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Parametric instabilities in magnetized bi-ion and dusty plasmas*

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Abstract. The excitation of low frequency modes of oscillations in a magnetized bi-ion or dusty plasma with parametric pumping of the magnetic field is analysed. The equation of motion governing the perturbed plasma is derived and parametrically excited transverse modes propagating along the magnetic field are found. With multiple ion species or charged dust present, a number of different circularly polarized modes can be excited. The stability of these modes is investigated as a function of the plasma parameters. The modulational instabilities of large amplitude normal modes, modified by the extra ion species or dust and propagating along the magnetic field, are also investigated.

Keywords. Parametric; bi-ion; dusty plasma; magnetized; modulational.

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1. Introduction

Magnetized plasmas in the laboratory and in space often have multiple ion species and/or dust components, and support strong low frequency (and hence typically long wavelength) hydromagnetic perturbations in the form of Alfvén and magnetoacoustic waves [1,2]. When the amplitudes of the waves are large, non-linear effects, in particular modulational and parametric instabilities [3] become important in their propagation. One such large amplitude wave is a magnetoacoustic wave propagating obliquely to the magnetic field, which may be excited by a magnetoacoustic shock wave, e.g. in a rapidly heated plasma, or in a supernova explosion. This magnetoacoustic pump wave modifies the background magnetic field. By considering perturbations to this pump, we can investigate the possibility of resonant modes excited by the pump, i.e., the magnetoacoustic oscillations can decay into Alfvén waves. This process was predicted by Montgomery and Harding [4] and Vahala

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and Montgomery [5], and subsequently pursued by several authors (e.g. Cramer [6]) for a single ion species.

It is known that the presence of multiple ion species and/or dust changes many plasma processes. In the case of dust, one of the most important effects is the collection of electrons and ions from the background plasma by the charged grains, influencing the propagation of plasma and electromagnetic waves [7,8]. For many dusty plasmas, the grain charge is negative and large ($\sim 10^2$–$10^3 e$), so that an appreciable proportion of the negative charge in the plasma may reside on the dust particles. Here we assume that all the dust grains are of the same size and mass, and have the same charge, and so the grains are treated as very massive ions. In this paper, we investigate the propagation of plane hydromagnetic waves parallel to the pumped magnetic field, modified due to the presence of multiple ion species or dust. We highlight the effect of multiple ion species and dust on the parametrically excited waves. Large amplitude parallel propagating modes are then considered, and modulational instabilities of these modes are analysed.

2. Wave equations

We invoke the standard multi-fluid plasma model, which includes the fluid momentum and continuity equations for the plasma ions and dust grains, and (inertialess) electrons, as well as Maxwell's equations ignoring the displacement current. The background magnetic field $\mathbf{B}_0$ is in the z-direction. The first species (assumed primary ionic) is denoted with a subscript '1', and the second species (either secondary ionic or dust) with a subscript '2'. We employ the parameters $\delta_1 = n_e/Z_1 n_1$ and $\delta_2 = n_e/Z_2 n_2$ which measure the distribution of charge amongst the species. We may write the total charge neutrality condition as $1/\delta_1 + 1/\delta_2 = 1$.

Ignoring collisions, but including the effects of pressure, the non-linear equations for the velocities, electric and magnetic fields reduce to:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\left(\frac{\mathbf{v}_1}{\delta_1} + \frac{\mathbf{v}_2}{\delta_2} \right) \times \mathbf{B} \right) - \frac{1}{\mu_0} \nabla \times \left(\frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{e n_e} \right) \tag{1}$$

$$\begin{aligned} \delta_1 \rho_1 \frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = {} & -(\delta_1 U_1^2 + \alpha_1^2)\nabla\rho_1 - \alpha_2^2 \nabla\rho_2 + \frac{\rho_2 \Omega_2}{B_0}(\mathbf{v}_1 - \mathbf{v}_2) \\ & \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \end{aligned} \tag{2}$$

$$\begin{aligned} \delta_2 \rho_2 \frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}t} = {} & -(\delta_2 U_2^2 + \alpha_2^2)\nabla\rho_2 - \alpha_1^2 \nabla\rho_1 + \frac{\rho_1 \Omega_1}{B_0}(\mathbf{v}_2 - \mathbf{v}_1) \\ & \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}. \end{aligned} \tag{3}$$

