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Half-monopoles in the Yang–Mills theory

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Abstract. Using a gauge-invariant characterization of monopoles defined via their centres, we investigate the generic topological field pattern for the three-dimensional Yang–Mills theory. This leads to field patterns with one-half winding number. After presenting the main features through the simpler case of half-vortices, we consider half-monopoles in detail.

Keywords. Monopole; Poincaré–Hopf index; one-half winding number.

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1. Introduction

Confinement in the Georgi–Glashow model has been demonstrated by dilute monopole gas calculation [1]. A monopole gas in the Yang–Mills theory is, however, non-dilute, with overlapping boundaries of the monopoles. So we have to specify the monopole in terms of an interior point to which the topology of the configuration can be traced. We refer to this interior point as the 'centre.'

Consider the 3×3 real symmetric matrix

$$S_{ij}(x) = B_i^a(x) B_j^a(x), \tag{1}$$

where $B_i^a = \varepsilon_{ijk}(\partial_j A_k^a - \frac{1}{2}\varepsilon^{abc}A_j^b A_k^c)$ is the $SO(3)$ magnetic field. The eigenvalue equation is

$$S_{ij}(x)\zeta_j^A(x) = \lambda^A(x)\zeta_i^A(x), \quad A = 1,2,3. \tag{2}$$

(There is no summation over A.) One or more of the eigenvectors $\zeta_i^A(x)$ will carry the topological information of the configuration [2,3]. Now, if an eigenvector has a non-zero winding number on being taken around a closed path, by shrinking the path we see that the eigenvector must be indeterminate in direction, and therefore singular, at some point: this is the 'centre.' Clearly, the eigenvalue must become degenerate at such a point of singularity [4]. A non-zero winding number in two or three dimensions must be associated with a point of double or triple degeneracy respectively, as we will see in our examples.

Consider, for example, the 't Hooft–Polyakov monopole [5,6]. In this case,

$$S_{ij} = \alpha(r^2)\delta_{ij} + \beta(r^2)x_i x_j, \tag{3}$$

where α and β are functions of distance r from the origin only. In particular, the Prasad–Sommerfield solution [7] makes it explicit that $\alpha(0) \neq 0$ and finite. The topological information is carried by the eigenvector x_i (the radial vector), with unit winding number. This has indeterminate direction at the origin $r = 0$, a point of triple degeneracy.

It is important to note that the entries of the matrix S are smooth in the coordinates x_i at the origin. The singularities are in the eigenvectors only.

As a consequence of rotational symmetry, the 't Hooft–Polyakov monopole has the exceptional feature that the entries of the matrix S are quadratic in the coordinates. Our aim is to analyse the generic case, i.e., we consider $S_{ij}(x)$ with linear terms in the Taylor expansion about the origin. This will lead us to the novel feature of one-half winding number.

We may appropriately subtract a multiple of the identity matrix from S_{ij}, and scale by an overall factor, since these changes do not affect the eigenvectors. The matrix after these changes will be referred to as T_{ij}. We will analyse the case $T_{ij} = v_i x_j + v_j x_i$, where v_i is constant, as opposed to the 't Hooft–Polyakov monopole case $T_{ij} = x_i x_j$.

2. Half-vortices

A half-vortex is a winding number-half configuration in two dimensions. To demonstrate it, we consider a 2×2 real symmetric matrix field $T_{ij}(x,y)$. The paradigm is provided by the matrix

$$T = \begin{pmatrix} 2x & y \\ y & 0 \end{pmatrix} . \tag{4}$$

The eigenvalues are $\lambda_{\pm} = x \pm r$, where $r = \sqrt{x^2 + y^2}$. Let us denote the corresponding normalised eigenfunctions by $\zeta_i^{\pm}$. One finds that ζ_i^{+} has the simple form

$$\begin{pmatrix} \zeta_1^{+} \\ \zeta_2^{+} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \tag{5}$$

in the polar coordinates. Here $\theta = \tan^{-1}(y/x)$.

Due to the occurrence of half the polar angle in (5), the eigenvector changes sign when taken round the origin once. Stated in another way, the phase of the complex vector $\zeta_1^{+} + i\zeta_2^{+} = \exp(i\theta/2)$ changes by π when we go around the origin once. The winding number is thus one-half. It can be checked that such a phase change takes place for the other eigenvector $\zeta_i^{-}(x)$ too: $\zeta_1^{-} + i\zeta_2^{-} = \exp(i(\theta/2 + \pi/2))$. The phenomenon is shown in figure 1.

In fact, even the most general 2×2 real symmetric matrix T_{ij} linear in the coordinates can be reduced to the paradigm considered in (4), essentially by a suitable linear transformation of the coordinates [8]. It is also possible to consider cases such that two winding number-half configurations give a winding number-one configuration, or a winding number-zero configuration, at large distances [8].

To bring out the relevance for the Yang–Mills theory, we regard the matrix in (4) as a block of the 3×3 matrix

$$T = \begin{pmatrix} 2x & y & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{6}$$

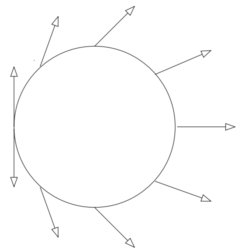


Figure 1. A winding number-half configuration in two dimensions: $\zeta_i^{\pm}$ changes sign when taken around any closed path enclosing the centre.

The eigenvector field configuration is then a vortex with one-half winding number centred on the z-axis and extending indefinitely along it.

