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Coherent states of general time-dependent harmonic oscillator

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Abstract. By introducing an invariant operator, we obtain exact wave functions for a general time-dependent quadratic harmonic oscillator. The coherent states, both in $\hat{x}$- and $\hat{p}$-spaces, are calculated. We confirm that the uncertainty product in coherent state is always larger than $\hbar/2$ and is equal to the minimum of the uncertainty product of the number states. The displaced wave packet for Caldirola–Kanai oscillator in coherent state oscillates back and forth with time about the center as for a classical oscillator. The amplitude of oscillation with no driving force decreases due to the dissipation in the system. However, the oscillation with resonant frequency oscillates with a large amplitude, even after a sufficient time elapse.

Keywords. Time-dependent harmonic oscillator; invariant operator; coherent state.

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1. Introduction

The study of exact quantum mechanical solutions for a general time-dependent harmonic oscillator has attracted considerable interest in the literature. A class of classical and quantum mechanical invariants for time-dependent one-dimensional harmonic oscillator was found [1]. The invariant operator is a powerful tool to derive exact solutions of time-dependent quantum mechanical Hamiltonian systems. For example, it can be used to describe the motion of a charged particle moving in a time-dependent electromagnetic field [2]. Invariant operators correspond physically to the initial points in the phase space of coordinates and momenta. Most of the time, if we know the exact invariant operator for the time-dependent Hamiltonian, we can construct a pair of the ladder operators, i.e., lowering and raising operators. The wave function can then be found easily, by making use of these ladder operators.

Glauber proposed standard coherent states for a harmonic oscillator which is the prototype for most of the coherent states [3,4]. The coherent states form a very convenient representation for problems of quantum mechanics and can be created from the ground state by a displacement operator and can be expanded in terms of the harmonic oscillator Hamiltonian eigenstates. The coherent state of the simple harmonic oscillator considered by Schrödinger [5] is close to a classical wave packet. The main aim of this paper is to introduce coherent states for generalized time-dependent harmonic oscillator and to use them to calculate various expectation values, uncertainty relation and other interesting quantities. Coherent state of a simple harmonic oscillator has been used frequently to describe the radiation field for laser light.

The coherent states of a harmonic oscillator with time-dependent frequency has been investigated in [6,7]. Coherent states of a charged particle in a time-dependent electromagnetic field are studied by Malkin *et al* [8] and Dodonov *et al* [9]. The topological and algebraic structure of the coherent states and some useful methods for employing the coherent states to study the quantum dynamic systems are described in [10]. The coherent states for general potentials have been investigated by Nieto and Simmons [11–16].

Even though the coherent states are not orthogonal, they constitute an over complete normalized set. Most of the physical quantities like minimum uncertainty state and Green function can be obtained from the coherent states. The quantum and classical correspondence can be established from coherent states which provide a general means of construction of Husimi–Wigner distributions [17]. The coherent state for quadratic Hamiltonians can be extended to non-stationary systems [18–20]. Other properties of coherent states are discussed in [10,21–23] in detail where they have been applied to a system of charged particles in a constant uniform magnetic field in an arbitrary gauge and in various external time-dependent electromagnetic field.

2. Invariant operator

We consider the Hamiltonian of the general time-dependent quadratic harmonic oscillator:

$$\hat{H}(\hat{x},\hat{p},t) = A(t)\hat{p}^2 + B(t)(\hat{x}\hat{p}+\hat{p}\hat{x}) + C(t)\hat{x}^2 + D(t)\hat{x} + E(t)\hat{p} + F(t), \tag{1}$$

where $A(t) - F(t)$ are time-dependent coefficients, mutually independent and differentiable with respect to time. Note that $A(t) \neq 0$. The equation of motion corresponding to the above Hamiltonian is

$$\frac{\mathrm{d}^2\hat{x}}{\mathrm{d}t^2} - \frac{\dot{A}}{A}\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} + \left(2\frac{\dot{A}B}{A} - 4B^2 + 4AC - 2\dot{B}\right)\hat{x} + \frac{\dot{A}E}{A} - 2BE + 2AD - \dot{E} = 0. \tag{2}$$

In order to investigate the quantum state of the time-dependent Hamiltonian systems, it is convenient to introduce a dynamically invariant operator. The invariant operator $\hat{I}$ can be determined from the relation:

$$\frac{\mathrm{d}\hat{I}}{\mathrm{d}t} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I},\ \hat{H}(\hat{x},\hat{p},t)] = 0. \tag{3}$$

By inserting eq. (1) into eq. (3), we obtain

$$\hat{I} = \frac{k}{4}\frac{1}{\rho^2(t)}(\hat{x}-x_p(t))^2 + \left[\rho(\hat{p}-p_p(t)) + \frac{1}{2A}(2B\rho(t)-\dot{\rho}(t))(\hat{x}-x_p(t))\right]^2, \tag{4}$$

where k is the arbitrary real constant and $\rho(t)$ is some time-dependent quantity that satisfies the following differential equation [24]:

$$\ddot{\rho}(t) - \frac{\dot{A}}{A}\dot{\rho}(t) + \left(2\frac{\dot{A}B}{A} - 4B^2 + 4AC - 2\dot{B}\right)\rho(t) - kA^2\frac{1}{\rho^3(t)} = 0 \tag{5}$$

and $x_p(t)$ and $p_p(t)$ are particular solutions [24,25] of the following classical equation of motion in coordinate and momentum space, respectively

$$\ddot{x}_p(t) - \frac{\dot{A}}{A}\dot{x}_p(t) + \left(2\frac{\dot{A}B}{A} - 4B^2 + 4AC - 2\dot{B}\right)x_p(t) = -\frac{\dot{A}E}{A} + 2BE - 2AD + \dot{E}, \tag{6}$$

$$\ddot{p}_p(t) - \frac{\dot{C}}{C}\dot{p}_p(t) + \left(4AC - 2\frac{\dot{C}B}{C} - 4B^2 + 2\dot{B}\right)p_p(t) = \frac{\dot{C}D}{C} + 2BD - 2CE - \dot{D}. \tag{7}$$

