbf{k}}(-\mathbf{q}) = n_{\rm F}(\mathbf{k}-\mathbf{q}/2)(1 - n_{\rm F}(\mathbf{k}+\mathbf{q}/2))$.

$$g_i^{-2}(-\mathbf{q}) = \frac{1}{V}\sum_{\mathbf{k}} \frac{n_{\rm F}(\mathbf{k}-\mathbf{q}/2) - n_{\rm F}(\mathbf{k}+\mathbf{q}/2)}{(\omega_i - (\mathbf{k}\cdot\mathbf{q}/m))^2} \tag{24}$$

$$\epsilon_{\rm RPA}(\mathbf{q},\omega) = 1 + \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}} \frac{n_{\rm F}(\mathbf{k}+\mathbf{q}/2) - n_{\rm F}(\mathbf{k}-\mathbf{q}/2)}{\omega - (\mathbf{k}\cdot\mathbf{q}/m)}. \tag{25}$$

As argued in earlier works, we have to interpret sum over modes with care so as to not lose the particle-hole mode, the collective mode being obvious. This is particularly important in two dimensions where we expect both to be present. Thus the sum over modes is defined as follows:

$$\sum_{i} f(\mathbf{q},\omega_i) = \frac{\int_0^{\infty} {\rm d}\omega W(\mathbf{q},\omega) f(\mathbf{q},\omega)}{\int_0^{\infty} {\rm d}\omega W(\mathbf{q},\omega)}. \tag{26}$$

Here the weight function is given by

$$W(\mathbf{q},\omega) = -\mathrm{Im}\left(\frac{1}{\epsilon_{\mathrm{RPA}}(\mathbf{q},\omega - i0^{+})}\right). \tag{27}$$

In our earlier work, we had suggested that in the above formula we have to use a dielectric function that is sensitive to significant qualitative changes in one-particle properties. The simple RPA dielectric function does not possess qualities we expect from a Wigner crystal. Thus we shall have to derive a new dielectric function using the localized basis rather than the plane-wave basis. In the case of the Fermi gas in two dimensions we find that the system is a Landau–Fermi liquid and there is no need to use a better dielectric function, the simple RPA suffices. The momentum distribution is always given by

$$\langle n_{\mathbf{k}}\rangle = \frac{1}{2}\left[1 + \mathrm{e}^{-2S^{0}_{B}(\mathbf{k})}\right] n_{\mathrm{F}}(\mathbf{k}) + \frac{1}{2}\left[1 - \mathrm{e}^{-2S^{0}_{A}(\mathbf{k})}\right](1 - n_{\mathrm{F}}(\mathbf{k})) \tag{28}$$

$$S^{0}_{A}(\mathbf{k}) = \sum_{\mathbf{q}} \langle A^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) A_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q})\rangle \tag{29}$$

$$S^{0}_{B}(\mathbf{k}) = \sum_{\mathbf{q}} \langle A^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) A_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q})\rangle. \tag{30}$$

The computation of the boson occupation number $\langle A^{\dagger}_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle$ is the key for evaluating one-particle properties.

In the case of a Fermi gas in two dimensions, with short-range repulsive interactions, we may use the simple RPA dielectric function. The integrals are somewhat complicated since in two dimensions, the angular parts are very troublesome unlike in three dimensions. Therefore we copy the results first derived by Stern [12].

$$\epsilon^{r}_{\mathrm{RPA}}(q,\omega) = 1 + \frac{mk_{\mathrm{F}}v_q}{2\pi q}\left\{\frac{q}{k_{\mathrm{F}}} - C_{-}\left[\left(\frac{q}{2k_{\mathrm{F}}} - \frac{m\omega}{k_{\mathrm{F}}q}\right)^{2} - 1\right]^{1/2} - C_{+}\left[\left(\frac{q}{2k_{\mathrm{F}}} + \frac{m\omega}{k_{\mathrm{F}}q}\right)^{2} - 1\right]^{1/2}\right\}, \tag{31}$$

$$\epsilon^{i}_{\mathrm{RPA}}(q,\omega) = \frac{mk_{\mathrm{F}}v_q}{2\pi q}\left\{D_{-}\left[1 - \left[\frac{q}{2k_{\mathrm{F}}} - \frac{m\omega}{k_{\mathrm{F}}q}\right]^{2}\right]^{1/2} - D_{+}\left[1 - \left[\frac{q}{2k_{\mathrm{F}}} + \frac{m\omega}{k_{\mathrm{F}}q}\right]^{2}\right]^{1/2}\right\}, \tag{32}$$

where

$$C_{\pm} = \mathrm{sgn}\left[\frac{q}{2k_{\mathrm{F}}} \pm \frac{m\omega}{k_{\mathrm{F}}q}\right], \quad D_{\pm} = 0, \quad \left|\frac{q}{2k_{\mathrm{F}}} \pm \frac{m\omega}{k_{\mathrm{F}}q}\right| > 1 \tag{33}$$

$$C_{\pm} = 0, \quad D_{\pm} = 1, \quad \left| \frac{q}{2k_{\mathrm{F}}} \pm \frac{m\omega}{k_{\mathrm{F}}q} \right| < 1 \tag{34}$$