Here $\rho_{1,2} = m_{1,2} n_{1,2}$ are the densities of each massive component of the plasma and $\Omega_{1,2}$ are the corresponding (signed) cyclotron frequencies. U_α are the individual sound speeds, and $\alpha_1^2 = Z_1 U_e^2 m_e/m_1$ and $\alpha_2^2 = Z_2 U_e^2 m_e/m_2$ are thermal speeds associated with the electron pressure.

Note that eqs (1)–(3) reduce, in the case of a single species plasma ($\delta_1 \to 1$, $|\delta_2| \to \infty$, assuming local charge neutrality is maintained), to the equations used in Mio *et al* [9], Mjølhus [10], Spangler and Sheerin [11], Mjølhus and Wyller [12], Ovenden *et al* [13] where non-linear Alfvén waves were investigated. Non-linear obliquely propagating waves in a bi-ion plasma have been investigated by McKenzie *et al* [14].

3. Parametric pumping

The background magnetic field is modulated periodically, with $\mathbf{B}^{(0)} = B_0(1 + \bar{\varepsilon}\cos(k_0 x)\cos(\omega_0 t))\hat{\mathbf{z}}$. The effect of this magnetic pumping is to modify the velocity, density and charge imbalance to order $\bar{\varepsilon}$, and we obtain the following dispersion equation for the pump wave:

$$\omega_0^2\,(\omega_0^2 - \omega_c^2) = W k_0^2\,(\omega_0^2 - X - Y k_0^2), \tag{4}$$

where $|\omega_c|$ is a hybrid cut-off frequency as $k_0 \to 0$. In the presence of a secondary species the fast magnetoacoustic wave gains an additional mode and the dispersion relation is dispersive.

We now test the stability of the pump wave solution. We search for transverse plane wave perturbations, propagating in the z-direction, with ion velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Assuming the velocities to have a spatial dependence of $\exp(ikz)$, we obtain

$$\begin{aligned}\delta_{01}\frac{\partial^2 v_{\pm 1}}{\partial t^2} \mp i\frac{v_{A1}^2 k^2}{\Omega_1}\frac{\partial v_{\pm 1}}{\partial t} \pm i\frac{\Omega_1\delta_{01}}{\delta_{02}}\left(\frac{\partial v_{\pm 1}}{\partial t} - \frac{\partial v_{\pm 2}}{\partial t}\right) + v_{A1}^2 k^2 v_{\pm 1} \\ = -\bar{\varepsilon}\left(b\cos(\omega_0 t) v_{A1}^2 k^2 v_{\pm 1} \pm i\frac{\Omega_1\delta_{01}k_0 V_{x2}}{\delta_{02}}\right. \\ \left.\times\left(\frac{\cos(\omega_0 t)}{\omega_0}\left(\frac{\partial v_{\pm 1}}{\partial t} - \frac{\partial v_{\pm 2}}{\partial t}\right) - \sin(\omega_0 t)(v_{\pm 1} - v_{\pm 2})\right)\right),\end{aligned} \tag{5}$$

where $v_+ = v'_x + iv'_y$ corresponds to a left-hand circularly polarised wave for positive frequencies and a right-hand circularly polarised wave for negative frequencies, while $v_- = v'_x - iv'_y$ corresponds to the reverse.

By taking the Fourier transform of both differential equations, we find the dispersion equation. The wave frequency is $\omega = \Omega + \bar{\varepsilon}\phi$, where Ω may be any one of ω_{Rs}, ω_{Rf}, ω_L for a negatively charged secondary species. The dispersion equation governing linear waves (with $\bar{\varepsilon} = 0$) propagating along the z-axis is given by

$$F_{\pm}(k,\omega) = -\delta_{01}\omega^2 + v_{A1}^2 k^2\left(1 \mp \frac{\omega}{\Omega_1}\right) \pm \frac{\Omega_1\delta_{01}}{\delta_{02}}\omega\left(1 - \frac{1 \mp \frac{\omega}{\Omega_1}}{1 \mp \frac{\omega}{\Omega_2}}\right) = 0. \tag{6}$$

