Topological field patterns in the Yang–Mills theory

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Abstract. We present a formalism where the topological configurations of pure Yang–Mills theory are characterised using gauge fields alone. Here, we obtain an expression for the charges of these topological $SO(3)/B_4^3/B_5$ gauge field configurations in terms of the Abelian vector potentials. In this formalism we analyse the 't Hooft–Polyakov monopole solution.

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1. Introduction

Study of monopoles and other topological configurations is essential for understanding the non-perturbative sector of QCD, in particular for addressing the problem of quark confinement. In this context, one would need a formalism where these topological configurations can be classified and their topological aspects discussed using the gauge fields alone. We have developed such a framework and analysed the 't Hooft–Polyakov monopole solution in this framework [1]. In this framework, the behaviour of fields near the origin/centre of the monopole is sufficient for discussing topological aspects of the configurations.

In the usual discussion of the ‘t Hooft–Polyakov monopole [2], the Poincaré–Hopf current is defined using the (normalised) Higgs equation as [3]

$$k_i = \frac{1}{2} \varepsilon_{ijk} e_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c$$

(1)

and thus it is essentially the Higgs field which gives the topological properties of the monopole. Such a characterisation is not suitable for studying the topological configurations of pure gauge theories.

2. Characterisation of topological configurations: A new framework

A novel way for characterising the topological features of monopoles using gauge fields alone has been proposed in [4]. Here we further develop along this line.

In this formalism the topological features of the $SO(3)$ gauge field configurations are related to the singularities of vector fields constructed using the Yang–Mills gauge potential.
Here, we start with a $3 \times 3$ matrix constructed using the non-Abelian magnetic field $B_a^i$. This real symmetric matrix is defined as

$$S_{ij} = B_a^i B_a^j$$

and it is gauge invariant. Using the normalised eigenvectors and eigenvalues of this matrix, one can have a complete gauge invariant description of the Yang–Mills potential. For this we consider the eigenvalue equation,

$$S_{ij}(x) \zeta^A_i(x) = \lambda^A(x) \zeta^A_j(x), \quad A = 1, 2, 3,$$

where $A$ labels the eigenvalues and corresponding eigenvectors and there is no summation over $A$. The Yang–Mills potential $A_a^i$ which has six gauge invariant degrees of freedom can be completely described using $\lambda^A(x)$ and $\zeta^A_i(x)$ which together have six degrees of freedom at each $x$.

The three normalised eigenvectors $\zeta^A_i(x)$ provide us an orthonormal frame at each point $x$. The topological features of the configurations of the Yang–Mills theory can be related to the singularities of these vector fields. (At the centre of the monopole, all the eigenvectors are degenerate.) Equivalently, by an appropriate local gauge transformation one can first make the $3 \times 3$ matrix $B_a^i$ symmetric and then its eigenvectors can be used as $\zeta^A_i(x)$. A third way to construct the eigenfunctions $\zeta^A_i(x)$ is to use the local gauge transformations to make the columns of $B_a^i$ mutually orthogonal and these columns, after normalisation provide $\zeta^A_i(x)$ [1,4,5]. We show that the topological aspects of the configurations can be related to the singularities of these vector fields $\zeta^A_i(x)$.

Instead of starting with the gauge invariant matrix $S_{ij}$ we could start with a gauge co-covariant matrix

$$S^{ab}(x) = B_a^i(x) B_b^i(x),$$

and use the associated normalised eigenfunctions $\xi^A_a$ and corresponding eigenvalues to describe the topological features of the configurations. The eigenvalue equation for $S^{ab}$ is given by

$$S^{ab}(x) \xi^A_b(x) = \lambda^A(x) \xi^A_a(x), \quad A = 1, 2, 3.$$ 

In the above equation there is no summation over $A$. It is easy to see that the eigenvalues are indeed the same for both the tensors and the eigenfunctions $\zeta^A_i$ are the same as $B_a^i \xi^A_a$ up to a normalisation. (For Yang–Mills field configurations, generically the $3 \times 3$ matrix $B_a^i(x)$ is invertible [6].)

