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Fermions in light front transverse lattice quantum chromodynamics

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Abstract. We briefly describe motivations for studying transverse lattice QCD. Presence of constraint equation for fermion field on the light front allows different methods to put fermions on a transverse lattice. We summarize our numerical investigation of two approaches using (a) forward and backward derivatives and (b) symmetric derivatives.

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1. Introduction

Our main motivation is to compute non-perturbatively physical observables relevant to high energy phenomena. It is well-known that Hamiltonian formulation of field theory provides the most direct route to wave functions. Thus *it is useful to keep time continuous*. Most high energy experiments probe the hadron structure very close to the light cone. Thus it behooves us to take $x^+ = x^0 + x^3$ as the 'time'. Thus we shall adopt the Hamiltonian framework, named *front form* by Dirac [1] where the initial conditions are specified on the hypersurface $x^+ = 0$. Then $x^- = x^0 - x^3$ is the longitudinal coordinate.

In the front form of dynamics, for an on-mass shell particle of four-momentum k^μ and mass m, longitudinal momentum $k^+ = k^0 + k^3 \geq 0$ and the dispersion relation is given by energy $k^- = ((k^\perp)^2 + m^2)/k^+$. Thus we have a relativistic dispersion relation with (a) no square root, (b) non-relativistic structure in the transverse plane, and (c) large energy for large $k^\perp$ and/or small k^+. One of the attractive features of the front form of dynamics is that longitudinal boost becomes a scale transformation and transverse boosts are Galilean boosts. Thus in a relativistic theory, internal motion and the motion associated with centre of mass motion can be separated out at the kinematical level and hence, one can construct boost invariant wave functions. Furthermore, rotations in the transverse plane remains kinematical.

Since vacuum processes receive contributions only from $k^+ = 0$, Fock vacuum becomes an eigenstate of the light front Hamiltonian in the interacting theory (with the exception of theories with spontaneous symmetry breaking) with a cut-off $k^+ \geq 0$. This is another attractive feature of light front dynamics.

When one tries to set up a non-perturbative calculational framework for gauge field theory, some difficult problems arise immediately. Two main features of gauge field theory are the symmetries associated with gauge invariance and Lorentz invariance. Because of infinitely many degrees of freedom involved, any practical calculation in quantum field theory necessitates a cut-off. In perturbation theory it is possible to maintain both the symmetries in a cut-off theory. It turns out that in non-perturbative calculations, it is impossible to do so. Formulation of gauge field theory on a space-time lattice provides a gauge invariant cut-off. It is hoped that one can recover Lorentz invariance on a finite lattice in an indirect way by studying the scaling behaviour of various observables. For studying non-perturbative phenomena, maintaining gauge invariance turned out to be more important than maintaining Lorentz invariance.

Since light front theories involving fermions and gauge bosons are inherently non-local in the longitudinal direction, it is natural to keep x^- continuous. Furthermore, it is known that conventional ultraviolet divergences come from small $x^\perp$ and hence it is natural to discretize transverse space. Then the question one faces is how to formulate a gauge invariant theory in the latticised transverse space with x^+ and x^- continuous. This can be achieved with the gauge choice $A^+ = 0$. Then A^- becomes a constrained variable which can be eliminated in favour of the dynamical gauge field variables. One can retain residual gauge invariance associated with x^- independent gauge transformations at each transverse location. The gauge choice $A^+ = 0$ is preferable also from the fact that the constrained fermion field, in this case, can be eliminated simply in favour of the dynamical (two component) fermion field and the dynamical gauge field. This gauge choice also provides straightforward parton interpretation for the bilocal current matrix elements that appear in deep inelastic scattering observables. Thus one arrives at the Hamiltonian for transverse lattice QCD [2].

2. Light front QCD on the transverse lattice

To set notation, we start from the QCD Lagrangian density in the continuum

$$\mathscr{L} = \bar{\psi}(i\gamma^\mu D_\mu - m_f)\psi + \frac{1}{2}\mathrm{Tr}(F_{\rho\sigma}F^{\rho\sigma}) \tag{1}$$

with $F^{\rho\sigma} = \partial^\rho A^\sigma - \partial^\sigma A^\rho + ig[A^\rho, A^\sigma],\ \ A^\rho = A^{\rho a}T^a$.