$$g^{-2}(-\mathbf{q}, \omega) = \frac{mk_{\mathrm{F}}}{2\pi q} \left\{ \frac{mC_{-}}{k_{\mathrm{F}}q} \left[\left(\frac{q}{2k_{\mathrm{F}}} - \frac{m\omega}{k_{\mathrm{F}}q} \right)^2 - 1 \right]^{-1/2} \left(\frac{q}{2k_{\mathrm{F}}} - \frac{m\omega}{k_{\mathrm{F}}q} \right) \right.$$
$$\left. - \frac{mC_{+}}{k_{\mathrm{F}}q} \left[\left(\frac{q}{2k_{\mathrm{F}}} + \frac{m\omega}{k_{\mathrm{F}}q} \right)^2 - 1 \right]^{-1/2} \left(\frac{q}{2k_{\mathrm{F}}} + \frac{m\omega}{k_{\mathrm{F}}q} \right) \right\} \tag{35}$$

$$\langle A^{\dagger}_{\mathbf{k}}(\mathbf{q}) A_{\mathbf{k}}(\mathbf{q}) \rangle = \frac{1}{V} \frac{1}{Z(q)} \int_0^{\infty} \mathrm{d}\omega W(q, \omega) \frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{(\omega + \frac{\mathbf{k}\cdot\mathbf{q}}{m})^2} g^2(-\mathbf{q}, \omega)$$
$$W(q, \omega) = \frac{\epsilon^{i}_{\mathrm{RPA}}(q, \omega)}{\epsilon^{r2}_{\mathrm{RPA}}(q, \omega) + \epsilon^{i2}_{\mathrm{RPA}}(q, \omega)} \tag{36}$$

$$Z(q) = \int_0^{\infty} \mathrm{d}\omega W(q, \omega). \tag{37}$$

In general we have

$$S^0_A(\mathbf{k}) = \frac{1}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q\, q \frac{1}{Z(q)} \int_0^{\infty} \mathrm{d}\omega W(q, \omega) f_A(k, q, \omega) g^2(-\mathbf{q}, \omega), \tag{38}$$

$$S^0_B(\mathbf{k}) = \frac{1}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q\, q \frac{1}{Z(q)} \int_0^{\infty} \mathrm{d}\omega W(q, \omega) f_B(k, q, \omega) g^2(-\mathbf{q}, \omega). \tag{39}$$

The rest of the details are relegated to Appendix A. The collective mode occurs when $\mathrm{Im}[\epsilon] = 0$, that is, for small enough q. This means that we have to treat this separately.

$$S^0_A(\mathbf{k}) = \frac{1}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q\, q \frac{1}{Z(q)} \int_0^{\infty} \mathrm{d}\omega W(q, \omega) f_A(k, q, \omega) g^2(-\mathbf{q}, \omega)$$
$$+ \frac{1}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q\, q f_A(k, q, \omega_{\mathrm{c}}) g^2(-\mathbf{q}, \omega_{\mathrm{c}}), \tag{40}$$
$$S^0_B(\mathbf{k}) = \frac{1}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q\, q \frac{1}{Z(q)} \int_0^{\infty} \mathrm{d}\omega W(q, \omega) f_B(k, q, \omega) g^2(-\mathbf{q}, \omega)$$
$$+ \frac{1}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q\, q f_B(k, q, \omega_{\mathrm{c}}) g^2(-\mathbf{q}, \omega_{\mathrm{c}}). \tag{41}$$

Here it is implicit that in W we assume that $\mathrm{Im}[\epsilon] \neq 0$. The dispersion of the collective mode may be found using *Mathematica*^{TM}. It is given below:

$$\omega_{\mathrm{c}}(q) = q(2\pi + mv_q) \frac{(\pi^2 q^2 + m\pi q^2 v_q + k^2_{\mathrm{F}} m^2 v^2_q)^{1/2}}{2m^2 \sqrt{\pi} v_q \sqrt{\pi + mv_q}}. \tag{42}$$

This dispersion is real and positive for all q and for all $v_q > 0$. Thus in the small q limit, where using just the RPA dielectric function is justified and is also the limit where close to the Fermi surface features of the momentum distribution is given exactly, we are justified in retaining only the coherent part. Thus we may write

$$S_A^0(\mathbf{k}) \approx \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}q q f_A(k, q, \omega_{\mathrm{c}}) g^2(-\mathbf{q}, \omega_{\mathrm{c}}), \tag{43}$$

$$S_B^0(\mathbf{k}) \approx \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}q q f_B(k, q, \omega_{\mathrm{c}}) g^2(-\mathbf{q}, \omega_{\mathrm{c}}). \tag{44}$$

To determine whether or not Fermi liquid theory breaks down, we have to compute

$$S_A^0(k_{\mathrm{F}}) \approx \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}q q f_A(k_{\mathrm{F}}, q, \omega_{\mathrm{c}}) g^2(-\mathbf{q}, \omega_{\mathrm{c}}), \tag{45}$$

$$S_B^0(k_{\mathrm{F}}) \approx \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}q q f_B(k_{\mathrm{F}}, q, \omega_{\mathrm{c}}) g^2(-\mathbf{q}, \omega_{\mathrm{c}}). \tag{46}$$