It can be shown in general that whenever the 3×3 matrix B_i^a is invertible and smooth, there exists a smooth A_i^a corresponding to it [9]. So for the case here, it is possible to construct A_i^a as a Taylor series expansion about the origin. Such a series will be presented for a different example below.

3. Half-monopoles

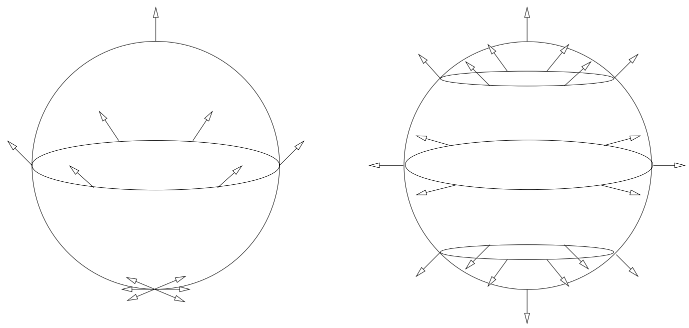
A half-monopole is a winding number-half configuration in three dimensions. In this case, the paradigm is provided by the 3×3 real symmetric matrix

$$T = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & 2z \end{pmatrix}. \tag{7}$$

Here the eigenvalues are $\lambda_{\pm} = r(\cos\theta \pm 1)$ and $\lambda_0 = 0$. In the spherical polar coordinates, the corresponding eigenvectors are

$$\zeta^+ = \begin{pmatrix} \sin(\theta/2)\ \cos\phi \\ \sin(\theta/2)\ \sin\phi \\ \cos(\theta/2) \end{pmatrix}, \quad \zeta^- = \begin{pmatrix} \cos(\theta/2)\ \cos\phi \\ \cos(\theta/2)\ \sin\phi \\ -\sin(\theta/2) \end{pmatrix}, \quad \zeta^0 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{8}$$

Comparing $\zeta_i^{\pm}$ with the radial unit vector $\hat{x}_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we notice that essentially the angle θ is replaced by $\theta/2$. This leads to one-half winding number. The phenomenon is illustrated for ζ_i^+ in figure 2. Note that the vector field ζ_i^+ is singular (indeterminate in direction) all along the negative z-axis. This is possible because T_{ij} has a double degeneracy (namely, $\lambda_+ = \lambda_0$) there. We thus have a vortex (of winding number one) along the negative z-axis terminating at the origin and giving rise to a monopole. Because the vector field is not continuous at the south pole of the sphere, one-half winding number is possible. (If the vector field were smooth everywhere on the sphere, the winding number would have been only integral.)



(A) (B)

Figure 2. (**A**) A winding number-half configuration in three dimensions. There is a vortex of winding number-one along the negative z-axis terminating at the centre. (**B**) A winding number-one configuration in three dimensions. The upper half of this configuration is mapped onto the entire sphere in **A** to give one-half winding number.

The eigenvector ζ_i^- corresponds to a vortex of unit winding number along the positive z-axis, terminating at the origin. Finally ζ_i^0 is a vortex of winding number-one extending indefinitely along the z-direction.

The Poincaré–Hopf index of the vector field ζ_i^A is given by $M = (1/4\pi)\oint_S \mathrm{d}S^i k_i^A$, where the integration is over a surface S enclosing the centre and k_i^A is the Poincaré–Hopf current [3,6,10]

$$k_i^A = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\zeta_l^A\partial_j\zeta_m^A\partial_k\zeta_n^A \quad (\text{no sum over } A). \tag{9}$$

For ζ_i^+, we have

$$k_i^+ = \hat{x}_i\frac{1}{4r^2}\sec\frac{\theta}{2}. \tag{10}$$

On integration over S, we indeed get the index to be $1/2$.

The Poincaré–Hopf current k_i^A can be expressed as the curl of an Abelian vector potential ω_i^A [3]:

$$k_i^A = \varepsilon_{ijk}\partial_j\omega_k^A - \text{Dirac string contributions}, \tag{11}$$

where

$$\omega_i^A = \frac{1}{2}\varepsilon^{ABC}\zeta_j^B\partial_i\zeta_j^C. \tag{12}$$

Here the indices A, B and C, having the values 1, 2 and 3, label the three eigenvectors. The Abelian vector potential corresponding to ζ_i^+ is

$$w_i^+ = -\hat{\phi}_i\frac{1}{2r}\mathrm{cosec}\frac{\theta}{2}. \tag{13}$$

This potential has the (unphysical) Dirac string along the positive z-axis.

Similarly, we get the Poincaré–Hopf index for ζ_i^- as $1/2$, and the index for ζ_i^0 as zero (the three-dimensional winding number is zero).

For presenting the Taylor series expansion of A_i^a about the origin, we use the symmetric gauge $(B)_{ia} = (B)_{ai}$ [3], where the matrix $(B)_{ia} = B_i^a$. We consider T_{ij} given in (7), with $2z$ changed to $-2z$ in the last element of the matrix. For this case, $\partial_i B_i^a = 0$ holds, making the solution easier to find. (This modified case also has winding number-half configurations; see [8].) Then the solution for the gauge field is [8]

$$A = \frac{1}{2} \begin{pmatrix} -xy/3 & z - y^2/3 & -y + yz \\ -z + x^2/3 & xy/3 & x - xz \\ y & -x & 0 \end{pmatrix} + \cdots, \tag{14}$$

where $(A)_{ia} = A_i^a$.

The argument for the finiteness of the energy of the half-monopole is to be found in [8].

4. Conclusion

There are smooth Yang–Mills potentials which give rise to monopole and vortex configurations of one-half winding number. They are the generic topological configurations of the Yang–Mills theory.

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