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Thermorheological effect on magnetoconvection in weak electrically conducting fluids under 1g or μg*

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Abstract. The thermorheological effect on magnetoconvection in fluids with weak electrical conductivity is studied numerically under 1g and μg conditions. The results with a non-linear thermorheological equation considered in the problem when compared with those of the classical approach with constant viscosity delineate the fact that the latter approach results in an over-prediction of the critical eigenvalue. The results have possible astrophysical applications involving sunspots as also in space applications under μg.

Keywords. Magnetoconvection; thermorheological effect; Rayleigh–Benard; Marangoni.

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1. Introduction

The Rayleigh–Benard and Marangoni instability problems with/without thermorheological effect have received widespread attention due to their implications in heat transfer and in many engineering applications (see [1–3] and references therein). The thermorheological effect in an exponential form or in a polynomial form through truncated Taylor series expansion has been examined by Torrance and Turcotte [4] and Busse and Frick [5]. The present paper aims at studying convective instability in a Newtonian liquid, which is assumed to be a poor conductor of electricity, in the presence of a transverse magnetic field by assuming a non-linear thermorheological equation of state. Thermal conductivity of liquids is not a very strong function of temperature (see [6]) and therefore it is taken to be invariant

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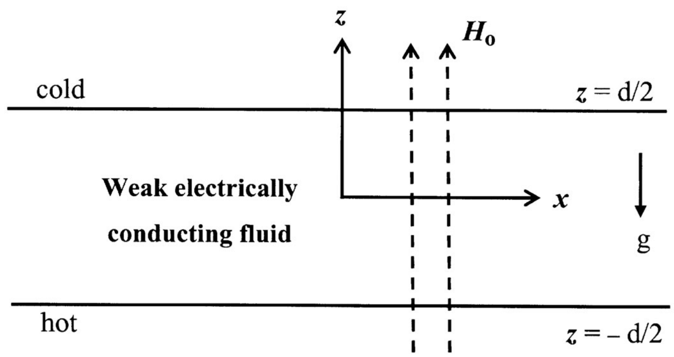

Figure 1. Schematic of flow configuration.

with increase in temperature. Higher order Rayleigh–Ritz method is employed to obtain the critical values.

2. Mathematical formulation and solution

Consider an infinite horizontal layer of a Boussinesquian liquid of weak electrical conductivity confined between two plates at a distance d apart. A Cartesian co-ordinate system is taken with the lower plate in the xy-plane and z-axis vertically upwards. The lower plate at $z = -(d/2)$ and the upper plate at $z = (d/2)$ are maintained at constant temperatures T_0 and T_1 (with $T_0 > T_1$) respectively. In addition to a temperature gradient, a vertical magnetic field is also imposed across the layer (see figure 1).

The system of equations of hydromagnetics describing the Rayleigh–Benard situation in a weak electrically conducting liquid is the following:

$$\nabla \cdot \vec{q} = 0, \tag{2.1}$$

$$\rho_0 \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}\right] = -\nabla p + \rho \vec{g} + \nabla \cdot [\mu_{\mathrm{f}}(T)(\nabla \vec{q} + \nabla \vec{q}^{\mathrm{Tr}})] - \mu^2 \sigma H_0^2 \vec{q} \tag{2.2}$$

$$\left[\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T\right] = \kappa \nabla^2 T, \tag{2.3}$$

where $\vec{q} = (u, v, w)$ is the velocity of the liquid, ρ the density, p the pressure, μ_{f} the fluid viscosity, $\vec{g} = (0, 0, -g)$ the gravitational acceleration, μ the magnetic permeability, σ the electrical conductivity, H_0 the applied, uniform transverse magnetic field, ρ_0 the density at the reference temperature T_0, T the temperature, κ the thermal diffusivity and Tr the transpose. The density equation of state is

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{2.4}$$

where $\alpha > 0$ is the constant coefficient of thermal expansion. The steady (quiescent) state solution is of the form

$$\vec{q}_b = 0, \quad T_b(z) = T_0 - \beta z, \quad \rho_b(z) = \rho_0[1 + \alpha\beta z],$$
$$\mu_{f_b} = \mu_{f_b}(z), \quad p_b = p_b(z), \tag{2.5}$$

where $\beta = (T_0 - T_1)/d$ is the basic temperature gradient.

A non-linear approximation (based on a truncated Taylor series expansion) characterizing the shear viscosity as a function of temperature yields the thermorheological equation of state in the form:

$$\mu_f(T) = \mu_1[1 - \delta(T - T_0)^2], \tag{2.6}$$

where $\delta > 0$ and μ_1 is the viscosity at $T = T_0$.