If $S_A^0(k_{\mathrm{F}}), S_B^0(k_{\mathrm{F}}) < \infty$ then the ground state is a Landau–Fermi liquid. If $S_A^0(k_{\mathrm{F}}) = S_B^0(k_{\mathrm{F}}) = \infty$ then the system is a non-Fermi liquid. For small q if we set $\omega_{\mathrm{c}} = v_{\mathrm{eff}} q$ we have,

$$f_A(k_{\mathrm{F}}, q, \omega_{\mathrm{c}}) \sim f_B(k_{\mathrm{F}}, q, \omega_{\mathrm{c}}) \sim 1/q^2. \tag{47}$$

Also,

$$g^2(-\mathbf{q}, \omega_{\mathrm{c}}) \sim q. \tag{48}$$

Thus the integrals in eqs (45) and (46) are infrared finite. This means that $S_A^0(k_{\mathrm{F}}), S_B^0(k_{\mathrm{F}}) < \infty$ and the system is a Landau–Fermi liquid. The details of the momentum distribution can be worked out but are not very important. A closed formula for the quasiparticle residue may be written down as shown below:

$$Z_{\mathrm{F}} = \frac{\mathrm{e}^{-2S_B^0(k_{\mathrm{F}})} + \mathrm{e}^{-2S_A^0(k_{\mathrm{F}})}}{2}. \tag{49}$$

6. One-dimensional system with long-range interactions

In this section, we consider electrons on a circle interacting via a two-body attractive long-range interaction with strength proportional to the separation between the electrons. In this case, we expect the system to be a Wigner crystal since the interaction is long-range and actually increases (in magnitude) with separation rather than decreases. This means that the electrons prefer to be as far apart from each other as possible to lower the energy leading to a crystalline ground state. Thus we have to be careful about the choice of the dielectric function. First, we postulate that $v_q = 2e^2/(qa)^2$ which corresponds to the gauge potential. Here a has dimensions of length and $e^2 > 0$ is dimensionless. From the form of this potential, one hopes that we need not concern ourselves with the issues that were relevant in the case of the Calogero–Sutherland model, namely the repulsion–attraction duality. Thus we may write as before

$$\langle A^{\dagger}_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle = \frac{1}{V}\frac{1}{Z(q)}\int_0^{\infty}\mathrm{d}\omega W(q,\omega)\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{(\omega+(\mathbf{k}\cdot\mathbf{q}/m))^2}g^2(-\mathbf{q},\omega),$$

$$W(q,\omega) = \frac{\epsilon^i(q,\omega)}{\epsilon^{r2}(q,\omega)+\epsilon^{i2}(q,\omega)}, \tag{50}$$

$$Z(q) = \int_0^{\infty}\mathrm{d}\omega W(q,\omega), \tag{51}$$

$$g^{-2}(-\mathbf{q},\omega) = \frac{1}{v_q}\frac{\partial}{\partial\omega}\epsilon(\mathbf{q},\omega). \tag{52}$$

As mentioned before, we have to be extra careful in making sure that we choose the right dielectric function. The RPA-dielectric function is not likely to suffice since its static structure factor (SSF) does not exhibit the features we expect from a Wigner crystal. In particular, we expect $S(2k_{\mathrm{F}}) = \infty$ as we shall see soon. To convince ourselves of this, we ascertain the properties of the RPA dielectric function with long-range interactions.

$$\epsilon^r_{\mathrm{RPA}}(q,\omega) = 1 + v_q\frac{m}{2\pi q}\log\left[\frac{(k_{\mathrm{F}}+q/2)^2-(m\omega/q)^2}{(k_{\mathrm{F}}-q/2)^2-(m\omega/q)^2}\right]. \tag{53}$$

The zero of the above dielectric function gives us the dispersion of the collective modes.

$$\omega_{\mathrm{c}}(q) = \frac{|q|}{m}\sqrt{\frac{(k_{\mathrm{F}}+q/2)^2-(k_{\mathrm{F}}-q/2)^2\exp(-2\pi q/mv_q)}{1-\exp(-2\pi q/mv_q)}}. \tag{54}$$

For $|q| \ll k_{\mathrm{F}}$ and $v_q = 2e^2/(aq)^2$ we find,

$$\omega_{\mathrm{c}}(q) \approx \frac{1}{m}\sqrt{\frac{e^2k_{\mathrm{F}}m}{a^2}}\sqrt{\frac{2}{\pi}} + \frac{a^2k_{\mathrm{F}}\sqrt{(e^2k_{\mathrm{F}}m/a^2)}\sqrt{(\pi/2)}q^2}{2e^2m^2} + O(q^4). \tag{55}$$

