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Cosmological models in general relativity

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Abstract. LRS Bianchi type-I space-time filled with perfect fluid is considered here with deceleration parameter as variable. The metric potentials A and B are functions of x as well as t. Assuming $B'/B = f(x)$, where prime denotes differentiation with respect to x, it was found that $A = (l'/l)B$ and $B = lS(t)$, where $l = f(x)$ and S is the scale factor which is a function of t only. The value of Hubble's constant H_0 was found to be less than half for non-flat model and is equal to 1.3 for a flat model.

Keywords. Cosmology; LRS Bianchi type-I; Hubble's constant.

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1. Introduction

Here, LRS Bianchi type-I space-time filled with perfect fluid with variable deceleration parameter are considered. In order to study the formation of galaxies and the isotropization of the universe, LRS Bianchi type-I model of space-time have been studied and it is less restrictive than spherically symmetric space-time. Jadav and Prasad [1] and Roy and Narayan [2] have studied the problem of cosmological models. Adopting LRS Bianchi type-I space-time filled with perfect fluid with variable deceleration parameters, the upper limit of Hubble's constant for various models have been found out. Physical properties of the solutions have been discussed for various models.

2. Field equations

LRS Bianchi type-I metric is given by

$$ds^2 = dt^2 - A^2 dx^2 - B^2(dy^2 + dz^2), \tag{1}$$

where A and B are functions of x and t. Einstein's field equations are

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij}. \tag{2}$$

The energy–momentum tensor

$$T_{ij} = (\rho + P)u_i u_j - P g_{ij}. \tag{3}$$

Together with co-moving coordinate $u_i u^j = 1$, we have

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{B'^2}{A^2B^2} = -8\pi p, \tag{4}$$

$$\dot{B}' - \frac{B'\dot{A}}{A} = 0, \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{B}{A^2B} + \frac{A'B'}{A^3B} = -8\pi p, \tag{6}$$

$$\frac{2B''}{A^2B} - \frac{2A'B'}{A^3B} + \frac{B'^2}{A^2B^2} - \frac{2\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} = 8\pi\rho. \tag{7}$$

The energy conservation equation is

$$\dot{\rho} + (p + \rho)\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) = 0, \tag{8}$$

where dots and primes indicate partial differentiation with respect to t and x respectively. A and B are metric potentials and they are functions of x as well as t.

From (5), on integration we have

$$A = \frac{B'}{l}, \tag{9}$$

where l is a function of x. From eqs (4), (6) and (9) we have

$$\frac{B'}{B}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ddot{B}}{B}\right) + \frac{\dot{B}}{B'}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{B'}{B}\right) + \frac{l^2}{B^2}\left(1 - \frac{B}{B'}\cdot\frac{l'}{l}\right) = 0. \tag{10}$$

Assuming $B'/B = f(x)$, we have from eq. (10),

$$B = lS(t), \tag{11}$$

where S is the scale factor which is a function of t. Thus, from eq. (9), we have

$$A = \frac{l'}{l}S. \tag{12}$$

The line element (1) is reduced to

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2[\mathrm{d}X^2 + e^{2X}(\mathrm{d}y^2 + \mathrm{d}z^2)], \tag{13}$$

where $X = \log l$. The mass-density, pressure and Ricci scalar are

$$8\pi\rho = \frac{3}{S^2}[\dot{S}^2 - 1], \tag{14}$$

$$8\pi p = \frac{1}{S^2}[1 - \dot{S}^2 - 2\ddot{S}], \tag{15}$$

$$R = \frac{6}{S^2}[\ddot{S} + \dot{S}^2 - 1]. \tag{16}$$

3. Solution of field equations

Case:

$$b = \frac{-aS}{\dot{S}^2}. \tag{17}$$

Non-flat model: The solution of the equation

$$S\ddot{S} + b\dot{S}^2 = 0 \tag{18}$$

gives

$$S = \frac{1}{2}at^2 + Ct + d, \tag{19}$$

where a, C and d are constants. The mass-density, pressure and Ricci scalar are given by

$$8\pi\rho = \frac{3}{S^2}[(C+at)^2 - 1], \tag{20}$$

$$8\pi p = -\frac{1}{S^2}[(C+at)^2 - 1 + 2a], \tag{21}$$

$$R = \frac{6}{S^2}[(C+at)^2 - 1 + a]. \tag{22}$$

From eq. (20)

$$\rho > 0 \quad \text{for} \quad t > \frac{1-C}{a}, \tag{23}$$

where $C < 1$, $P < 0$ and $R > 0$. ρ and R decrease as t increases.

$$\text{Expansion } \theta = \frac{3(C+at)}{S}. \tag{24}$$

Expansion decreases as t increases.