For a negatively charged secondary species there are three modes of excitation. There are two right-hand modes ω_{Rf} and ω_{Rs} (with a resonance at the heavy species cyclotron resonance), and a single left-hand mode ω_L with a cut-off. A parametric interaction occurs between the excited linear fields due to the pump fields. There will be a resonant parametric interaction when $\omega + \omega_0$ or $\omega - \omega_0$ is near a natural mode of the system. It is found that the only choice for this resonance is for a left-hand polarised mode to interact with a right-hand polarised mode, or vice versa.

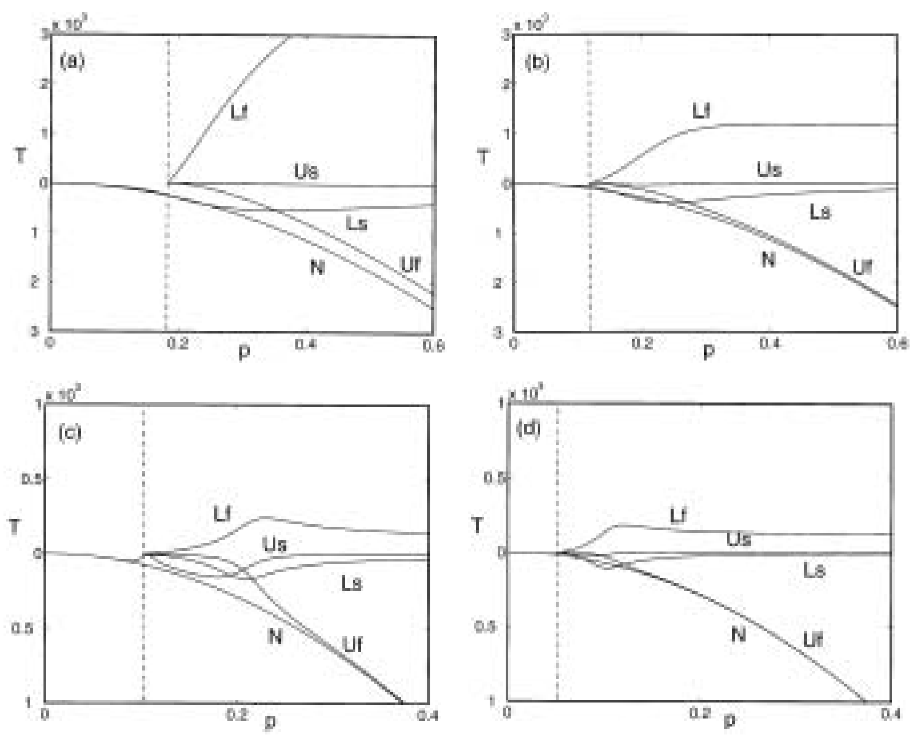

Figure 1. The squared normalised frequency change $T = (\phi/\Omega_1)^2$ plotted against pump frequency $p = \omega_0/\Omega_1$. In all four plots the second species is positive, with $\beta_1 = 0.8, B_1 = 1.5$. In (**a**)–(**c**) we have $\Omega_2/\Omega_1 = 0.1$ and $(U_2/U_1)^2 = 1/4$. In (**d**) we have $\Omega_2/\Omega_1 = 0.05$ and $(U_2/U_1)^2 = 1/8$. The number densities vary in (**a**)–(**d**), with $\delta_{01} = 1.1$ in (**a**), $\delta_{01} = 1.02$ in (**b**), and $\delta_{01} = 1.002$ in (**c**) and (**d**). Also, 'Uf, Us, Lf, Ls' denote the upper-fast, the upper-slow, the lower-fast and the lower-slow combinations, respectively. 'N' denotes the single species result.