Longitudinal derivative $\partial^\pm = 2(\partial/\partial x^\pm)$, gamma matrices $\gamma^\pm = \gamma^0 \pm \gamma^3$, projection operators $\Lambda^\pm = \frac{1}{4}\gamma^\mp\gamma^\pm$ and fermion field components $\psi^\pm = \Lambda^\pm\psi$. Choose the gauge condition $A^+ = 0$. Then A^- becomes a constrained variable. Keeping the variable x^+, x^- continuous, discretize the transverse space, $\mathbf{x} = (x^1, x^2)$. Replace the continuous gauge fields $A_r(x^\perp, x^-, x^+)$, $r = 1, 2$ by the lattice link variable $U_r(\mathbf{x}, x^-, x^+)$ which connects $\mathbf{x}$ to $\mathbf{x} + a\hat{\mathbf{r}}$.

The constraint equations are $i\partial^+\psi^- = [i\alpha_r D_r + \gamma^0 m]\psi^+$ where D_r is an appropriately defined covariant lattice derivative and $(\partial^+)^2 A^{-\alpha} = 2g(\frac{1}{a^2}\left(J^{+\alpha}_{\mathrm{LINK}} - J^{+\alpha}_q\right))$. The dynamical field ψ^+ can essentially be represented by two components [3] such that

$$\psi^+(x^-, x^\perp) = \begin{bmatrix} \eta(x^-, x^\perp) \\ 0 \end{bmatrix}, \tag{2}$$

where η is a two-component field. The currents

$$J_q^{+\alpha}(\mathbf{x}) = \eta^\dagger(\mathbf{x}) T^\alpha \eta(\mathbf{x}) \ ,$$
$$J_{\rm LINK}^{+\alpha}(\mathbf{x}) = \sum_r \frac{1}{g^2} {\rm Tr}\{T^\alpha [U_r(\mathbf{x}) i \overset{\leftrightarrow}{\partial^+} U_r^\dagger(\mathbf{x}) + U_r^\dagger(\mathbf{x}-a\hat{\mathbf{r}}) i \overset{\leftrightarrow}{\partial^+} U_r(\mathbf{x}-a\hat{\mathbf{r}})]\}. \quad (3)$$

After eliminating the constraint fields we arrive at the transverse lattice Hamiltonian

$$P^- = P_1^- + P_2^-, \quad (4)$$

where P_1^- arises from the elimination of ψ^- (hence sensitive to how fermions are put on the transverse lattice) and P_2^- contains Wilson plaquette term and the terms arising from the elimination of A^-. Explicitly

$$\begin{aligned} P_2^- = \int {\rm d}x^- a^2 \sum_{\mathbf{x}} \Bigg\{ & -\frac{1}{g^2a^2} \\ & \times \sum_{r\neq s} \{{\rm Trace}[U_r(\mathbf{x}) U_s(\mathbf{x}+a\hat{\mathbf{r}}) U_{-r}(\mathbf{x}+a\hat{\mathbf{r}}+a\hat{\mathbf{s}}) U_{-s}(\mathbf{x}+a\hat{\mathbf{s}}) - 1]\} \\ & -\frac{g^2}{2a^2} J_{\rm LINK}^{+\alpha} \left(\frac{1}{\partial^+}\right)^2 J_{\rm LINK}^{+\alpha} - \frac{g^2}{2a^2} J_q^{+\alpha} \left(\frac{1}{\partial^+}\right)^2 J_q^{+\alpha} \\ & + \frac{g^2}{a^2} J_{\rm LINK}^{+\alpha} \left(\frac{1}{\partial^+}\right)^2 J_q^{+\alpha} \Bigg\} . \end{aligned} \quad (5)$$