This plasmon-like gap $\omega_0 \equiv (1/m)\sqrt{(e^2k_{\mathrm{F}}m/a^2)}\sqrt{2/\pi}$ in the collective mode is present due to the characteristic $1/q^2$ nature of the potential. But this is also present in the three-dimensional electron gas and is not a sign of an insulator since the latter is not at high densities. A gap in the *one-particle* Green function could be taken as a sign of insulating behavior [13]. However, in our approach we are unable to compute the full Green function as yet. Thus we must resort to a more indirect approach. For a Wigner crystal, the SSF must exhibit certain singularities. Thus we have to use the generalized-RPA that is sensitive to qualitative changes in single-particle properties. The new dielectric function will involve the full momentum distribution which has to be determined self-consistently using the above sea-boson equations. In our earlier work we suggested that the new dielectric function should also involve fluctutations in the momentum distribution. However, it now appears that, that is fortunately not needed. The number–number correlation function is vanishingly small in the thermodynamic limit as shown in another preprint and this means we may simply write,

$$\epsilon(\mathbf{q},\omega) = 1 + \frac{v_q}{L}\sum_k \frac{\bar{n}_{k+q/2} - \bar{n}_{k-q/2}}{\omega - \xi_{k+q/2} + \xi_{k-q/2}} \tag{56}$$

$$\xi_k = \frac{k^2}{2m} - \sum_{q\neq 0} \frac{v_q}{L}\bar{n}_{k-q} \tag{57}$$

and the momentum distribution is determined self-consistently using the sea-boson equation (eq. (28)). This is too difficult to solve analytically and hence we have to resort to a numerical solution. In order to simplify proceedings even further, we use only the collective mode. The particle–hole mode which is due to a non-zero Im$[\epsilon]$ is needed if one is interested in features of the momentum distribution away from the Fermi surface more accurately. However, we shall hope that this is not given too badly even at these regions far from the Fermi points.

$$\langle A^{\dagger}_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle = \frac{1}{V}\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{(\omega_{\rm c}(q) + (k\cdot q/m))^2} g^2(-\mathbf{q},\omega_{\rm c}) \tag{58}$$

$$g^{-2}(-\mathbf{q},\omega_{\rm c}) = \frac{m\omega_{\rm c}}{\pi q}\left[\frac{1}{\omega_{\rm c}^2 - (v_{\rm F}q + \epsilon_q)^2} - \frac{1}{\omega_{\rm c}^2 - (v_{\rm F}q - \epsilon_q)^2}\right] \tag{59}$$

$$S^0_A(k) = \frac{1}{L}\sum_q \frac{n_{\rm F}(k-q)}{(\omega_{\rm c}(q) + (k\cdot q/m) - \epsilon_q)^2} g^2(-\mathbf{q},\omega_{\rm c}) \tag{60}$$

$$S^0_B(k) = \frac{1}{L}\sum_q \frac{(1 - n_{\rm F}(k+q))}{(\omega_{\rm c}(q) + (k\cdot q/m) + \epsilon_q)^2} g^2(-\mathbf{q},\omega_{\rm c}). \tag{61}$$

To proceed further, we have to ascertain the nature of the collective modes $\omega_{\rm c}$. If we use the RPA-dielectric function, we find a constant dispersion (plasmon) for small $|q|$. However, we have found that this choice is inconsistent because if we use the momentum distribution obtained from this to solve for the dielectric function and recompute the collective mode, we obtain a completely different answer, namely, $\omega_{\rm c}(q) = v_s|q|$. Therefore it is critical that we get the dispersion right. It appears then that we have to use the form given in the appendix which is not easy to simplify. A systematic approach for obtaining the dispersion of the collective modes has been suggested by Sen and Baskaran [10]. Since the plane-wave basis is not appropriate for deriving a formula for the dielectric function of a Wigner crystal, we shall follow this approach. First, we would like to ascertain the lattice structure in the small a limit. In this limit, the potential energy dominates over the kinetic energy. If we assume that the electrons are all on a circle of perimeter L then to minimize the potential energy, we have to maximize the separation. This leads to an equally spaced set of lattice points with lattice constant $l_{\rm c}$ such that $Nl_{\rm c} = L$. Thus we have $l_{\rm c} = 1/\rho_0 = \pi/k_{\rm F}$. Thus we assume that the electrons all lie on a circle with equal spacing between them. Therefore we expect the structure factor to diverge for a momentum $q_0 = 2\pi/l_{\rm c} = 2k_{\rm F}$. From the Bijl-Feynman formula $S(q) = \epsilon_q/\omega_{\rm c}(q)$ we may suspect that a choice of $\omega_{\rm c}$ that vanishes at $q = 2k_{\rm F}$ is needed. The form of the dispersion is given in the appendix. For $\pi/N \ll |ql_{\rm c}| \ll 2\pi$ it seems that $\omega_q \approx \omega_0$. For $|ql_{\rm c}| \ll \pi/N$ we have to be more careful. And of course we must have $\omega_q = 0$ for $ql_c = \pm 2\pi$. But since in the thermodynamic limit $\pi/N \approx 0$ we may choose (hopefully) $\omega_q \approx \omega_0$. In figures 1 and 2, the momentum distribution obtained from these formulas has been plotted. In fact, we may write down a closed formula for the momentum distribution.