We introduce lowering and raising operators given by [26]

$$\hat{a} = \sqrt{\frac{1}{\hbar k^{1/2}}}\left\{\left[\frac{\sqrt{k}}{2}\frac{1}{\rho} + i\frac{1}{2A}(2B\rho-\dot{\rho})\right](\hat{x}-x_p) + i\rho(\hat{p}-p_p)\right\}, \tag{8}$$

$$\hat{a}^\dagger = \sqrt{\frac{1}{\hbar k^{1/2}}}\left\{\left[\frac{\sqrt{k}}{2}\frac{1}{\rho} - i\frac{1}{2A}(2B\rho-\dot{\rho})\right](\hat{x}-x_p) - i\rho(\hat{p}-p_p)\right\}. \tag{9}$$

The above ladder operators satisfy the boson commutation relation:

$$[\hat{a},\ \hat{a}^\dagger] = 1. \tag{10}$$

From eqs (8) and (9), we can see that $\hat{x}$ and $\hat{p}$ can be represented in terms of lowering and raising operators as

$$\hat{x} = \rho\sqrt{\frac{\hbar}{k^{1/2}}}(\hat{a}+\hat{a}^\dagger) + x_p, \tag{11}$$

$$\hat{p} = \frac{\sqrt{\hbar}k^{1/2}}{2i\rho}[(1-iZ)\hat{a} - (1+iZ)\hat{a}^\dagger] + p_p, \tag{12}$$

where

$$Z = \frac{\rho}{A\sqrt{k}}(2B\rho - \dot{\rho}). \tag{13}$$

The eigenvalue equation for invariant operator can be written as

$$\hat{I}|\phi_n(t)\rangle = \lambda_n|\phi_n(t)\rangle, \tag{14}$$

where the subscript n takes values $1, 2, 3, \ldots$. Since eq. (4) can be rewritten in terms of lowering and raising operators as

$$\hat{I} = \hbar\sqrt{k}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{15}$$

the eigenvalue λ_n is given by

$$\lambda_n = \hbar\sqrt{k}\left(n + \frac{1}{2}\right). \tag{16}$$

To obtain the eigenstate $|\phi_n(t)\rangle$ of the invariant operator, first of all, we operate lowering operator to ground eigenstate

$$\hat{a}|\phi_0(t)\rangle = 0. \tag{17}$$

Then, we obtain the ground eigenstate in $\hat{x}$-space as

$$\langle\hat{x}|\phi_0(t)\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}} \times \exp\left\{\frac{i}{\hbar}p_p\hat{x} - \frac{1}{2\rho\hbar}\left[\frac{\sqrt{k}}{2}\frac{1}{\rho} + \frac{i}{2A}(2B\rho - \dot{\rho})\right](\hat{x} - x_p)^2\right\}. \tag{18}$$

To get nth eigenstate, we operate $\hat{a}^\dagger$ to the ground eigenstate n times

$$|\phi_n(t)\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|\phi_0(t)\rangle. \tag{19}$$

Then, we obtain in $\hat{x}$-space

$$\langle\hat{x}|\phi_n(t)\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left[\sqrt{\frac{k^{1/2}}{2\rho^2\hbar}}(\hat{x} - x_p)\right] \times \exp\left\{\frac{i}{\hbar}p_p\hat{x} - \frac{1}{2\rho\hbar}\left[\frac{\sqrt{k}}{2}\frac{1}{\rho} + \frac{i}{2A}(2B\rho - \dot{\rho})\right](\hat{x} - x_p)^2\right\}. \tag{20}$$

3. Wave function

The wave function $|\psi_n(t)\rangle$ can be obtained easily from the eigenstate of invariant operator, since it differs only by some time-dependent phase factor, $\exp[i\varepsilon_n(t)]$, from the eigenstate of the invariant operator [27].

$$|\psi_n(t)\rangle = |\phi_n(t)\rangle \exp[i\varepsilon_n(t)]. \tag{21}$$

The Schrödinger equation for the Hamiltonian in eq. (1), is

$$i\hbar \frac{\partial |\psi_n(t)\rangle}{\partial t} = \hat{H} |\psi_n(t)\rangle. \tag{22}$$

Inserting eq. (21) into eq. (22) and operating $\langle \psi_n(t)|$ from left, we obtain

$$\hbar \dot{\varepsilon}_n(t) = \left\langle \phi_n(t) \left| \left(i\hbar \frac{\partial}{\partial t} - \hat{H} \right) \right| \phi_n(t) \right\rangle. \tag{23}$$

The expectation value of the Hamiltonian in the state $|\phi_n(t)\rangle$ is easily calculated as

$$\begin{aligned} \langle \phi_n(t)|H(\hat{x},\hat{p},t)|\phi_n(t)\rangle = & A\left[\frac{\hbar\sqrt{k}}{4\rho^2}(1+Z^2)(2n+1)+p_p^2\right] \\ & +B[-\hbar Z(2n+1)+2x_p p_p] \\ & +C\left[\frac{\rho^2\hbar}{\sqrt{k}}(2n+1)+x_p^2\right] \\ & +Dx_p+Ep_p+F. \end{aligned} \tag{24}$$