Shear $\sigma = 0$ and hence $\sigma/\theta = 0$. This shows that the space-time is isotropic. From Hubble's parameter equation, we have an epoch time

$$t_0 = \frac{1}{H_0} - \frac{C}{a} + \frac{\sqrt{a^2 + H_0^2(C^2 - 2ad)}}{aH_0}. \tag{25}$$

From eq. (25) we have

$$at_0 + C = \frac{a}{H_0} - 1 + \frac{\sqrt{a^2 + H_0^2(C^2 - 2ad)}}{aH_0} > 0 \tag{26}$$

and hence $(a/H_0) - 1 > 0$, i.e.

$$H_0 < a. \tag{27}$$

From eq. (25) $t_0 > 0$ for

$$H_0 < \frac{a}{2d} \tag{28i}$$

and

$$C^2 > 2ad, \tag{28ii}$$

i.e.

$$H_0 < \frac{C}{2d}. \tag{29}$$

From Ricci scalar equation we have

$$(C+at)^2 > (\sqrt{1-a})^2. \tag{30}$$

It is evident from eq. (23) and (30) that $a < 1$ and thus

$$H_0 < 1. \tag{31}$$

Case:

$$b = \frac{-aS}{\dot{S}^2}. \tag{17}$$

Flat model: The solution of the equation

$$S\ddot{S} + b\dot{S}^2 = 0 \tag{18}$$

gives

$$S = \frac{1}{2}at^2 + Ct + d, \tag{32}$$

where a, C and d are constants. For flat model Ricci scalar, $R = 0$ and hence

$$(C+at)^2 = (1-a). \tag{33}$$

The mass-density and pressure are given by

$$8\pi\rho = \frac{3}{S^2}[(C+at)^2 - 1], \tag{34}$$

$$8\pi p = \frac{1}{S^2}[(C+at)^2 - 1]. \tag{35}$$

ρ and P are greater than zero if $C + at > 1$ and hence from (33)

$$a < 0. \tag{36}$$

From Hubble's parameter

$$H = \frac{\dot{S}}{S}.$$

We have an epoch time t_0 which is

$$t_0 = \frac{1}{H_0} + \frac{C}{a^1} \neq \frac{\sqrt{(a^1)^2 + H_0^2(C^2 + 2a^1 d)}}{a^1 H_0}, \tag{37}$$

where $a^1 = -a$ and it is greater than zero if

$$\frac{C}{H_0} > d. \tag{38}$$

At any time

$$t = \frac{C-2}{a^1}, \quad \text{where} \quad C > 2. \tag{39}$$

We have from eq. (33),

$$\left(C - a^1 \frac{C-2}{a^1}\right)^2 = 1 + a^1,$$

i.e., $a^1 = 3$.

Thus from eq. (32), S is given by

$$\begin{aligned} S &= d + \frac{(C-2)}{a^1}\frac{(3C-2)}{2} \\ &= d + \frac{(C-2)(3C-2)}{2a^1} \end{aligned} \tag{40}$$

which is positive.

Case:

$$b = \frac{-atS}{\dot{S}^2}. \tag{41}$$

Non-flat model: The solution of the equation

$$S\ddot{S} + b\dot{S}^2 = 0 \tag{18}$$

is given by

$$S = \frac{1}{6}at^3 + Ct + d, \tag{42}$$

where a, C and d are constants.

The mass-density, pressure and Ricci scalar are given by

$$8\pi\rho = \frac{3}{S^2}\left[\left(C + \frac{1}{2}at^2\right)^2 - 1\right], \tag{43}$$

$$8\pi p = \frac{-1}{S^2}\left[\left(C+\frac{1}{2}at^2\right)^2 - 1 + 2at\right], \tag{44}$$

$$R = \frac{6}{S^2}\left[\left(C+\frac{1}{2}at^2\right)^2 - 1 + at\right]. \tag{45}$$

From eq. (43) $\rho > 0$ for $t > \sqrt{(2/a)(1-C)}$, where $C < 1$. From eq. (44), $P < 0$. From eq. (45), $R > 0$.

$$\text{Expansion } \theta = \frac{3(C+\frac{1}{2}at^2)}{(\frac{1}{6}at^3 + Ct + d)} \tag{46}$$

and it decreases as t increases.

Shear $\sigma = 0$ and hence $\sigma/\theta = 0$. This shows that the space-time is isotropic. At any intermediate time $t = \sqrt{(2/a)(2-C)}$, we have from $R > 0$ that

$$2^2 - 1 + \sqrt{2a(2-C)} > 0, \quad \text{where } C < 2$$

and hence

$$2 > \left[1 - \sqrt{2a(2-C)}\right]^{1/2}. \tag{47}$$

From eq. (47), it is evident that

$$1 > \sqrt{2a(2-C)}. \tag{48}$$

From Hubble's parameter H which is

$$H = \frac{\dot{S}}{S},$$

we have a cubic equation in t_0 which is

$$t_0^3 - \frac{3t_0^2}{H_0} + \frac{6ct_0}{a} - \frac{6}{aH_0}(C - \mathrm{d}H_0) = 0. \tag{49}$$