In figure 1 we plot $T := (\phi/\Omega_1)^2$ vs. pump frequency $p = \omega_0/\Omega_1$ for the upper 'U' and lower 'L' branches of the pump dispersion relation for each mode of interaction. This figure is for a warm plasma in which the second heavy species is positively charged. The same basic features are present in the negatively charged case. All curves labeled 'f', i.e., 'Uf' and 'Lf', correspond to the combination between ω_{R} and ω_{Lf}, that is the *fast* interaction. The curves labeled 's', i.e., 'Us' and 'Ls', are for the *slow* interaction between ω_{R} and ω_{Ls}. In figure 1a the second species has a substantial number density in the plasma, i.e., $\delta_{01} = 1.1$. The 'Uf' interaction is unstable and monotonically decreases, approaching the single species result 'N' for large ω_0. The 'Ls' interaction is also unstable, however it experiences a minumum, about which it turns over and approaches zero. Next, we see that the 'Us' interaction is small and positive, and hence it is weakly stable. Finally, we see that the 'Lf' interaction, which is only present in a *warm* plasma, is strongly stable. All these curves in fact lie above the 'N' curve, which shows that instabilities are reduced throughout.

We decrease the number density of the second species in figures 1b and 1c to $\delta_{01} = 1.02$ and $\delta_{02} = 1.002$, respectively. In (b) 'Lf' rises then levels out near $\omega_0 \sim 0.3\Omega_1$. In (c) it actually experiences a maximum and then approaches zero for large ω_0. Moreover, in (c) the 'Us' interaction has actually become unstable in the frequency region just above the cut-off frequency. In figure 1d we increase the mass of the second species, while keeping all other parameters the same as in (c). The interesting behaviours have all shifted to lower frequencies, since Ω_2 has decreased. Note that the 'Ls' interaction actually becomes more unstable than 'N' near its minimum. Hence, the introduction of a second species adds a range of extra behaviours, even with a small number density or large mass.

4. Modulational instabilities

In this section we investigate the stability of the parallel propagating waves. By considering the low and high β regimes we test for modulational and decay instabilities, and compare the results to the one species case. We now refer to the parallel propagating waves as the 'pump' wave. The linear solution is an exact solution of the non-linear equations. The velocity expansions may be written as

$$\mathbf{v}_1(z,t) = \overline{\varepsilon\mathbf{v}}_{\perp 1}(z,t) + \varepsilon' \mathbf{v}'_{\perp 1}(z,t) + \varepsilon' \mathbf{v}'_{z1}(z,t), \tag{7}$$

$$\mathbf{v}_2(z,t) = \overline{\varepsilon\mathbf{v}}_{\perp 2}(z,t) + \varepsilon' \mathbf{v}'_{\perp 2}(z,t) + \varepsilon' \mathbf{v}'_{z2}(z,t), \tag{8}$$

where $\bar{\varepsilon}$ refers to the parallel propagating pump and ε' refers to the excited fields. Substituting these fields into the non-linear equations, we obtain the equation of motion governing the transverse velocities for species 1:

$$\begin{aligned} \delta_{01}\rho_{01}\frac{\partial v'_{\pm 1}}{\partial t} \pm i\rho_{02}\Omega_2(v'_{\pm 1} - v'_{\pm 2}) - \frac{B_0}{\mu_0}\frac{\partial B'_{\pm}}{\partial z} &= \bar{\varepsilon}\left(\delta_{01}\rho_{01}v'_{z1}\chi_1\frac{\partial}{\partial z} - \left(\rho'_1 + \frac{\Omega_2}{\Omega_1}\rho'_2\right)\chi_1\frac{\partial}{\partial t}\right. \\ &\qquad \left. \pm \frac{i\rho_{02}\Omega_2}{B_0}(v'_{z1} - v'_{z2}) \mp i\Omega_2\rho'_2(\chi_1 - \chi_2)\right)\bar{B}_{\pm}, \end{aligned} \tag{9}$$

where $\chi_{1,2}$ are the ratios of the pump velocities to the transverse pump magnetic fields. Here the second heavy species causes this differential equation to contain non-derivative terms that would otherwise be absent. Taking the Fourier transform over both space and time, we obtain the dispersion equation, giving a tenth-order polynomial. In the work of Wong and Goldstein [15] for a single ion species, the dispersion equation was found to be a sixth-order polynomial.