When inserting eq. (24) into eq. (23), we obtain $\varepsilon_n(t)$ as

$$\begin{aligned} \varepsilon_n(t) = & -\left(n+\frac{1}{2}\right)\int_0^t \frac{A(t')\sqrt{k}}{\rho^2(t')}\mathrm{d}t' \\ & -\frac{1}{\hbar}\int_0^t \left[\mathcal{L}_p(x_p(t'),\dot{x}_p(t'),t') - \frac{E^2(t')}{4A(t')} + F(t')\right]\mathrm{d}t', \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathcal{L}_p(x_p(t'),\dot{x}_p(t'),t') = & \frac{1}{4A(t')}\dot{x}_p^2(t') - \frac{B(t')}{A(t')}x_p(t')\dot{x}_p(t') \\ & -\left(C(t') - \frac{B^2(t')}{A(t')}\right)x_p^2(t'). \end{aligned} \tag{26}$$

Thus, inserting eqs (20) and (25) into eq. (21), the exact wave function can be evaluated in $\hat{x}$-space as

$$\langle\hat{x}|\psi_n(t)\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left[\sqrt{\frac{k^{1/2}}{2\rho^2\hbar}}(\hat{x}-x_p)\right]$$
$$\times\exp\left\{\frac{i}{\hbar}p_p\hat{x}-\frac{1}{2\rho\hbar}\left[\frac{\sqrt{k}}{2}\frac{1}{\rho}+\frac{i}{2A}(2B\rho-\dot{\rho})\right](\hat{x}-x_p)^2\right\}$$
$$\times\exp\left[-i\left(n+\frac{1}{2}\right)\int_0^t\frac{A(t')\sqrt{k}}{\rho^2(t')}\mathrm{d}t'\right.$$
$$\left.-\frac{i}{\hbar}\int_0^t\left[\mathcal{L}_p(x_p(t'),\dot{x}_p(t'),t')-\frac{E^2(t')}{4A(t')}+F(t')\right]\mathrm{d}t'\right]. \tag{27}$$

For $x_p(t)=0$ and $p_p(t)=0$, this agrees with eq. (30) of ref. [26].

The $\hat{p}$-space representation of the wave function can be obtained by Fourier transformation of eq. (27),

$$\langle\hat{p}|\psi_n(t)\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\langle\hat{x}|\psi_n(t)\rangle\mathrm{e}^{-i\hat{p}\hat{x}/\hbar}\mathrm{d}\hat{x}. \tag{28}$$

Making use of eq. (27), the above equation can be evaluated as

$$\langle\hat{p}|\psi_n(t)\rangle = \sqrt[4]{\frac{2k^{1/2}}{\hbar\pi}}(-i)^n\frac{1}{\sqrt{2^n n!}}\left\{\frac{[\sqrt{k}/\rho-i(2B\rho-\dot{\rho})/A]^n}{[\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A]^{n+1}}\right\}^{1/2}$$
$$\times H_n\left[\sqrt{\frac{2}{\hbar}}\frac{\sqrt[4]{k}(\hat{p}-p_p)}{\sqrt{k/\rho^2+(2B\rho-\dot{\rho})^2/A^2}}\right]$$
$$\times\exp\left[-\frac{i}{\hbar}x_p(\hat{p}-p_p)-\frac{\rho(\hat{p}-p_p)^2}{\hbar[\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A]}\right]$$
$$\times\exp\left[-i\left(n+\frac{1}{2}\right)\int_0^t\frac{A(t')\sqrt{k}}{\rho^2(t')}\mathrm{d}t'\right.$$
$$\left.-\frac{i}{\hbar}\int_0^t\left[\mathcal{L}_p(x_p(t'),\dot{x}_p(t'),t')-\frac{E^2(t')}{4A(t')}+F(t')\right]\mathrm{d}t'\right]. \tag{29}$$

The fluctuation for canonical variables can be evaluated as

$$\Delta\hat{x} = \sqrt{\langle\psi_n(t)|\hat{x}^2|\psi_n(t)\rangle-(\langle\psi_n(t)|\hat{x}|\psi_n(t)\rangle)^2}$$
$$= \rho\sqrt{\frac{\hbar}{k^{1/2}}(2n+1)}, \tag{30}$$
$$\Delta\hat{p} = \sqrt{\langle\psi_n(t)|\hat{p}^2|\psi_n(t)\rangle-(\langle\psi_n(t)|\hat{p}|\psi_n(t)\rangle)^2}$$
$$= \frac{1}{2\rho}\sqrt{\hbar k^{1/2}(1+Z^2)(2n+1)}. \tag{31}$$

By multiplying the above two equations, we obtain the following uncertainty relation:

$$\Delta\hat{x}\ \Delta\hat{p} = \hbar\sqrt{1+Z^2}\left(n+\frac{1}{2}\right). \tag{32}$$

As expected, the above value is always larger than $\hbar/2$.

4. Standard coherent state

We will denote the coherent state as $|\alpha\rangle$. The coherent state can be represented in terms of $|\phi_n(t)\rangle$ in a simple form [28]

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_n \frac{\alpha^n}{\sqrt{n!}}|\phi_n(t)\rangle. \tag{33}$$

Inserting eq. (20) into eq. (33) gives coherent state in $\hat{x}$-space as

$$\begin{aligned}\langle\hat{x}|\alpha\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}}\exp\Bigg\{&\alpha\sqrt{\frac{k^{1/2}}{\rho^2\hbar}}(\hat{x}-x_p)\\ &-\frac{1}{4\rho\hbar}\left[\frac{\sqrt{k}}{\rho}+\frac{i}{A}(2B\rho-\dot{\rho})\right](\hat{x}-x_p)^2\\ &+\frac{i}{\hbar}p_p\hat{x}-\frac{1}{2}|\alpha|^2-\frac{1}{2}\alpha^2\Bigg\}.\end{aligned} \tag{34}$$