The isotriplet scalar $\xi^A_a(x)$ which is constructed from Yang–Mills field strength can be used to describe the topological configurations. Using these eigenvectors $\xi^A_a(x)$ we define the winding number (or topological charge) of any topological configuration in the Yang–Mills theory as

$$Q^A = \frac{1}{8\pi} \int d\sigma^i \epsilon_{ijk} e^{abc} \xi^A_a \partial_j \xi^A_b \partial_k \xi^A_c$$

$$= \frac{1}{4\pi} \int d\sigma^i k^A_i.$$ 

Here there is no summation over $A$ and $k^A_i$ is the Poincaré–Hopf current.
Since the $3 \times 3$ orthogonal matrix $\xi^A(x)$ can be treated as a local gauge transformation, we can construct a pure gauge potential $\omega^A_i$ using them. Thus we have

$$\omega^A_i = \frac{1}{2} \epsilon^{ABC} \xi^B_a \partial^i \xi^C_a,$$

satisfying

$$\epsilon_{ijk} \left( \partial_j \omega^A_k - \frac{1}{2} \epsilon^{ABC} \omega^B_j \omega^C_k \right) = 0.$$  

In terms of $\omega^A_i$, the Poincaré–Hopf current $k^A_i$ can be rewritten as

$$k^A_i = \frac{1}{2} \epsilon_{ijk} \epsilon^{ABC} \xi^B_{ab} \xi^C_{ac} \partial_j \xi^A_a \partial_k \xi^A_c$$

$$= \frac{1}{2} \epsilon_{ijk} \epsilon^{ABC} \omega^B_j \omega^C_k.$$  

Here we have used the facts $\det \xi^A_a = 1$ and $\epsilon^{ABC} \epsilon^{ABD} \epsilon^{ACE} = \epsilon^{ADE}$ (no sum over $A$). From eqs (9) and (8) it is easy to see that the Poincaré–Hopf current $k^A_i$ can be expressed as a curl of $\omega^A_i$ (it satisfies $\partial_i k^A_i = 0$, everywhere except at the points where $\xi^A_a(x)$ is singular), i.e.,

$$k^A_i = \epsilon_{ijk} \partial_j \omega^A_k.$$  

Here we note that the Poincaré–Hopf current which defines the topological charge is expressed as an Abelian curl of a vector potential ($\omega^A_i$).

We can also construct Abelian vector potentials from the Yang–Mills potential by a gauge transformation using $\xi^A_i$. Thus we have

$$a^A_i = \frac{\xi^A_a A^a_i}{\xi^A_a} + \frac{1}{2} \epsilon^{ABC} \xi^B_a \partial_i \xi^C_a$$

$$= A^A_i + \omega^A_i.$$  

$\tilde{A}^A_i$ in the above equation is similar to the Abelian vector potential of ’t Hooft [2]. But unlike the ’t Hooft potential, $\tilde{A}^A_i$ are constructed using the three eigenvectors $\xi^A_a$. Gauge invariant vector potentials $a^A_i$ can be regarded as three Abelian vector potentials constructed from the non-Abelian Yang–Mills potential. These Abelian potentials ($a^A_i$) carry all the topological information about the Yang–Mills theory and thus in terms of $a^A_i$ we have an ‘Abelianised’ description of the Yang–Mills potential.