$$\bar{n}_k = \frac{1}{2}\left(1 + \exp\left[-\frac{m\omega_0}{k_F^2 - k^2}\right]\right) n_F(k) + \frac{1}{2}\left(1 - \exp\left[-\frac{m\omega_0}{k^2 - k_F^2}\right]\right)(1 - n_F(k)). \quad (62)$$

The striking feature of this momentum distribution is that it is perfectly flat at $|k| = k_F$. In other words, not only is the slope zero but also all the derivatives of the momentum distribution vanish at $|k| = k_F$. This is a striking prediction. This may be contrasted with the smooth Gaussian function of Gori-Giorgi and Ziesche [14] (eq. (B1) in their Appendix B). But they consider three-dimensional systems which may be different from the one studied here. One particle spectral functions are accessible to tunneling experiments or angle-resolved photoemission spectroscopy (ARPES). A more difficult problem may be to experimentally realize a 1d electron system with long-range gauge interactions.

The formula below for the static structure factor is derived in the appendix.

$$S(q) = \frac{1}{N}\frac{\sin^2(qNl_c/2)}{\sin^2(ql_c/2)} e^{-q^2/m\omega_0}. \quad (63)$$

This may be further simplified in the thermodynamic limit as follows. Consider

$$\delta(x) \approx \frac{\sin(Nx)}{\pi x}. \quad (64)$$

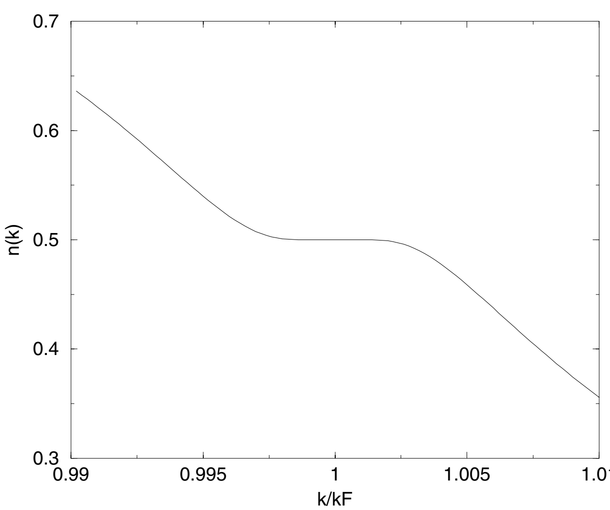

Figure 1. Momentum distribution of a Wigner crystal (with zoom).

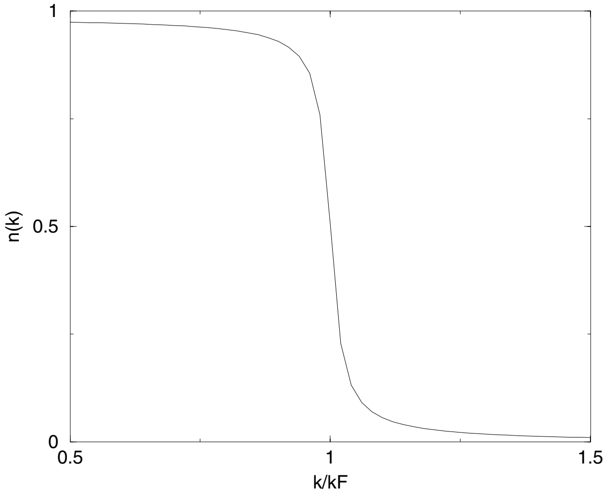

Figure 2. Momentum distribution of a Wigner crystal (no zoom).

Then we may write,

$$S(q) = \delta(q)\frac{2\pi}{l_{\rm c}} + \delta(q-2k_{\rm F})\frac{2\pi}{l_{\rm c}}{\rm e}^{-4k_{\rm F}^2/m\omega_0}. \tag{65}$$

As we can see here the structure factor diverges at $|q| = 2k_{\rm F}$ which means the system is a Wigner crystal (a true Wigner crystal, since the divergence is from a delta-function).

7. Conclusions

To conclude, we have finally shown how to reproduce Luttinger liquids using sea-bosons. We have also shown how systems with both long-range and short-range interactions may be studied with almost equal ease and that too in any number of dimensions. Using the same formalism we may also probe short-wavelength physics as illustrated by the momentum distribution of the Wigner crystal that contains no momentum cut-offs. Finally we showed that the two-dimensional analog of the Luttinger model is a Landau–Fermi liquid.

Appendix A

$$f_A(k,q,\omega) = \int_0^{2\pi} {\rm d}\theta \frac{\theta(k_{\rm F}^2 - k^2 - q^2 + 2kq\ \cos(\theta))}{(\omega + (kq/m)\cos(\theta) - \epsilon_q)^2}, \tag{A1}$$

$$f_B(k,q,\omega) = \int_0^{2\pi} {\rm d}\theta \frac{\theta(k^2 + q^2 - k_{\rm F}^2 + 2k\,q\ \cos(\theta))}{(\omega + (kq/m)\cos(\theta) + \epsilon_q)^2}. \tag{A2}$$