Using eq. (29) and by a similar procedure as for the $\hat{x}$-space, we obtain the coherent state in $\hat{p}$-space as

$$\begin{aligned}\langle\hat{p}|\alpha\rangle = &\sqrt[4]{\frac{2k^{1/2}}{\hbar\pi}}\left[\frac{\sqrt{k}}{\rho}+\frac{i}{A}(2B\rho-\dot{\rho})\right]^{-1/2}\\ &\times\exp\left\{-i\left[\frac{1}{\hbar}x_p+\frac{2\alpha\sqrt[4]{k}}{\sqrt{\hbar}[\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A]}\right](\hat{p}-p_p)\right\}\\ &\times\exp\left\{-\frac{\rho(\hat{p}-p_p)^2}{\hbar[\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A]}\right\}\\ &\times\exp\left\{-\frac{1}{2}|\alpha|^2+\frac{1}{2}\alpha^2\frac{\sqrt{k}/\rho-i(2B\rho-\dot{\rho})/A}{\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A}\right\}.\end{aligned} \tag{35}$$

Since the coherent state is an eigenstate of the lowering operator, the eigenvalue equation for $\hat{a}$ can be written as

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{36}$$

Multiplying both sides of the above equation with $\langle x'|$ from left, we obtain [29]

$$\langle x'|\alpha\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}}\exp\left\{-\frac{1}{2\rho\hbar}\left[\frac{\sqrt{k}}{2\rho}(\langle\hat{x}\rangle - x')^2 + \frac{i}{A}(2B\rho-\dot{\rho})\left(\frac{1}{2}x'^2 - \langle\hat{x}\rangle x'\right)\right] + \frac{i}{\hbar}\langle\hat{p}\rangle x' + i\delta_x\right\}, \tag{37}$$

where $\langle\hat{x}\rangle = \langle\alpha|\hat{x}|\alpha\rangle$ and $\langle\hat{p}\rangle = \langle\alpha|\hat{p}|\alpha\rangle$, and δ_x represents some phase. If we choose δ_x as

$$\delta_x = \frac{i}{4}\left[\alpha^2 - \alpha^{*2} + \frac{2}{\rho}\sqrt{\frac{k^{1/2}}{\hbar}}x_p(\alpha-\alpha^*)\right] - \frac{1}{4A\rho\hbar}(2B\rho-\dot{\rho})x_p^2, \tag{38}$$

then, eq. (37) can be represented as

$$\langle x'|\alpha\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}}\exp\left\{\alpha\sqrt{\frac{k^{1/2}}{\rho^2\hbar}}(x'-x_p) - \frac{1}{4\rho\hbar}\left[\frac{\sqrt{k}}{\rho} + \frac{i}{A}(2B\rho-\dot{\rho})\right](x'-x_p)^2 + \frac{i}{\hbar}p_p x' - \frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^2\right\}. \tag{39}$$

One can verify that the above equation is similar to eq. (34).

Multiplying both sides of eq. (36) with $\langle p'|$ from left, we obtain

$$\langle p'|\alpha\rangle = \sqrt[4]{\frac{2k^{1/2}}{\hbar\pi}}\left[\frac{\sqrt{k}}{\rho} + \frac{i}{A}(2B\rho-\dot{\rho})\right]^{-1/2} \times\exp\left\{-\frac{1}{\hbar[k/\rho^2 + (2B\rho-\dot{\rho})^2/A^2]} \times\left[\sqrt{k}(\langle\hat{p}\rangle - p')^2 - \frac{i\rho}{A}(2B\rho-\dot{\rho})(p'^2 - 2p'\langle\hat{p}\rangle)\right] - \frac{i}{\hbar}\langle\hat{x}\rangle p' + i\delta_p\right\}, \tag{40}$$

where δ_p is another phase. Let us choose δ_p in the following form:

$$\delta_p = \frac{x_p p_p}{\hbar} + \left[\frac{k}{\rho^2} + \frac{1}{A^2}(2B\rho-\dot{\rho})^2\right]^{-1} \times\left\{-\frac{i}{4}\left\{\alpha^2\left[\frac{\sqrt{k}}{\rho} - \frac{i}{A}(2B\rho-\dot{\rho})\right]^2 - \alpha^{*2}\left[\frac{\sqrt{k}}{\rho} + \frac{i}{A}(2B\rho-\dot{\rho})\right]^2\right\}\right.$$

$$+\frac{\sqrt[4]{k}\,p_p}{\sqrt{\hbar}}\left\{\alpha\left[\frac{\sqrt{k}}{\rho}-\frac{i}{A}(2B\rho-\dot{\rho})\right]+\alpha^*\left[\frac{\sqrt{k}}{\rho}+\frac{i}{A}(2B\rho-\dot{\rho})\right]\right\}$$
$$+\frac{\rho p_p^2}{\hbar A}(2B\rho-\dot{\rho})\Bigg\}. \tag{41}$$

Then, eq. (40) becomes

$$\langle p'|\alpha\rangle=\sqrt[4]{\frac{2k^{1/2}}{\hbar\pi}}\left[\frac{\sqrt{k}}{\rho}+\frac{i}{A}(2B\rho-\dot{\rho})\right]^{-1/2}$$
$$\times\exp\left\{-i\left[\frac{1}{\hbar}x_p+\frac{2\alpha\sqrt[4]{k}}{\sqrt{\hbar}[\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A]}\right](p'-p_p)\right\}$$
$$\times\exp\left\{-\frac{\rho(p'-p_p)^2}{\hbar[\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A]}\right\}$$
$$\times\exp\left\{-\frac{1}{2}|\alpha|^2+\frac{1}{2}\alpha^2\frac{\sqrt{k}/\rho-i(2B\rho-\dot{\rho})/A}{\sqrt{k}/\rho+i(2B\rho-\dot{\rho})/A}\right\}. \tag{42}$$

The above equation coincides with eq. (35).