The Fock expansion of η in the light front quantization with transverse directions discretized on a two-dimensional square lattice is given by

$$\begin{aligned} \eta(x^-, \mathbf{x}) = \sum_\lambda \chi_\lambda \int & \frac{{\rm d}k^+}{2(2\pi)\sqrt{k^+}} [b(k^+, \mathbf{x}, \lambda) {\rm e}^{-(i/2)k^+x^-} \\ & + d^\dagger(k^+, \mathbf{x}, -\lambda) {\rm e}^{(i/2)k^+x^-}], \end{aligned} \quad (6)$$

where χ_λ is the Pauli spinor and $\lambda = 1,2$ denotes two helicity states. $\mathbf{x}$ now denotes the transverse lattice points. The presence of constraint equation for fermion field allows different methods to put fermions on the transverse lattice. For details see ref. [4].

3. Fermions with forward and backward derivatives

With forward derivative for ψ^+ and backward derivative for ψ^-, we arrive at

$$\begin{aligned} P^-_{1\,fb} = \int {\rm d}x^- a^2 \sum_{\mathbf{x}} m^2 \Bigg\{ & \eta^\dagger \frac{1}{i\partial^+} \eta \\ & + m\eta^\dagger(\mathbf{x}) \sum_r \hat{\sigma}_r \frac{1}{a} \frac{1}{i\partial^+} [U_r(\mathbf{x}) \eta(\mathbf{x}+a\hat{\mathbf{r}}) - \eta(\mathbf{x})] \\ & + m \sum_r [\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) U_r^\dagger(\mathbf{x}) - \eta^\dagger(\mathbf{x})] \hat{\sigma}_r \frac{1}{a} \frac{1}{i\partial^+} \eta(\mathbf{x}) \end{aligned}$$

$$+\frac{1}{a^2}\sum_r [\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}})U_r^\dagger(\mathbf{x}) - \eta^\dagger(\mathbf{x})]$$
$$\left.\times\hat{\sigma}_r\frac{1}{i\partial^+}\hat{\sigma}_s[U_s(\mathbf{x})\eta(\mathbf{x}+a\hat{\mathbf{s}}) - \eta(\mathbf{x})]\right\}. \quad (7)$$

In the free limit the fermionic Hamiltonian becomes

$$P^-_{fb} = P^-_0 + P^-_{hf}, \quad (8)$$

where the helicity non-flip term

$$P^-_0 = \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} \left[m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+}\eta(\mathbf{x}) - \frac{1}{a^2}\sum_r \eta^\dagger(\mathbf{x}) \sum_r \frac{1}{i\partial^+}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}})]\right], \quad (9)$$

and the helicity flip term

$$p^-_{hf} = \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} \frac{1}{a^2}\eta^\dagger(\mathbf{x}) \sum_r (am\hat{\sigma}_r)\frac{1}{i\partial^+}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}})]. \quad (10)$$

$\hat{\sigma}_1 = \sigma_2$ and $\hat{\sigma}_2 = -\sigma_1$. Sign of the linear mass term changes if we switch the forward and backward derivatives.

3.1 *Numerical investigation*

Let us first consider the Hamiltonian without any helicity flip term. Considering the Fourier transform in transverse space

$$\eta(x^-,\mathbf{x}) = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(x^-)$$

we get the helicity non-flip term,

$$P^-_0 = \int \mathrm{d}x^- \int \frac{\mathrm{d}^2k}{(2\pi)^2}\phi^\dagger_{\mathbf{k}}(x^-)\frac{1}{i\partial^+}\phi_{\mathbf{k}}(x^-)\left[m^2 + \sum_r k_r^2\left(\frac{\sin\, k_r a/2}{k_r a/2}\right)^2\right] \quad (11)$$

which is free from fermion doubling.