In the domain $K = k/k_0 < 1$ a modulational instability exists, which is also seen in the one species description. This is the result of an interaction between the slow left-hand mode and the fast acoustic mode. In figures 2a and 2b we plot the growth rate for a range of values of δ_{01}. This modulational instability is found to decrease in growth rate and shift to low values of wave number as δ_{01} moves away from 1. For $K \gtrsim 1$ there exists a very narrow instability. It is found that this shifts to higher values of wave number as the number density of the second species is increased.

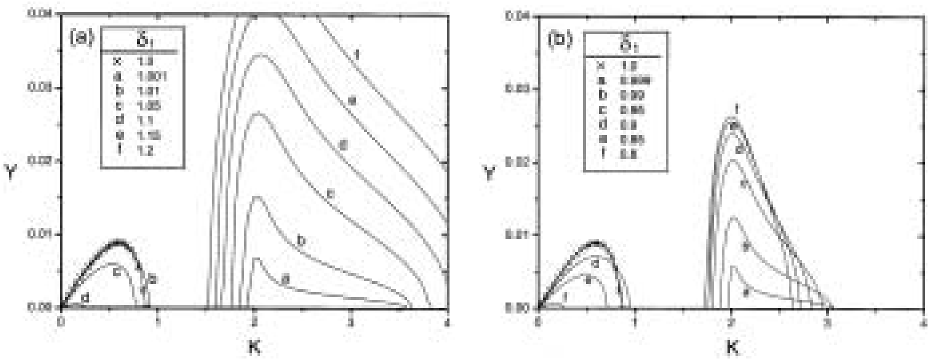

Figure 2. The normalised growth rate $\mathrm{Im}(\omega)/\Omega_1$ for the full dispersion relation plotted against the normalised wave number $K = k/k_0$, with $\kappa = k_0 v_{A1}/\Omega_1 = 0.3$ and $\omega_0(k_0)$ the upper (positive) left-hand mode. (**a**) $\Omega_2/\Omega_1 = 0.04$ and $\delta_{01} = 1.05$. (**b**) $\Omega_2/\Omega_1 = -0.04$ and $\delta_{01} = 0.9$, for a range of δ_{01}. Here $\beta_1 = 0.1, \beta_2 = 0, B_1 = 0.4$ and $\eta^2 = 0.04$.

In the region $K > 1$ we see a strong decay instability. This is absent when $\delta_{01} = 1$, and hence this reveals an extra instability. It is the result of the two slow acoustic modes associated with the additional heavy species interacting. The choice of low β causes this instability to be quite wide. This is because the positive and negative slow acoustic modes are then quite close to one another for a large range of K. By comparing (a) to (b) we see that the growth rate is both higher and wider in the positive case. Moreover, the width of the instability for the negative case actually begins to decrease when $\delta_{01} \ll 1$.

For high values of β, all four acoustic modes increase in magnitude. As in Wong and Goldstein [15] we only find instabilities for high values of β when the pump wave is the upper (negative) right-hand mode. Again a modulational instability appears. This time it is the result of an interaction between the fast acoustic mode and the right-handed transverse mode. In the single species case this is the *only* instability found. However, we find two further instabilities, both of which are quite narrow. Firstly, the slow right-handed mode interacts with the fast acoustic mode in the region $K \gtrsim 1$. Secondly, the right-handed transverse mode interacts with the slow acoustic mode in a region such that $K \gg 1$. If the second species is positive then these narrow decay instabilities are no longer present. Also, if the pump wave is chosen to be the slow mode (right or left handed), no additional instabilities with large growth rates are found.

5. Conclusions

The parametric excitation of parallel propagating modes by a magnetic pump in a bi-ion or dusty plasma has been considered. A number of modes has been found to be excited, and the growth rate was calculated. The stability of the parallel modes was then investigated, with respect to the modulational instability. Several new instabilities were found, related to the excitation of different acoustic modes. These instabilities should be relevant to the stability of large-amplitude low-frequency waves in plasmas containing multiple species and charged dust, such as laboratory plasmas and space plasmas.

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