Making use of eqs (11), (12) and (36), the expectation values for canonical variables in coherent state are calculated as, respectively,

$$\langle\hat{x}\rangle=\rho\sqrt{\frac{\hbar}{k^{1/2}}}(\alpha+\alpha^*)+x_p, \tag{43}$$
$$\langle\hat{p}\rangle=\frac{\sqrt{\hbar k^{1/2}}}{2i\rho}[(1-iZ)\alpha-(1+iZ)\alpha^*]+p_p, \tag{44}$$
$$\langle\hat{x}^2\rangle=\frac{\rho^2\hbar}{\sqrt{k}}(\alpha^2+\alpha^{*2}+2\alpha^*\alpha+1)$$
$$+2\rho\sqrt{\frac{\hbar}{k^{1/2}}}(\alpha+\alpha^*)x_p+x_p^2, \tag{45}$$
$$\langle\hat{p}^2\rangle=\frac{\sqrt{\hbar k^{1/2}}}{i\rho}[(1-iZ)\alpha-(1+iZ)\alpha^*]p_p$$
$$-\frac{\hbar\sqrt{k}}{4\rho^2}[(1-iZ)^2\alpha^2+(1+iZ)^2\alpha^{*2}$$
$$-(1+Z^2)(2\alpha^*\alpha+1)]+p_p^2. \tag{46}$$

By using eqs (43) and (44), the eigenvalue, α, of the lowering operator is obtained as

$$\alpha=\frac{1}{2\rho}\sqrt{\frac{k^{1/2}}{\hbar}}\,(1+iZ)(\langle\hat{x}\rangle-x_p)+i\rho\sqrt{\frac{1}{\hbar k^{1/2}}}(\langle\hat{p}\rangle-p_p). \tag{47}$$

Using eqs (43) and (45), the fluctuation for $\hat{x}$ can be written as

$$(\Delta\hat{x})_\alpha = \sqrt{\langle\hat{x}^2\rangle - (\langle\hat{x}\rangle)^2} = \sqrt{\frac{\rho^2\hbar}{k^{1/2}}}. \tag{48}$$

And using eqs (44) and (46), the fluctuation for $\hat{p}$ can be represented as

$$(\Delta\hat{p})_\alpha = \sqrt{\langle\hat{p}^2\rangle - (\langle\hat{p}\rangle)^2} = \frac{\sqrt{\hbar k^{1/2}}}{2\rho}\sqrt{1+Z^2}. \tag{49}$$

Multiplying eqs (48) and (49), we obtain the uncertainty relation in coherent state as

$$(\Delta\hat{x})_\alpha\ (\Delta\hat{p})_\alpha = \frac{1}{2}\hbar\sqrt{1+Z^2}. \tag{50}$$

Note that the above value is exactly the same with the minimum value ($n=0$) of eq. (32).

5. Even and odd coherent states

The power-series expansions of the even and odd coherent states in time-dependent harmonic oscillator are given by [30,31]

$$\langle\hat{x}|\alpha_e\rangle = \frac{1}{\sqrt{\cosh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{(2n)!}}\langle\hat{x}|\phi_{2n}(t)\rangle, \tag{51}$$

$$\langle\hat{x}|\alpha_o\rangle = \frac{1}{\sqrt{\sinh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}}\langle\hat{x}|\phi_{2n+1}(t)\rangle. \tag{52}$$

Using eq. (20), we can easily expand eqs (51) and (52) as

$$\langle\hat{x}|\alpha_e\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}}\frac{1}{\sqrt{\cosh|\alpha|^2}}\exp\left\{-\frac{1}{4\rho\hbar}\left[\frac{\sqrt{k}}{\rho}+\frac{i}{A}(2B\rho-\dot{\rho})\right]\right. \\ \left.\times(\hat{x}-x_p)^2+\frac{i}{\hbar}p_p\hat{x}-\frac{1}{2}\alpha^2\right\}\cosh\left[\alpha\sqrt{\frac{k^{1/2}}{2\rho^2\hbar}}(\hat{x}-x_p)\right], \tag{53}$$

$$\langle\hat{x}|\alpha_o\rangle = \sqrt[4]{\frac{k^{1/2}}{2\rho^2\hbar\pi}}\frac{1}{\sqrt{\sinh|\alpha|^2}}\exp\left\{-\frac{1}{4\rho\hbar}\left[\frac{\sqrt{k}}{\rho}+\frac{i}{A}(2B\rho-\dot{\rho})\right]\right. \\ \left.\times(\hat{x}-x_p)^2+\frac{i}{\hbar}p_p\hat{x}-\frac{1}{2}\alpha^2\right\}\sinh\left[\alpha\sqrt{\frac{k^{1/2}}{2\rho^2\hbar}}(\hat{x}-x_p)\right]. \tag{54}$$

In the derivation of the above two equations, we used the following relations [32]:

$$\sum_{n=0}^{\infty}(-1)^n H_{2n}(x)\frac{z^{2n}}{(2n)!} = \exp(z^2)\cos(\sqrt{2}xz),$$
$$\sum_{n=0}^{\infty}(-1)^n H_{2n+1}(x)\frac{z^{2n+1}}{(2n+1)!} = \exp(z^2)\sin(\sqrt{2}xz). \tag{55}$$