For numerical investigation we use discretized light cone quantization (DLCQ) [5] for the longitudinal dimension ($-L \leq x^- \leq +L$) and implement antiperiodic boundary conditions to avoid zero modes. We employ two types of boundary conditions on the transverse lattice: (a) Fixed boundary condition (FBC): For each transverse direction, we choose $2n+1$ lattice points ranging from $-n$ to $+n$ where fermions are allowed to hop. To implement fixed boundary condition we add two more points at the two ends and demand that

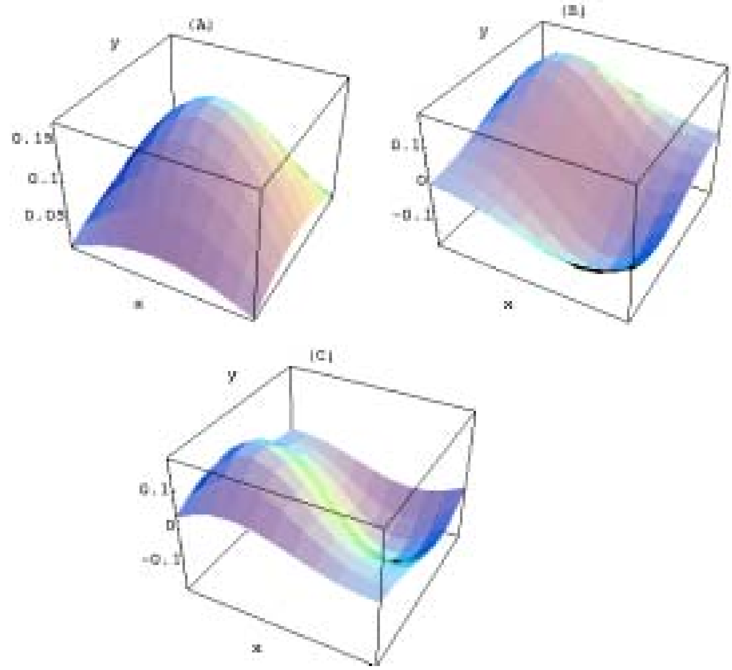

Figure 1. Lowest three wave functions for P_0^-.

the fermion remains fixed at these lattice points for which the allowed momentum components are $k_p = (\pi/(2n+2))p$ with $p = 1,2,3,...,2n+1$. (b) Periodic boundary condition (PBC): Again, for each transverse direction, we choose $2n+1$ lattice points as discussed above. We identify the $(2n+2)$th lattice point with the first lattice point so that we get $k_p = \pm(2\pi/(2n+1))p$, $p = 0,1,2,...,n$. At finite volume, with fixed boundary condition, we do get a nodeless wave function. All the excited states have nodes characteristic of sine waves as shown in figure 1. The lowest momentum allowed in PBC is zero and we get a flat lowest wave function.

Now, consider the full Hamiltonian (eq. (8)) including the helicity flip term. The eigenvalue equation now reads

$$M^2 = m^2 + \frac{4}{a^2}\sum_r \sin^2\frac{k_r a}{2} \pm \frac{4m}{a}\sqrt{\sum_r \sin^4\frac{k_r a}{2}} \tag{12}$$

and is *free from fermion doubling* for physical fermions, i.e., $m < 1/a$, the ultraviolet cutoff. With fixed boundary condition the lowest eigenstate has non-vanishing transverse momentum in finite volume. In the absence of helicity flip term, positive and negative helicity fermions are degenerate. The helicity flip term lifts the degeneracy in finite volume. The splitting vanishes in the infinite volume limit (figure 2). For the periodic boundary condition, the lowest state has exactly zero transverse momentum and we get two degenerate fermions corresponding to two helicity states for all n.

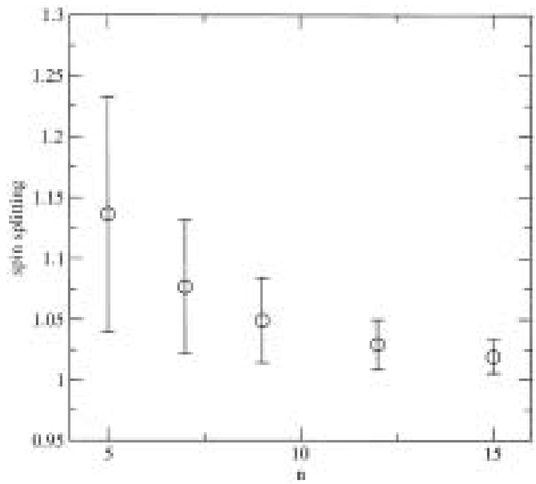

Figure 2. Spin splitting of the ground state caused by the spin (helicity) dependent interaction as a function of n (with FBC).