We now assume that the initial state is slightly perturbed. Following the classical procedure of linear stability analysis and taking d as the unit length, d^2/κ as the unit time and βd as the unit temperature, the linearized dimensionless equations governing small perturbations turn out to be

$$\mathrm{Pr}^{-1}\frac{\partial}{\partial t}(\nabla^2 w') = R\nabla_1^2 T' - 4Vz\nabla^2\left(\frac{\partial w'}{\partial z}\right) + 2V\left[\nabla_1^2 w' - \frac{\partial^2 w'}{\partial z^2}\right]$$
$$+(1 - Vz^2)\nabla^4 w' - Q\nabla^2 w' \tag{2.7}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T' = w', \tag{2.8}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla_1^2 + \frac{\partial^2}{\partial z^2}$ and the primes refer to perturbed quantities. The dimensionless parameters appearing in eqs (2.7) and (2.8) are the following:

$$\mathrm{Pr} = \frac{\nu}{\kappa}, \quad R = \frac{\alpha\beta g d^4}{\nu\kappa}, \quad V = \delta\beta^2 d^2, \quad Q = \frac{\mu^2\sigma H_0^2 d^2}{\mu_1},$$

which are respectively, the Prandtl number, the Rayleigh number, the thermorheological parameter and the Chandrasekhar number. It should be mentioned that the term $2V[\nabla_1^2 w' - (\partial^2 w'/\partial z^2)]$ (see [7]) appearing in the vertical component of the vorticity transport equation disappears when the thermorheological equation of state is a linear function of temperature.

We now restrict ourselves to the problem of stationary instability. In other words, we shall not consider the possibility of the existence of over-stable motions for the problem at hand. Nevertheless, it can be substantiated with recourse to a single term Rayleigh–Ritz method that the principle of exchange of stabilities is valid for the problem under consideration. We thus seek solutions of the form

$$[w', T'] = [w(z), T(z)]\exp[i(lx + my)], \tag{2.9}$$

where l and m are the horizontal components of the wave number a (a is used in relation to $a^2 = l^2 + m^2$) and $w(z)$ and $T(z)$ are the amplitudes of the perturbations of velocity and temperature respectively.

Substituting eq. (2.9) into eqs (2.7) and (2.8) and using D to denote the non-dimensional derivative operator $(\mathrm{d}/\mathrm{d}z)$, we obtain

$$(1 - Vz^2)(D^2 - a^2)^2 w - 4Vz(D^2 - a^2)Dw - 2V(D^2 + a^2)w$$
$$-Q(D^2 - a^2)w - Ra^2 T = 0, \tag{2.10}$$

$$(D^2 - a^2)T + w = 0. \tag{2.11}$$

We consider the following boundary conditions

$$\left.\begin{array}{l} w = D^2 w + \frac{Ma^2 T}{1 - Vz^2} = DT = 0 \text{ at } z = \frac{1}{2} \\ w = Dw = T = 0 \text{ at } z = -\frac{1}{2} \end{array}\right\}, \tag{2.12}$$

where $M = (\sigma_1 \beta d^2/\rho_0 \nu \kappa)$ is the Marangoni number and σ_1 is the surface tension gradient (see [8]).

Since the aforementioned asymmetric boundary conditions make the problem analytically intractable, we employ higher order Rayleigh–Ritz technique (HORT) to compute the eigenvalues. To this end, we expand $w(z)$ and $T(z)$ in a series of trial functions as

$$w(z) = \sum_{i=1}^{n} \alpha_i w_i(z), \quad T(z) = \sum_{i=1}^{n} \beta_i T_i(z), \tag{2.13}$$

where α_i and β_i are constants. Applying HORT to eqs (2.10) and (2.11), one obtains the following system of homogeneous equations with space varying coefficients:

$$\left.\begin{array}{l} A_{ji}\alpha_i + B_{ji}\beta_i = 0 \\ C_{ji}\alpha_i + D_{ji}\beta_i = 0 \end{array}\right\}, \tag{2.14}$$

where

$$\begin{aligned} A_{ji} &= \langle D^2 w_j D^2 w_i \rangle + a^4 \langle w_j (1 - Vz^2) w_i \rangle - 2a^2 \langle w_j (1 - Vz^2) D^2 w_i \rangle \\ &\quad - V \langle w_j z^2 D^4 w_i \rangle - 4V \langle w_j z D^3 w_i \rangle + 4Va^2 \langle w_j z D w_i \rangle \\ &\quad - 2V \langle w_j D^2 w_i \rangle - 2Va^2 \langle w_j w_i \rangle + Q\{\langle Dw_j Dw_i \rangle + a^2 \langle w_j w_i \rangle\} \\ B_{ji} &= \frac{M}{(1 - (V/4))} a^2 D w_j \left(\frac{1}{2}\right) T_i \left(\frac{1}{2}\right) - Ra^2 \langle w_j T_i \rangle, \\ C_{ji} &= \{\langle T_j w_i \rangle\}, \quad D_{ji} = -\{\langle DT_j DT_i \rangle + a^2 \langle T_j T_i \rangle\}, \end{aligned}$$

where use has been made of the inner product $\langle p(z) q(z) \rangle = \int_{-1/2}^{1/2} p(z) q(z) \mathrm{d}z$.