The expectation values of the variables in even coherent states are evaluated as

$$\langle\hat{x}\rangle_e = x_p, \tag{56}$$
$$\langle\hat{p}\rangle_e = p_p, \tag{57}$$
$$\langle\hat{x}^2\rangle_e = \frac{\rho^2\hbar}{\sqrt{k}}(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\tanh|\alpha|^2 + 1) + x_p^2, \tag{58}$$
$$\langle\hat{p}^2\rangle_e = -\frac{\hbar\sqrt{k}}{4\rho^2}[(1-iZ)^2\alpha^2 + (1+iZ)^2\alpha^{*2} - (1+Z^2)(2|\alpha|^2\tanh|\alpha|^2 + 1)] + p_p^2, \tag{59}$$

and, the expectation values in odd coherent states are

$$\langle\hat{x}\rangle_o = x_p, \tag{60}$$
$$\langle\hat{p}\rangle_o = p_p, \tag{61}$$
$$\langle\hat{x}^2\rangle_o = \frac{\rho^2\hbar}{\sqrt{k}}(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\coth|\alpha|^2 + 1) + x_p^2, \tag{62}$$
$$\langle\hat{p}^2\rangle_o = -\frac{\hbar\sqrt{k}}{4\rho^2}[(1-iZ)^2\alpha^2 + (1+iZ)^2\alpha^{*2} - (1+Z^2)(2|\alpha|^2\coth|\alpha|^2 + 1)] + p_p^2. \tag{63}$$

Using eqs (56)–(59), we can easily derive the uncertainty product in even coherent state as

$$(\Delta\hat{x})_{\alpha,e}(\Delta\hat{p})_{\alpha,e} = \sqrt{\langle\hat{x}^2\rangle_e - (\langle\hat{x}\rangle_e)^2}\sqrt{\langle\hat{p}^2\rangle_e - (\langle\hat{p}\rangle_e)^2} = \frac{\hbar}{2}\Pi_e^{1/2}, \tag{64}$$

where

$$\Pi_e = 2Z[(Z+i)\alpha^2 + (Z-i)\alpha^{*2}](2|\alpha|^2\tanh|\alpha|^2 + 1) + (1+Z^2)(2|\alpha|^2\tanh|\alpha|^2 + 1)^2 - [(1-iZ)^2\alpha^4 + (1+iZ)^2\alpha^{*4} + 2(1-Z^2)|\alpha|^4]. \tag{65}$$

The uncertainty product in odd coherent state can be evaluated using eqs (60)–(63) as

$$(\Delta\hat{x})_{\alpha,o}(\Delta\hat{p})_{\alpha,o} = \sqrt{\langle\hat{x}^2\rangle_o - (\langle\hat{x}\rangle_o)^2}\sqrt{\langle\hat{p}^2\rangle_o - (\langle\hat{p}\rangle_o)^2} = \frac{\hbar}{2}\Pi_o^{1/2}, \tag{66}$$

where

$$\Pi_o = 2Z[(Z+i)\alpha^2 + (Z-i)\alpha^{*2}](2|\alpha|^2 \coth|\alpha|^2 + 1) \\ +(1+Z^2)(2|\alpha|^2 \coth|\alpha|^2+1)^2 \\ -[(1-iZ)^2\alpha^4 + (1+iZ)^2\alpha^{*4} + 2(1-Z^2)|\alpha|^4]. \quad (67)$$

If we designate α in terms of the magnitude $|\alpha|$ and phase θ:

$$\alpha = |\alpha|\mathrm{e}^{i\theta}, \quad (68)$$

eqs (65) and (67) can be rewritten as

$$\Pi_e = 4Z|\alpha|^2[Z\cos(2\theta) - \sin(2\theta)](2|\alpha|^2\tanh|\alpha|^2+1) \\ -2|\alpha|^4\{(1-Z^2)[1+\cos(4\theta)] + 2Z\sin(4\theta)\} \\ +(1+Z^2)(2|\alpha|^2\tanh|\alpha|^2+1)^2, \quad (69)$$

$$\Pi_o = 4Z|\alpha|^2[Z\cos(2\theta) - \sin(2\theta)](2|\alpha|^2\coth|\alpha|^2+1) \\ -2|\alpha|^4\{(1-Z^2)[1+\cos(4\theta)] + 2Z\sin(4\theta)\} \\ +(1+Z^2)(2|\alpha|^2\coth|\alpha|^2+1)^2. \quad (70)$$

6. Applications to the time-dependent harmonic oscillator

Our theory may be applied to various kinds of time-dependent Hamiltonian systems. As an example, let us see the periodically driven Caldirola–Kanai oscillator [33,34]. For this system, we can express the Hamiltonian as

$$\hat{H} = \mathrm{e}^{-\gamma t}\frac{\hat{p}^2}{2m} + \mathrm{e}^{\gamma t}\frac{1}{2}m\omega_0^2\hat{x}^2 - \mathrm{e}^{\gamma t}F(t)\hat{x}, \quad (71)$$

where m is the mass, γ the damping constant and $F(t)$ the arbitrary time-dependent driving force. Then, eq. (5) becomes

$$\ddot{\rho} + \gamma\dot{\rho} + \omega_0^2\rho - \frac{k}{4m^2}\mathrm{e}^{-2\gamma t}\frac{1}{\rho^3} = 0. \quad (72)$$

The solution of the above equation can be written as

$$\rho(t) = \sqrt{\frac{k^{1/2}}{2m\omega_d}}\mathrm{e}^{-\gamma t/2}, \quad (73)$$

where ω_d is given by

$$\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}. \quad (74)$$

The particular solutions x_p and p_p satisfy the following relations:

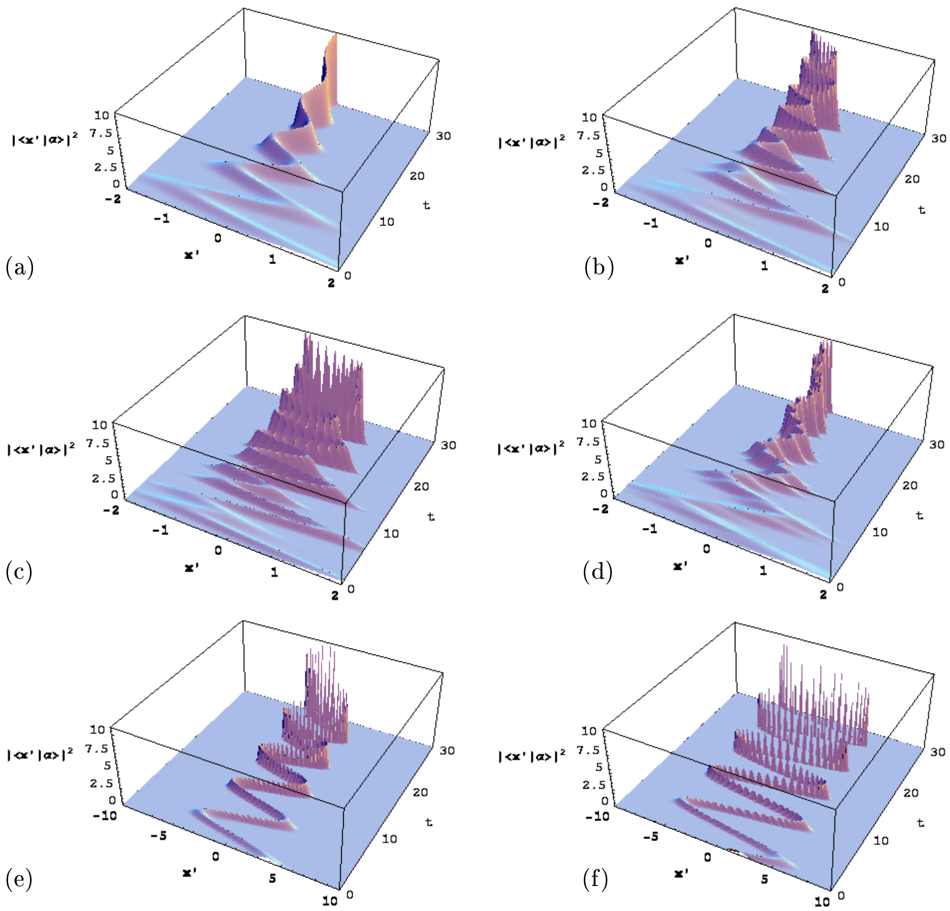

Figure 1. Coordinate space probability density in coherent state for the periodically driven Caldirola–Kanai oscillator as a function of position x' and time t. We used $\omega_0 = 1$, $\omega_1 = 2$, $\theta = 0$, $\gamma = 0.2$, $m = 1$, $\alpha = \mathrm{e}^{-i\omega t}$ and $\hbar = 1$. The value of (F_0, ω_1) is given by (0,0) for (**a**), (1,2) for (**b**), (2,2) for (**c**), (2,5) for (**d**), $(0.5, \sqrt{\omega_0^2 - \gamma^2/2})$ for (**e**) and $(1, \sqrt{\omega_0^2 - \gamma^2/2})$ for (**f**).

$$\ddot{x}_p + \gamma \dot{x}_p + \omega_0^2 x_p = \frac{F(t)}{m}, \tag{75}$$

$$\ddot{p}_p - \gamma \dot{p}_p + \omega_0^2 p_p = e^{\gamma t} \dot{F}(t). \tag{76}$$

The solutions of the above equations depend on $F(t)$. We consider the system driven by the following periodic force:

$$F(t) = F_0 \cos(\omega_1 t + \theta), \tag{77}$$

where F_0 is the amplitude of the driving force, ω_1 the frequency and θ the arbitrary phase. Equation (77) has classical particular solutions which are given by

$$x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + \gamma^2\omega_1^2}} \cos(\omega_1 t + \theta - \delta), \tag{78}$$

$$p_p(t) = -\frac{F_0\omega_1}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + \gamma^2\omega_1^2}} \mathrm{e}^{\gamma t} \sin(\omega_1 t + \theta - \delta), \tag{79}$$

where phase δ is given by

$$\delta = \tan^{-1} \frac{\gamma\omega_1}{\omega_0^2 - \omega_1^2}. \tag{80}$$

We show coordinate and momentum space probability densities in coherent state in figures 1 and 2, respectively, for various values of F_0 and ω_1. Figures 1a and 2a show that the unperturbed wave packets oscillate back and forth with time about $x' = 0$ and $p' = 0$. This behavior is very similar to the motion of classical oscillator. The amplitude of oscillation of the coordinate wave packet decreases with time due to the damping constant γ, while that of the momentum wave packet increases with time. However, from figures 1b and 1c, we can see that the oscillation in coordinate space can be maintained by a driving force.

There are two resonant frequencies for this system [35]. One is amplitude resonant frequency which is given by $\sqrt{\omega_0^2 - \gamma^2/2}$ and the other is the velocity resonant frequency which is the same with a natural frequency ω_0. In figures 1e, 1f, 2e and 2f, we show the probability densities in coherent state for amplitude resonant frequency. Figures 1e and 1f show that the decrease in the amplitude for the resonant oscillator can be negligible.

Now we consider another application which is the damped harmonic oscillator with time-variable frequency $\omega(t)$. For this case, the Hamiltonian is given by

$$\hat{H} = \mathrm{e}^{-\gamma t} \frac{\hat{p}^2}{2m} + \mathrm{e}^{\gamma t} \frac{1}{2} m\omega^2(t)\hat{x}^2. \tag{81}$$

We choose exponentially increasing frequency for $\omega(t)$ [36]:

$$\omega(t) = \omega_0 \mathrm{e}^{\xi t}, \tag{82}$$

where ω_0 is the initial frequency and ξ the positive real constant. Then, eq. (5) becomes

$$\ddot{\rho} + \gamma\dot{\rho} + \omega_0^2 \mathrm{e}^{2\xi t}\rho - \frac{k}{4m^2} \mathrm{e}^{-2\gamma t} \frac{1}{\rho^3} = 0. \tag{83}$$