4. Fermions with symmetric derivatives

With symmetric derivative for both ψ^+ and ψ^-, we have

$$\begin{aligned}
P^-_{1\,sd} = & \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x}) \\
& - \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} \left\{ m \frac{1}{2a} \eta^\dagger(\mathbf{x}) \sum_r \hat{\sigma}_r \frac{1}{i\partial^+} \right. \\
& \times [U_r(\mathbf{x}) \eta(\mathbf{x}+a\hat{\mathbf{r}}) - U_{-r}(\mathbf{x}) \eta(\mathbf{x}-a\hat{\mathbf{r}})] \\
& - m \frac{1}{2a} \sum_r [\eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}}) \hat{\sigma}_r U_r(\mathbf{x}-a\hat{\mathbf{r}}) \\
& \left. - \eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) \hat{\sigma}_r U_{-r}(\mathbf{x}+a\hat{\mathbf{r}})] \frac{1}{i\partial^+} \eta(\mathbf{x}) \right\} \\
& - \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} \frac{1}{4a^2} \sum_r [\eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}}) U_r(\mathbf{x}-a\hat{\mathbf{r}}) \\
& - \eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) U_{-r}(\mathbf{x}+a\hat{\mathbf{r}})] \\
& \times \frac{1}{i\partial^+} [U_r(\mathbf{x}) \eta(\mathbf{x}+a\hat{\mathbf{r}}) - U_{-r}(\mathbf{x}) \eta(\mathbf{x}-a\hat{\mathbf{r}})]. \qquad (13)
\end{aligned}$$

In the free field limit the Hamiltonian (eq. (13)) becomes

$$\begin{aligned}
P^-_{sd}(\mathbf{x}) = & \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} \left\{ m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x}) \right. \\
& + \frac{1}{4a^2} \sum_r [\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}})]
\end{aligned}$$

$$\times \frac{1}{i\partial^+}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - \eta(\mathbf{x}-a\hat{\mathbf{r}})]\Bigg\}. \tag{14}$$

Observe that the above Hamiltonian involves only next to nearest neighbour interactions, so that odd and even lattice points are decoupled. As a result, we get four independent sublattices for two transverse directions each with effective lattice spacing $2a$. So, a momentum component in each sublattice is bounded by $\pi/2a$ in magnitude. Going through the Fourier transform in each sublattice of the transverse space, we arrive at the free particle dispersion relation for the light front energy in each sector

$$k_{\mathbf{k}}^{-} = \frac{1}{k^+}\left[m^2 + \frac{1}{a^2}\sum_r \sin^2\, k_r a\right]. \tag{15}$$

Because of the momentum bound of $\pi/2a$, doublers cannot arise from $ka = \pi$. However, because of the decoupling of odd and even lattices, one can get two zero transverse momentum fermions one each from the two sublattices. Thus, for two transverse dimensions, we get four zero transverse momentum fermions as follows: (1) even lattice points in x, even lattice points in y, (2) even lattice points in x, odd lattice points in y, (3) odd lattice points in x, even lattice points in y, and (4) odd lattice points in x, odd lattice points in y. Thus we expect a four-fold degeneracy of zero transverse momentum fermions.

4.1 *Numerical investigation*

For each transverse direction, we have $2n+1$ lattice points where the fermions are allowed to hop. To implement the fixed boundary condition, we need to consider $2n+5$ lattice points. For one sublattice the allowed values of k_p are $k_p = p\pi/(2n+4)a$, with $p = 1,2,3,\ldots,n+1$. For the other sublattice the allowed values of momenta $k_p = p\pi/(2n+2)a$ with $p = 1,2,3,\ldots,n$.