Define

$$u(x;A,B) \equiv \int \mathrm{d}x \frac{1}{(A+B\cos[x])^2}. \tag{A3}$$

From $Mathematica^{TM}$ we find,

$$u(x;A,B) = -2A \frac{\mathrm{arctanh}\left[\frac{(A-B)\tan\left[\frac{x}{2}\right]}{\sqrt{B^2-A^2}}\right]}{(A^2-B^2)\sqrt{B^2-A^2}} + \frac{B\sin[x]}{(B^2-A^2)(A+B\cos[x])}. \tag{A4}$$

Integrating by parts we find

$$f_A(k,q,\omega) = \int_0^{2\pi} \mathrm{d}\theta \sin\theta \ \ \delta\left(\frac{k_{\mathrm{F}}^2-k^2-q^2}{2kq} + \cos(\theta)\right) \times u\left(\theta; \omega - \epsilon_q, \frac{kq}{m}\right), \tag{A5}$$

$$f_B(k,q,\omega) = \int_0^{2\pi} \mathrm{d}\theta \sin\theta \ \delta\left(\frac{k^2+q^2-k_{\mathrm{F}}^2}{2kq} + \cos(\theta)\right) \times u\left(\theta; \omega + \epsilon_q, \frac{kq}{m}\right). \tag{A6}$$

This may be rewritten as

$$\begin{aligned} f_A(k,q,\omega) = &\int_0^{\pi} \mathrm{d}\theta \sin\theta \ \delta\left(\frac{k_{\mathrm{F}}^2-k^2-q^2}{2kq} + \cos(\theta)\right) u\left(\theta; \omega - \epsilon_q, \frac{kq}{m}\right) \\ &- \int_0^{\pi} \mathrm{d}\theta \sin\theta \ \delta\left(\frac{k_{\mathrm{F}}^2-k^2-q^2}{2kq} - \cos(\theta)\right) u\left(\theta + \pi; \omega - \epsilon_q, \frac{kq}{m}\right) \end{aligned} \tag{A7}$$

$$\begin{aligned} f_B(k,q,\omega) = &\int_0^{\pi} \mathrm{d}\theta \sin\theta \ \delta\left(\frac{k^2+q^2-k_{\mathrm{F}}^2}{2kq} + cos(\theta)\right) u\left(\theta; \omega + \epsilon_q, \frac{kq}{m}\right) \\ &- \int_0^{\pi} \mathrm{d}\theta \sin\theta \ \delta\left(\frac{k^2+q^2-k_{\mathrm{F}}^2}{2kq} - \cos(\theta)\right) u\left(\theta + \pi; \omega + \epsilon_q, \frac{kq}{m}\right) \end{aligned} \tag{A8}$$

$$\theta_0 = \arccos\left[\frac{k^2+q^2-k_{\mathrm{F}}^2}{2kq}\right] \tag{A9}$$

$$\theta_0' = \arccos\left[\frac{-k^2-q^2+k_{\mathrm{F}}^2}{2kq}\right] \tag{A10}$$

$$f_A(k,q,\omega) = u\left(\theta_0;\omega-\epsilon_q,\frac{kq}{m}\right)\left[\theta\left(\frac{k_{\rm F}^2-k^2-q^2}{2kq}+1\right)\right.$$
$$\left.-\theta\left(\frac{k_{\rm F}^2-k^2-q^2}{2kq}-1\right)\right] - u\left(\theta_0'+\pi;\omega-\epsilon_q,\frac{kq}{m}\right)$$
$$\times\left[\theta\left(1-\frac{k_{\rm F}^2-k^2-q^2}{2kq}\right)-\theta\left(-1-\frac{k_{\rm F}^2-k^2-q^2}{2kq}\right)\right] \tag{A11}$$

$$f_B(k,q,\omega) = u\left(\theta_0';\omega+\epsilon_q,\frac{kq}{m}\right)\left[\theta\left(\frac{k^2+q^2-k_{\rm F}^2}{2kq}+1\right)\right.$$
$$\left.-\theta\left(\frac{k^2+q^2-k_{\rm F}^2}{2kq}-1\right)\right] - u\left(\theta_0+\pi;\omega+\epsilon_q,\frac{kq}{m}\right)$$
$$\times\left[\theta\left(1-\frac{k^2+q^2-k_{\rm F}^2}{2kq}\right)-\theta\left(-1-\frac{k^2+q^2-k_{\rm F}^2}{2kq}\right)\right]. \tag{A12}$$

Appendix B

Here we use the approach suggested by Sen [10] to derive a formula for the collective modes. The formula (eq. (56)) although probably right is not very illuminating, for it is hard to see how the structure factor derived from this formula possesses the features we expect, namely, a divergence at $|q| = 2k_{\rm F}$. Thus we would like to derive a formula for the dielectric function where this feature is manifest. To do this we adopt the localized basis rather than the plane-wave basis. In real space, the Hamiltonian we are studying is written as follows:

$$H = \sum_{i=0}^{N-1}\frac{p_i^2}{2m} - \frac{e^2}{a^2}\sum_{i>j}|x_i-x_j|. \tag{B1}$$