Because this system acted without a driving force, the particular two solutions are zero:

$$x_p(t) = 0, \quad p_p(t) = 0. \tag{84}$$

By introducing the following abbreviations:

$$\nu = \frac{\gamma}{2\xi}, \quad z = \frac{\omega_0}{\xi} \mathrm{e}^{\xi t}, \tag{85}$$

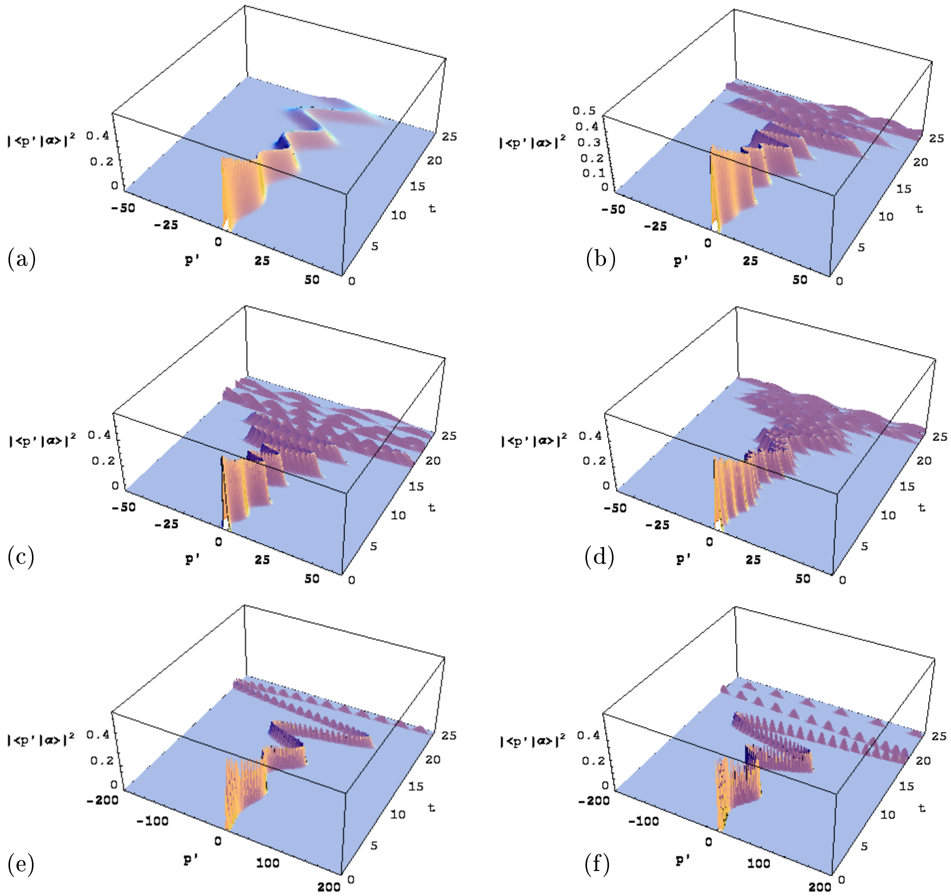

Figure 2. Momentum space probability density in coherent state for periodically driven Caldirola–Kanai oscillator as a function of position x' and time t. We used $\omega_0 = 1$, $\gamma = 0.2$, $m = 1$, $\alpha = \mathrm{e}^{-i\omega t}$ and $\hbar = 1$. The value of (F_0, ω_1) is given by (0,0) for (**a**), (1,2) for (**b**), (2,2) for (**c**), (2,5) for (**d**), $(0.5, \sqrt{\omega_0^2 - \gamma^2/2})$ for (**e**) and $(1, \sqrt{\omega_0^2 - \gamma^2/2})$ for (**f**).

the solution to eq. (83) can be represented as [36]

$$\rho(t) = \mathrm{e}^{-\gamma t/2}[c_1 J_\nu^2(z) + c_2 J_\nu(z) N_\nu(z) + c_3 N_\nu^2(z)]^{1/2}, \tag{86}$$

with the choice of k in the form

$$k = (4c_1c_3 - c_2^2)m^2 \left\{ J_\nu(z) \left\{ -\frac{\gamma}{2} N_\nu(z) + \frac{1}{2}\mathrm{e}^{\xi t}\omega_0 [N_{\nu-1}(z) - N_{\nu+1}(z)] \right\} \right.$$
$$\left. -N_\nu(z) \left\{ -\frac{\gamma}{2} J_\nu(z) + \frac{1}{2}\mathrm{e}^{\xi t}\omega_0 [J_{\nu-1}(z) - J_{\nu+1}(z)] \right\} \right\}^2 . \tag{87}$$

By direct differentiation of the above equation, we can check that k is a time-constant quantity: $\mathrm{d}k/\mathrm{d}t = 0$. In terms of eqs (84) and (86), the coherent state of this system can be thoroughly described.

7. Summary

We used invariant operator method to obtain quantum mechanical solution of general time-dependent quadratic harmonic oscillator. Using raising and lowering operators, we calculated the wave functions of the system, which are expressed in terms of Hermite polynomials. We obtained coherent state in both $\hat{x}$- and $\hat{p}$-spaces. The even and odd coherent states are represented in terms of hyperbolic functions. In addition, we calculated expectation values of the canonical variables, $\hat{x}$ and $\hat{p}$. We verified that the uncertainty product in coherent state is the same as the minimum of the uncertainty product in the number state and is always larger than $\hbar/2$.

We applied our theory to the driven Caldirola–Kanai oscillator and time-dependent harmonic oscillator with time-variable frequency. The displaced wave packet for Caldirola–Kanai oscillator in the coherent state is shown to oscillate back and forth with time about the center as for the classical oscillator. The amplitude of oscillation for $\hat{x}$-space wave packet is decreased with time due to the dissipation in the system while that of the $\hat{p}$-space increased.

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