But, to implement PBC, we need to consider $2n+3$ lattice points in each direction. Thus the allowed momentum values for the two sublattices are $k_p = \pm 2\pi p/(2n+2)a$, $p = 0,1,2,\ldots,(n+1)/2$ and $k_p = \pm\pi p/na$, $p = 0,1,2,\ldots,(n-1)/2$. For details see [4]. In figure 3 we present the lowest four eigenvalues as a function of n for FBC. At finite volume, the four states do not appear exactly degenerate even though the even–odd and odd–even states are always degenerate because of the hypercubic (square) symmetry in the transverse plane. The four states become degenerate in the infinite volume limit (see figure 3). The lowest four eigenstates are nodeless representing particle states. All other states in the spectrum have one or more nodes. With periodic boundary condition, for any n we get four degenerate eigenvalues corresponding to zero transverse momentum fermions. Corresponding wave functions are flat in transverse coordinate space.

Though we do not have doublers for Hamiltonian with forward–backward derivatives, we face the doublers with symmetric derivatives. To remove the doublers we consider two approaches, namely, staggered fermion and Wilson fermion.

5. Staggered fermion on the light front transverse lattice

In the free theory, the two linear mass terms cancel each other. But since they are present in the interacting theory we keep them to investigate the staggered fermions. The Hamiltonian for free fermions is already spin diagonal to begin with except the linear mass term.

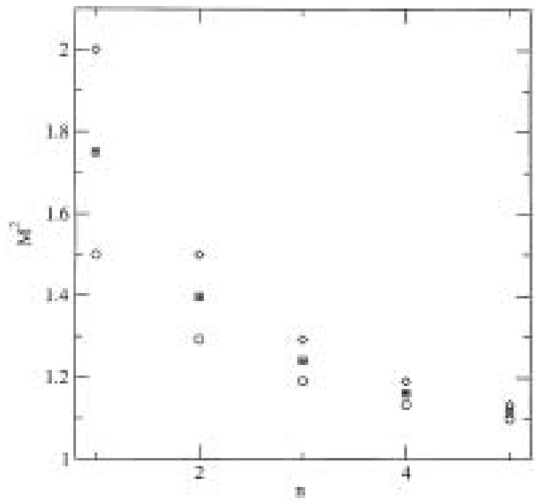

Figure 3. First four eigenvalues (with FBC) as a function of n.

In analogy with the Euclidean staggered formulation [6], define the spin diagonalization transformation

$$\eta(x_1,x_2) = (\hat{\sigma}^1)^{x_1}(\hat{\sigma}^2)^{x_2}\chi(x_1,x_2). \tag{16}$$

After spin diagonalization, the full Hamiltonian in the free field limit becomes

$$\begin{aligned} P^-_{sf} = \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} \Bigg\{ & m^2 \chi^\dagger(\mathbf{x}) \frac{1}{i\partial^+}\chi(\mathbf{x}) \\ & + \frac{1}{4a^2}\sum_r [\chi^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \chi^\dagger(\mathbf{x}-a\hat{\mathbf{r}})]\frac{1}{i\partial^+}[\chi(\mathbf{x}+a\hat{\mathbf{r}}) - \chi(\mathbf{x}-a\hat{\mathbf{r}})] \\ & - \frac{1}{2a} m \chi^\dagger(\mathbf{x}) \frac{1}{i\partial^+}\sum_r \phi(\mathbf{x},r)[\chi(\mathbf{x}+a\hat{\mathbf{r}}) - \chi(\mathbf{x}-a\hat{\mathbf{r}})] \\ & - \frac{1}{2a} m \sum_r [\chi^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \chi^\dagger(\mathbf{x}-a\hat{\mathbf{r}})]\phi(\mathbf{x},r)\frac{1}{i\partial^+}\chi(\mathbf{x}) \Bigg\}, \end{aligned} \tag{17}$$