We have seen that in this new framework the Poincaré–Hopf current and hence the topological charge (winding number) associated with configurations of Yang–Mills theory can be defined using the Abelian vector fields $\xi^A_i$. These vector fields are the eigenfunctions of $S^{ab}$ which is constructed using the Yang–Mills potential alone. We have also seen that the Poincaré–Hopf current for configurations of non-Abelian fields can be expressed as the Abelian curl of a vector potential $\omega^A_i$ and this $\omega^A_i$ is also constructed using gauge fields alone. By a gauge transformation of Yang–Mills potential using the vector fields $\xi^A_a$, we have obtained three Abelian vector potentials and they contain all the topological information of the configurations of Yang–Mills fields.
3. ’t Hooft–Polyakov monopole

We now illustrate this formalism by applying it to analyse the ’t Hooft–Polyakov monopole solution. For the case of ’t Hooft–Polyakov monopole, generic form of $S_{ab}$ is

$$S_{ab} = \alpha(r)\delta_{ab} + \beta(r)x^a x^b. \quad (12)$$

Here $\alpha$ and $\beta$ depend only on radial distance $r$. The eigenvectors for this matrix are

$$\xi^1_a = \frac{r^a}{r}, \quad \xi^2_a = \hat{\theta}^a, \quad \xi^3_a = \hat{\phi}^a. \quad (13)$$

It can be readily seen that at $r = 0$ all the eigenvectors are degenerate and this point is identified as the centre (or origin) of the monopole. At all other points $\xi^2_a$ and $\xi^3_a$ are degenerate. (This double degeneracy is because of the spherical symmetry of ’t Hooft–Polyakov solution.)

Using eq. (13) in eq. (6), we find the Poincaré–Hopf current $k^i_1$ corresponding to the eigenfunctions of the ’t Hooft–Polyakov monopole as

$$k^1_i = \hat{x}_i \frac{1}{r^2},$$

$$k^2_i = \hat{x}_i \frac{\cot \theta}{r^2},$$

$$k^3_i = 0, \quad (14)$$

where $\hat{x}_i$ is the $i$th component of unit radial vector. Here $k^1_i$ precisely corresponds to the magnetic field of a Dirac monopole. The corresponding magnetic charge is

$$\frac{1}{4\pi} \int ds^i k^1_i = 1. \quad (15)$$

Though $k^2_i \neq 0$, we find that the corresponding magnetic charge is zero. This current $k^2_i$ corresponds to a radial flux from the region $z < 0$ to the region $z > 0$. One can also calculate the pure gauge $\omega^a_i$ for the ’t Hooft–Polyakov monopole using eqs (7) and (13). We find them to be

$$\omega^1_i = -\hat{\phi} \frac{\cot \theta}{r}, \quad (16)$$

$$\omega^2_i = \hat{\theta} \frac{1}{r}, \quad \omega^3_i = -\hat{\phi} \frac{1}{r}. \quad (17)$$

It is easy to see that $\omega^1_i$ is the same as the average of two Dirac monopole potentials (after suitable scaling), one with Dirac string along the positive $z$-axis and the other with Dirac string along the negative $z$-axis. $\omega^2_i$ gives magnetic flux without any monopole and $\omega^3_i$ do not give magnetic flux.

Next we construct the Abelian vector potentials $a^A_i$ for ’t Hooft–Polyakov monopole. Using eq. (13) in eq. (11), we get
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\begin{align}
  a_1^i &= -\frac{\phi_i}{r} \cot \theta i, \\
  a_2^i &= \frac{K(r)}{r} i, \\
  a_3^i &= -\frac{\theta_i}{r} K(r) i. \tag{18}
\end{align}

In $a_1^i$, we recover the Dirac monopole potential while $a_2^i$ and $a_3^i$ vanish asymptotically. Divergences of $a_i^i$ here are such that the energy of monopole is finite.

4. Conclusion

We have obtained a gauge invariant/covariant description of monopoles using the gauge fields alone. In this framework, winding number (monopole charge) is given by the curl of an Abelian vector potential. By a local gauge transformation gauge invariant, Abelian vector potentials are obtained and they characterise all the topological features. Also using this framework, we have analysed the 't Hooft–Polyakov monopole solution. Using this formalism, generic topological configurations can be constructed and studied [5].

References

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