In view of the boundary conditions given in eq. (2.12), the following trial functions are chosen at discretion (see [9]):

$$w_i = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)^{i+1}, \quad T_i = \left(z(z-1) - \frac{3}{4}\right)^i.$$

As a particular case, we now explain the procedure for determining the expression for the Rayleigh number using a single-term Rayleigh–Ritz method (i.e., $i = j = 1$). Multiplying eq. (2.10) by w and eq. (2.11) by T, integrating with respect to z between the limits $z = -\frac{1}{2}$ and $z = +\frac{1}{2}$, taking $w = Aw_1$ and $T = BT_1$ (in which A and B are constants), we obtain the expression for Rayleigh number R in the form

$$R = \frac{X_1 Y_1}{a^2 \langle w_1 T_1 \rangle^2} + \frac{Ma^2 T_1(1/2) D w_1(1/2)}{a^2 \langle w_1 T_1 \rangle (1 - (V/4))}, \tag{2.15}$$

where

$$X_1 = \langle (D^2 w_1)^2 \rangle - V\langle w_1 z^2 D^4 w_1 \rangle + a^4 \langle w_1(1 - Vz^2) w_1 \rangle$$
$$-2a^2 \langle w_1(1 - Vz^2) D^2 w_1 \rangle - 4V\langle w_1 z D^3 w_1 \rangle + 4Va^2 \langle w_1 z D w_1 \rangle$$
$$+2V\langle (Dw_1)^2 \rangle - 2Va^2 \langle w_1^2 \rangle + Q(\langle (Dw_1)^2 \rangle + a^2 \langle w_1^2 \rangle)$$

and

$$Y_1 = \langle (DT_1)^2 \rangle + a^2 \langle T_1^2 \rangle.$$

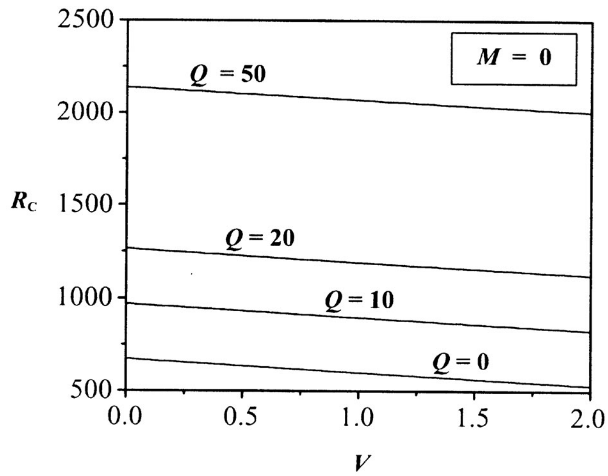

Figure 2. Plot of critical Rayleigh number R_C vs. thermorheological parameter V for different values of Chandrasekhar number Q.

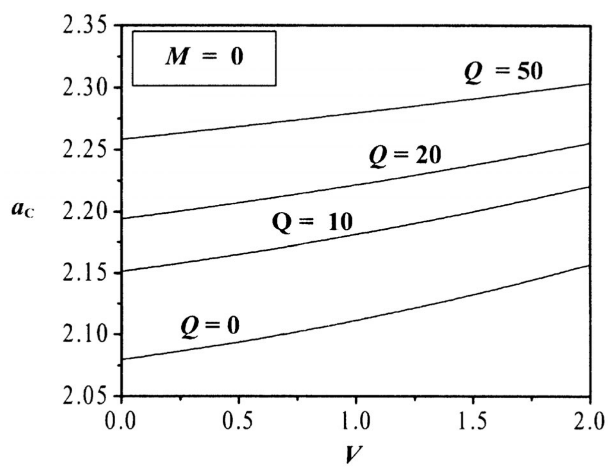

Figure 3. Plot of critical wave number a_C vs. V for different values of Q.