where $\phi(\mathbf{x},r) = 1$ for $r = 1$ and $\phi(\mathbf{x},r) = (-1)^{x_1}$ for $r = 2$. Since the Hamiltonian is now spin diagonal we put single component field at each transverse site. Thus we have thinned the degrees of freedom by half. Interpret the remaining two species as two flavours. Apart from the linear mass term in eq. (17), all the other terms have the feature that fermion fields on the even and odd lattices do not mix. Let us denote (see figure 4) the even–even lattice points by 1, odd–odd lattice points by $1'$, odd–even lattice points by 2 and even–odd lattice points by $2'$, and the corresponding fields by η_1, $\eta_{1'}$ etc. with proper phase factors determined by eq. (16). Define the fields $u_1 = (1/\sqrt{2})(\eta_1 + \eta_{1'})$, $u_2 = (1/\sqrt{2})(\eta_2 + \eta_{2'})$, $\tilde{d}_1 = (1/\sqrt{2})(\eta_1 - \eta_{1'})$, $\tilde{d}_2 = (1/\sqrt{2})(\eta_2 - \eta_{2'})$ at the centre of the block and $d = \hat{\sigma}^1\tilde{d}$. So, we can now write the Hamiltonian (eq. (17)) as

$$P^-_{sf} = \int \mathrm{d}x^- a^2 \sum_x \Bigg\{ m^2 f^\dagger \frac{1}{i\partial^+} f$$

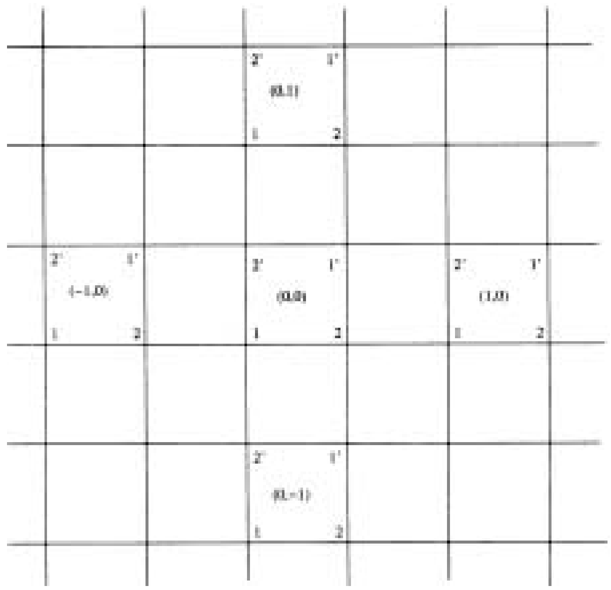

Figure 4. Staggered distribution.

$$
\begin{aligned}
&+\frac{1}{4}\left[\hat{\nabla}_r f^\dagger \frac{1}{i\partial^+}\hat{\nabla}_r f + a^2\hat{\nabla}_r^2 f^\dagger \frac{1}{i\partial^+}\hat{\nabla}_r^2 f\right.\\
&\left.+\frac{ia}{2}\left(\hat{\nabla}_r f^\dagger \frac{1}{i\partial^+}\sigma^r T^r \hat{\nabla}_r^2 f + \hat{\nabla}_r^2 f^\dagger \frac{1}{i\partial^+}\sigma^r T^r \hat{\nabla}_r f\right)\right]\\
&-\frac{1}{2}m\left(f^\dagger \frac{1}{i\partial^+}\hat{\sigma}^r \hat{\nabla}_r f + \frac{a}{2} f^\dagger \frac{1}{i\partial^+}\sigma^3 T^r \hat{\nabla}_r^2 f + \text{h.c.}\right)\Bigg\}, \qquad (18)
\end{aligned}
$$

where the flavour isospin doublet

$$f = \begin{bmatrix} u \\ d \end{bmatrix},$$

and $\hat{\nabla}_r$ and $\hat{\nabla}_r^2$ are respectively first- and second-order block derivatives, $T^1 = -i\sigma^2$, $T^2 = -i\sigma^1$ matrices in the flavour space. Note that, the irrelevant flavour mixing terms survive only in the interacting theory and get canceled among themselves in the free theory.