Solving eq. (49) we have

$$t_0 = \frac{1}{H_0}, \quad a = 2CH_0^2 \quad \text{and} \quad d = \frac{a}{3H_0^3}, \tag{50}$$

where $t_0 =$ epoch time. Thus from eq. (48)

$$4H_0^2C(2-C) < 1. \tag{51}$$

Again for $C < 1$, $H_0^2C(2-C) > H_0^2$ and hence from eq. (51) we have

$$H_0 < \frac{1}{2}. \tag{52}$$

Flat model: $R = 0$ gives

$$\left(C + \frac{1}{2}at^2\right)^2 - 1 + at = 0. \tag{53}$$

At any intermediate time, $t = \sqrt{(2/a)(2-C)}$, where $C < 2$, we have from eq. (53)

$$(3 + at)(3 - at) = 0$$

and hence

$$H_0^2 C(2-C) = \frac{9}{4}. \tag{54}$$

For $C < 1$, $H_0^2 C(2-C) > H_0^2$ and hence $9/4 > H_0^2$, i.e.

$$H_0 < 1.5. \tag{55}$$

Case:

$$b = -\frac{KS}{\dot{S}^3}. \tag{56}$$

Non-flat model: The solution of the equation

$$S\ddot{S} + b\dot{S}^2 = 0 \tag{18}$$

gives

$$S = K_2 + \frac{(K_1 + 2Kt)^{3/2}}{3K}, \tag{57}$$

where K, K_1 and K_2 are constants.

The mass-density, pressure and Ricci scalar are given by

$$8\pi\rho = \frac{3}{S^2}[(K_1 + 2Kt) - 1], \tag{58}$$

$$8\pi p = \frac{1}{S^2}\left[1 - (K_1 + 2Kt) - \frac{2K}{(K_1 + 2Kt)^{1/2}}\right], \tag{59}$$

$$R = \frac{6}{S^2}\left[(K_1 + 2Kt) - 1 + \frac{K}{(K_1 + 2Kt)^{1/2}}\right]. \tag{60}$$

From eq. (58), $\rho > 0$ for $t > (1 - K_1)/2K$, where $K_1 < 1$. From eqs (59) and (60), $P < 0$ and $R > 0$.

$$\text{Expansion } \theta = \frac{3(K_1 + 2Kt)^{1/2}}{K_2 + \frac{(K_1+2Kt)^{3/2}}{3K}}. \tag{61}$$

Shear $\sigma = 0$ and hence $\sigma/\theta = 0$. Thus the space-time is isotropic.

4. Physical behaviour of the solution

As t increases, the mass-density ρ, Ricci scalar R and expansion θ decrease.

Evaluation of constants and Hubble's constant: From Hubble's parameter

$$H = \frac{\dot{S}}{S}.$$

We have

$$Z_0^3 = 3K\left(\frac{Z_0}{H_0} - K_2\right), \tag{62}$$

where $Z_0 = \sqrt{K_1 + 2Kt_0}$ and t_0 is an epoch time. For the solution of eq. (62) let us assume $Z_0 = K_2$ and hence

$$K_2^2 = 3K\left(\frac{1}{H_0} - 1\right). \tag{63}$$

From eq. (60), $Z^3 - Z + K > 0$, where $Z = \sqrt{K_1 + 2Kt}$.

At any intermediate time, $t = (3 - K_1)/2K$, where $K_1 < 3$, we have from eq. (60)

$$\sqrt{27} - \sqrt{3} + K > 0. \tag{64}$$

From $Z_0 = \sqrt{K_1 + 2Kt_0} = K_2$, we have

$$K_1 + 2K\frac{3 - K_1}{2K} = K_2^2,$$

$$K_2^2 = 3. \tag{65}$$

Thus from eq. (63)

$$3 = 3K\left(\frac{1}{H_0} - 1\right) \quad \text{or} \quad K = \frac{1}{\frac{1}{H_0} - 1}. \tag{66}$$

From eq. (64)

$$3.67 + \frac{1}{\frac{1}{H_0} - 1} > 0,$$

$$\frac{3.67}{H_0} - 3.67 + 1 > 0.$$

Hence

$$H_0 < 1.3. \tag{67}$$

From (66) in order that $K > 0$, $H_0 < 1$. Thus value of Hubble's constant is less than unity.

For flat model: Equation $R = 0$ gives at any intermediate time, $t = (3 - k_1)/2k$, where $k_1 < 3$,

$$3.67 + \frac{1}{\frac{1}{H_0} - 1} = 0. \tag{68}$$

Hence we have,

$$H_0 = 1.3. \tag{69}$$

The mass-density and pressure are given by

$$8\pi\rho = 3(8\pi p) = \frac{3}{S^2}[(k_1 + 2kt) - 1], \tag{70}$$

where

$$S = \frac{(k_1 + 3kt)^{3/2}}{3k} + k_2 \tag{71}$$

and k, k_1 and k_2 are constants. In non-flat model, eq. (65) gives $k_2 = \sqrt{3}$ and from (69) using $H_0 = 1.3$, we have from (66), $k = -4.3$. As a result S is positive for $t > (k_1 - (3k_2k')^{2/3})/2k_1$; where $k' = 4.3$ such that $(3k_2k')^{2/3} < k_1 < 3$.

The study of the results of the three deceleration parameter models of the universe showed that Hubble's constant is less than $\frac{1}{2}$ for the non-flat model, and it is 1.3 for the flat model. The scale factor is not found to be linearly related to time as is the case with supernova cosmology model [3].

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