We assume that particles are on a circle and $|x|$ is the chord length. We would like to compute the dielectric function using this model. The density operator in momentum space is

$$\rho_{\mathbf{q}} = \sum_{m=0}^{N-1}{\rm e}^{iq(ml_{\rm c}+\tilde{x}_m)}. \tag{B2}$$

Here $x_m = ml_{\rm c}+\tilde{x}_m$ is measured along the circumference of the circle (see figure 3). The average density is given by

$$\langle\rho_{\mathbf{q}}\rangle = \sum_{m=0}^{N-1}{\rm e}^{iqml_{\rm c}}{\rm e}^{-\frac{1}{2}q^2\langle\tilde{x}_m^2\rangle}. \tag{B3}$$

It can be shown that (see below) $\langle\tilde{x}_n^2\rangle = 1/m\omega_0$, independent of the index n. Thus we have,

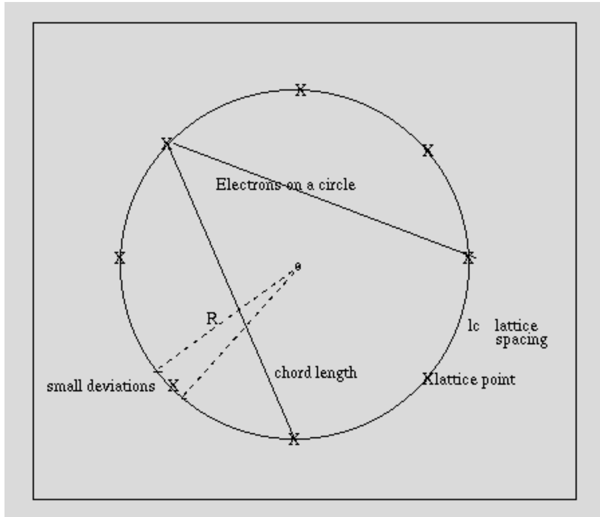

Figure 3. Schematic diagram of electrons on a circular lattice.

$$\langle \rho_{\mathbf{q}} \rangle = \frac{1 - \mathrm{e}^{iqNl_c}}{1 - \mathrm{e}^{iql_c}} \mathrm{e}^{-(q^2/2m\omega_0)}. \tag{B4}$$

In those instances where $\langle \rho_q \rangle \neq 0$ the static structure factor is given by

$$\begin{aligned} S(q) \equiv \frac{|\langle \rho_{\mathbf{q}} \rangle|^2}{N} &= \frac{1}{N} \left(\left| \frac{1 - \mathrm{e}^{iqNl_c}}{1 - \mathrm{e}^{iql_c}} \right| \right)^2 \mathrm{e}^{-(q^2/m\omega_0)} \\ &= \frac{1}{N} \frac{\sin^2(qNl_c/2)}{\sin^2(ql_c/2)} \, \mathrm{e}^{-(q^2/m\omega_0)}. \end{aligned} \tag{B5}$$

The rest of the details are as follows: We write $\tilde{x}_m(t) \approx R\tilde{\theta}_m(t)$. In terms of the small angles $\tilde{\theta}_i$ the Hamiltonian in eq. (B1) may be written as follows:

$$\begin{aligned} H = &-\frac{1}{2mR^2} \sum_{m=0}^{N-1} \frac{\partial^2}{\partial \tilde{\theta}_m^2} - \frac{R\, e^2}{2a^2} \sum_{m \neq m'} \left[\left(\cos\left(2\pi \frac{m}{N} + \tilde{\theta}_m\right) \right.\right. \\ &\left. - \cos\left(2\pi \frac{m'}{N} + \tilde{\theta}_{m'}\right) \right)^2 + \left(\sin\left(2\pi \frac{m}{N} + \tilde{\theta}_m\right) \right. \\ &\left.\left. - \sin\left(2\pi \frac{m'}{N} + \tilde{\theta}_{m'}\right) \right)^2 \right]^{1/2}. \end{aligned} \tag{B6}$$

We may expand the above Hamiltonian in powers of the angle and retain only the leading terms to arrive at the following Hamiltonian in the harmonic approximation.

$$H = \sum_{n=0}^{N-1} \frac{\tilde{p}_n^2}{2m} + \sum_{n \neq n'} A(n,n') \, (\tilde{x}_n - \tilde{x}_{n'})^2 + \sum_{n \neq n'} B(n,n')(\tilde{x}_n - \tilde{x}_{n'}), \tag{B7}$$

where

$$A(n,n') = \frac{\pi e^2}{4La^2}\left|\sin\left(\pi\frac{(n-n')}{N}\right)\right|, \tag{B8}$$

$$B(n,n') = -\frac{e^2}{2a^2}\mathrm{sgn}\left(\sin\left(\pi\frac{(n-n')}{N}\right)\right)\cos\left(\pi\frac{(n-n')}{N}\right). \tag{B9}$$

Despite appearences to the contrary, the extremum of the potential is at $\tilde{x}_n \equiv 0$. Since $A > 0$, this extremum is also a minimum. One has to compute now the various correlation function of the system. The primary one of interest is

$$G_{11}(nt;n't') = \langle T\tilde{x}_n(t)\tilde{x}_{n'}(t')\rangle. \tag{B10}$$