6. Wilson fermion

Since doublers in the light front transverse lattice arise from the decoupling of even and odd lattice sites, a term that will couple these sites will remove the zero momentum doublers. However, conventional doublers now may arise from the edges of the Brillouin zone. A second derivative term couples the even and odd lattice sites and also removes the conventional doublers. Thus, the term originally proposed by Wilson to remove the doublers arising from $ka = \pi$ in the conventional lattice theory will do the job [7]. To remove doublers, add an irrelevant term to the Lagrangian density

$$\delta\mathscr{L}(\mathbf{x}) = \frac{\kappa}{a}\sum_r \bar{\psi}(\mathbf{x})[U_r(\mathbf{x})\psi + a\hat{\mathbf{r}}) - 2\psi(\mathbf{x}) + U_{-r}(\mathbf{x})\psi(\mathbf{x}-a\hat{\mathbf{r}})], \tag{19}$$

where κ is the Wilson parameter. In the free limit the resulting Hamiltonian goes over to

$$\begin{aligned} P_w^- = \int \mathrm{d}x^- a^2 \sum_{\mathbf{x}} \Bigg[& \mu^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+}\eta(\mathbf{x}) \\ & + \frac{1}{2a}\sum_r [\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}})]\frac{1}{i\partial^+}\frac{1}{2a}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - \eta(\mathbf{x}-a\hat{\mathbf{r}})] \\ & + \frac{\kappa^2}{a^2}\sum_r [\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta^\dagger(\mathbf{x}) + \eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}})] \\ & \times \frac{1}{i\partial^+}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}})] \\ & - 2\frac{\mu\kappa}{a}\sum_r \eta^\dagger(\mathbf{x})\frac{1}{i\partial^+}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}})]\Bigg], \end{aligned} \tag{20}$$

where $\mu^2 = (m + 4(\kappa/a))^2$. Using the Fourier transform in the transverse space, we get

$$\begin{aligned} & \int \mathrm{d}x^- \int \frac{\mathrm{d}^2k}{(2\pi)^2)}\phi_{\mathbf{k}}^\dagger(x^-)\frac{1}{i\partial^+}\phi_{\mathbf{k}}(x^-)\left[\mu^2 + \sum_{\hat{\mathbf{r}}} k_r^2 \left(\frac{\sin k_r a}{k_r a}\right)^2\right. \\ & \left. + 2a\mu\kappa\sum_{\hat{\mathbf{r}}} k_r^2\left(\frac{\sin k_r a/2}{k_r a/2}\right)^2 + a^2\kappa^2\sum_{\hat{\mathbf{r}}} k_r^4 \left(\frac{\sin k_r a/2}{k_r a/2}\right)^4\right]. \end{aligned}$$

Note that, as anticipated, Wilson term removes the doublers.

Since the Wilson term connects even and odd lattices, the extra fermions that appear at zero transverse momentum are removed. For large n, we get the expected spectra but, numerical results suggest that the finite volume effect is larger for small κ.

7. Doubling and symmetry on the light front transverse lattice

In lattice gauge theory in the Euclidean or equal time formalism, there has to be explicit chiral symmetry breaking in the kinetic part of the action or Hamiltonian. Translated to the light front formalism, this would then require helicity flip in the kinetic part since chirality *is* helicity even for a massive fermion in front form. A careful observation of all the above methods that get rid of fermion doublers on the light front transverse lattice reveals that this is indeed true. In particular, we draw attention to the even–odd helicity flip transformation

$$\eta(x_1, x_2) \to (\hat{\sigma}_1)^{x_1}(\hat{\sigma}_2)^{x_2}\eta(x_1, x_2). \tag{21}$$

Hamiltonian invariant under this transformation shows fermion doubling and the Hamiltonian which is not invariant under the above transformation is free of doublers.

8. Summary

After a brief introduction to the salient features of light front dynamics (non-relativistic structure in the transverse plane, kinematic boosts, triviality of the vacuum, etc.) motivations for studying transverse lattice QCD were given. Presence of constraint equation on the light front allows different methods to put fermions on a transverse lattice. In this brief presentation we described two methods using (a) forward and backward derivatives in which case there are no doublers and (b) symmetric derivatives which lead to doublers and two approaches to remove them. We have summarized our numerical investigation of the two methods. We have identified an even–odd helicity flip symmetry on the light front transverse lattice which is relevant for fermion doubling. For details see ref. [4].

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