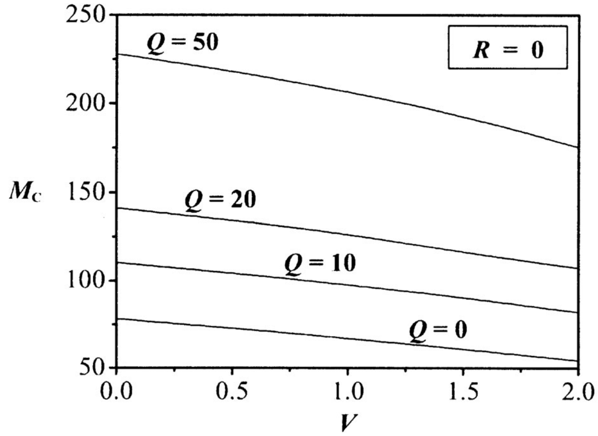

Figure 4. Plot of critical Marangoni number M_C vs. V for different values of Q.

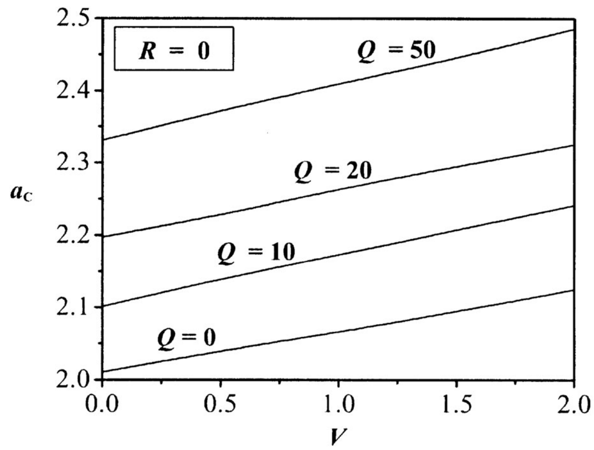

Figure 5. Plot of a_C vs. V for different values of Q.

3. Results and discussion

The effect of thermorheological parameter on the threshold of thermal instability is studied using the higher order Rayleigh–Ritz method. The viscosity is assumed to be reliant on temperature. The convergence of the solution obtained by Rayleigh–Ritz method is assured by the fifth approximation. Our results in respect of Rayleigh–Benard and Marangoni convection agree quite well with those of Sparrow *et al* [10] and Pearson [8] respectively, for a Newtonian fluid with $V = 0$. We note here that μg corresponds to $R = 0$ and 1g to $M = 0$. We now move on to discuss the results in figures 2–5. It is appropriate to note here that magnetoconvection is intimately connected to astrophysical applications, especially in sunspots (see [11,12]).

Before we embark on a discussion of the results obtained in the study, we present below the values of various physical quantities with respect to Mazola corn oil, which is a weak electrically conducting liquid (WECL) [13–15]:

$$\begin{aligned}
&\sigma(T) = [0.21 + 0.03(T - 10^\circ)] \times 10^{-10}\ \text{mho/m},\\
&\mu_f(T) = [8.2 - 0.235(T - 12^\circ)] \times 10^{-2}\ \text{kg/ms},\\
&\alpha = 6.6 \times 10^{-4}/^\circ\text{C},\\
&\rho = 9.1 \times 10^{2}\ \text{kg/m}^3,\\
&\kappa = 1.23 \times 10^{-7}\ \text{m}^2/\text{s}.
\end{aligned}$$

Clearly with respect to this WECL, σ is quite small and one may deal with magnetoconvection in the form of a Hartmann formulation. The Prandtl number for these liquids is nearly 1000 and this in a way justifies the assumption in the present problem of the 'Principle of Exchange of Stabilities'.

Figure 2 is a plot of R_C vs. V and figure 3 is a plot of a_C vs. V for different values of Q and $M = 0$. The stabilizing influence of the magnetic field is apparent from both the figures. It is clear that R_C decreases and a_C increases with increase in V. Figure 4 is a plot of M_C vs. V for different values of Q and $R = 0$. The stabilizing influence of the magnetic field is apparent in the case of μg also. Figure 5 is a plot of a_C vs. V for different values of Q and $R = 0$. It is clear that a_C increases with increase in V. From figures 2–5, it is apparent that the magnetic field diminishes the effect of thermorheological parameter V on convection in the case of both μg and 1g. In the case of μg, however, there is a more marked influence of V on convection. This result points to the fact that temperature-sensitive conducting fluids (with poor electrical conductivity) are most suitable for space-based applications as well as terrestrial-based applications. It would be interesting to consider temperature-sensitive and magnetically-sensitive fluids (known as ferrofluids) for space-based applications [16–19,3]. This work is under progress.

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