The other is

$$G_{21}(nt;n't') = \langle T\tilde{p}_n(t)\tilde{x}_{n'}(t')\rangle. \tag{B11}$$

Thus we have

$$i\frac{\partial}{\partial t}G_{11}(nt;n't') = \frac{i}{m}G_{21}(nt;n't'), \tag{B12}$$

$$i\frac{\partial}{\partial t}G_{21}(nt;n't') = \delta_{n,n'}\delta(t-t') - 4i\sum_{j\neq n}A(n,j)(G_{11}(nt;n't') - G_{11}(jt;n't')). \tag{B13}$$

This may be solved by a Fourier transform.

$$G_{ij}(nt;n't') = \frac{1}{-i\beta}\sum_p \mathrm{e}^{z_p(t-t')}\frac{1}{N}\sum_q \mathrm{e}^{iql_c(n-n')}\tilde{G}_{ij}(q,z_p). \tag{B14}$$

Thus we have

$$iz_p\ \tilde{G}_{11}(q,z_p) = \frac{i}{m}\tilde{G}_{21}(q,z_p) \tag{B15}$$

$$iz_p\tilde{G}_{21}(q,z_p) = 1 + 4i(\tilde{A}(q) - \tilde{A}(0))\tilde{G}_{11}(q,z_p) \tag{B16}$$

$$\tilde{A}(q) = \sum_j A(j)\mathrm{e}^{iql_c j} \tag{B17}$$

$$\tilde{G}_{11}(q,z_p) = \left(imz_p^2 - 4i(\tilde{A}(q) - \tilde{A}(0))\right)^{-1}. \tag{B18}$$

Thus,

$$G_{11}(nt;n't') = \frac{1}{N}\sum_q \mathrm{e}^{iql_c(n-n')}\frac{1}{\beta}\sum_p \mathrm{e}^{z_p(t-t')}\left(mz_p^2 - 4(\tilde{A}(q) - \tilde{A}(0))\right)^{-1} \tag{B19}$$

$$\tilde{A}(q) = \frac{c_0}{2i}\sum_{j=0}^{N-1}\left(\mathrm{e}^{i((\pi/N)+ql_c)j} - \mathrm{e}^{i(-(\pi/N)+ql_c)j}\right)$$
$$= \frac{c_0}{2i}\left(\frac{1+\mathrm{e}^{iql_cN}}{1-\mathrm{e}^{i((\pi/N)+ql_c)}} - \frac{1+\mathrm{e}^{iql_cN}}{1-\mathrm{e}^{i(-(\pi/N)+ql_c)}}\right) \quad \text{(B20)}$$

$$\tilde{A}(0) = \frac{c_0}{2i}\left(\frac{2}{1-\mathrm{e}^{i(\pi/N)}} - \frac{2}{1-\mathrm{e}^{-i(\pi/N)}}\right) \approx \frac{k_{\mathrm{F}}e^2}{2\pi a^2}. \quad \text{(B21)}$$

Here $c_0 = \pi e^2/4La^2$. The dispersion of the collective mode is then given by

$$\omega_q = \sqrt{\frac{4}{m}}\left(\tilde{A}(0) - \tilde{A}(q)\right)^{1/2}. \quad \text{(B22)}$$

If $q = 0$ or $q = 2k_{\mathrm{F}}$ then $\omega_q = 0$. Notice that unless $|ql_c| \ll \pi/N$, we have $\tilde{A}(q) \approx 0$ for $|q| \neq 0$ since it is vanishingly small in the thermodynamic limit. Thus we have to take the large N limit first in which case $\tilde{A}(q) \approx 0$ for $\pi/N \ll |ql_c| \ll 2\pi$. Thus we may set $\omega_q \approx \omega_0$ with impunity in this case. For a more thorough analysis, one has to compute the full dielectric function from the dynamical density–density correlation function and use it to compute the full momentum distribution that is accurate even away from the Fermi surface. However, we shall be content at features close to the Fermi surface. For the static structure factor we have to compute the equal time version of the correlation function. We find that ω_q is in general, complex. This means that the eigenmodes also have a finite lifetime. Thus we have,

$$G_{11}(nt;n't) = \frac{1}{N}\sum_q \mathrm{e}^{iql_c(n-n')}\frac{1}{2\pi m}\int_{-\infty}^{\infty}\mathrm{d}z_p\left(z_p^2 + \omega_c^2(q)\right)^{-1}$$
$$= \frac{1}{2mN}\sum_q \frac{\mathrm{e}^{iql_c(n-n')}}{\omega_c(q)}. \quad \text{(B23)}$$

From eq. (B22) it is clear that for $\pi/N \ll |ql_c| < 2\pi$ we have $\omega_q \approx \omega_0$ since $\tilde{A}(q) \approx 0$ for q in this region. Since in the thermodynamic limit, this is all of q, we shall write,

$$\langle x_n(t)x_{n'}(t)\rangle = \frac{1}{2mN}\sum_q \frac{\mathrm{e}^{iql_c(n-n')}}{\omega_0} = \delta_{n,n'}\frac{1}{m\omega_0}. \quad \text{(B24)}$$

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