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# Second invariant for two-dimensional classical super systems

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**Abstract.** Construction of superpotentials for two-dimensional classical super systems (for $N \geq 2$) is carried out. Some interesting potentials have been studied in their super form and also their integrability.

**Keywords.** Second constant of motion; super integrable system; Dirac's constraints.

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## 1. Introduction

There have been considerable efforts to identify integrable dynamical systems having both finite and infinite degrees of freedom by employing different techniques [1–11]. It is known that the invariance of evolution of equation under Lie transformations and generalisation lead to the integrability of the system. The Lie symmetry approach has been extensively used for the case of non-linear partial differential equation by various authors [3,4–7,12,13]. In classical dynamical systems, one expects the existence of additional invariants and subsequently tries to construct additional invariants if possible. Several authors have studied the problem of constructing an invariant (also called second constant of motion since first one is the total energy) for a two-dimensional system. With the same spirit, in this paper we make use of super time variable which involves the usual time $t$ and Grassmannian variables $\theta_i$ and $\bar{\theta}_i$. Corresponding super position variable is given by

$$Z_i = Z_i(t, \theta\bar{\theta}) = q_i(t) + i\bar{\theta}_i\psi_i(t) + i\theta_i\bar{\psi}_i(t) + \theta_i\bar{\theta}_i A_i(t), \quad i = 1, 2. \tag{1}$$

Equation (1) is associated to the super symmetry [14–17] transformations which emerges as one of the powerful creations in theoretical physics, where bosons turn into fermions and vice versa – a very interesting property for the description of fundamental interaction in particle physics. Though there have been several attempts to test the integrability of two-dimensional classical systems, not much effort has been made to test the integrability for $N \geq 2$ super space in two dimensions. The super space formulation for $N = 1$ and $N = 2$ pseudo-mechanical systems has been studied in the context of dynamical invariance (super) symmetry associated with Pauli systems in the presence of magnetic mono-poles in one spatial dimension [3,18].

We obtain a time independent Euler–Lagrangian equation for super space Lagrangian on applying the Dirac's constraints [18]. Here we address ourselves to the construction of second-order invariant involving the position, velocity and Grassmannian degree of freedom [3,19] for a couple of systems. The recipe could be prescribed to construct any integrable system in two spatial dimensions and could be generalised to classical systems with an arbitrary space and Grassmann finite degree of freedom. The method which we adopt [1–3,20] here is the one used earlier for the study of both time dependent and time independent classical systems in two dimensions. In §2, we derive the equation of motion and the general structure of the super potential of a dynamical system which involves second class Dirac's constraints. We show how this constraints algebra help us to reduce the system to a time independent one. In §3, we obtain a set of coupled linear equations using graded bracket relation for the canonical conjugate variable where the existence of the second-order invariant for the super dynamical system under consideration is being assumed. A consistent solution of these equations yields the systems which are integrable. Alternatively, knowing the coefficients in the polynomial structure of the invariants, one can obtain a class of super potentials which furnish the integrable systems. In §4, we discuss some of the interesting potentials in their super forms and their integrability. Finally in §5, we describe the summary and conclusions, and the potentials under study in its super form.

## 2. Equations of motion and the constrained Hamiltonian systems

Let us consider the action of a system which is described by the integral equation

$$T = \int \mathrm{d}t\, L, \tag{2}$$

where

$$L = \int \mathrm{d}\theta_i \mathrm{d}\bar{\theta}_i \mathscr{L}(Z_i, D_i Z_i, \bar{D}_i Z_i), \tag{3}$$

where $\mathscr{L}$ is the super Lagrangian density, $Z_i$ are the superposition variables given in eq. (1).

Let us define super derivatives $D_i$ and $\bar{D}_i$ as

$$D_i = \partial_{\bar{\theta}_i} - i\theta_i \partial_t, \tag{4}$$

and

$$\bar{D}_i = \partial_{\theta_i} - i\bar{\theta}_i \partial_t, \tag{5}$$

which satisfy the relation

$$[D_i, \bar{D}_i]_+ = -2i\delta_{ij}\partial_t \tag{6}$$

and

$$\mathscr{L}(Z_i, D_iZ_i, \bar{D}_iZ_i) = \frac{1}{2}\bar{D}_iZ_iD_iZ_i - W(Z_i), \tag{7}$$

where $W(Z_i)$ is the superpotential which can be written by making Taylor series expansion as

$$W(Z_1, Z_2) = W(q_1, q_2) + i\theta_i(W'_{q_i}\bar{\psi}_i) + i\bar{\theta}_i(W'_{q_i}\psi_i) + \theta_i\bar{\theta}_i(W'_{q_i}A_i + W''_{q_i}\bar{\psi}_i\psi_i), \tag{8}$$

where $W'_{q_i} = \partial W(q_1, q_2)/\partial q_i$ etc. The corresponding Euler–Lagrange equation becomes

$$\frac{\partial \mathscr{L}}{\partial Z_i} - D_i\frac{\partial \mathscr{L}}{\partial(\bar{D}_iZ_i)} - \bar{D}_i\frac{\partial \mathscr{L}}{\partial(D_iZ_i)} = 0, \tag{9}$$

which reduces to the following equation after using eq. (7):

$$\frac{1}{2}[D_i, \bar{D}_i]_- Z_i = \frac{\partial W}{\partial Z_i}. \tag{10}$$

Using eqs (1), (7), (8) and (10) we obtain

$$A_i = W'_{q_i}(q_1, q_2), \tag{11}$$

$$\dot{\bar{\psi}}_i = iW''_{q_i}\bar{\psi}_i, \tag{12}$$

$$\dot{\psi}_i = -iW''_{q_i}\psi_i, \tag{13}$$

and

$$\begin{aligned}\ddot{q}_i &= -W''_{q_i}A_i - W'''_{q_i}\bar{\psi}_i\psi_i = -W''_{q_i}W'_{q_i} - W'''_{q_i}\bar{\psi}_i\psi_i \\ &= -\frac{1}{2}(W^{2'}_{q_i})' - (W''_{q_i}\bar{\psi}_i\psi_i)' = -\frac{\partial V}{\partial q_i}(W'_{q_i}, W''_{q_i}, \psi_i, \bar{\psi}_i).\end{aligned} \tag{14}$$

Hence the potential $V$ becomes

$$V = \frac{1}{2}W'^2_{q_i} + W''_{q_i}\bar{\psi}_i\psi_i - \frac{1}{2}(\bar{\psi}_i\dot{\psi}_i - \dot{\bar{\psi}}_i\ \psi_i). \tag{15}$$

Alternatively, using eqs (7), (8) and (10) in eqs (2) and (3) we get the Lagrangian as

$$L = \frac{1}{2}\dot{q}_i^2 + \frac{1}{2}A_i^2 - W'_{q_i}A_i + \frac{1}{2}(\bar{\psi}_i\dot{\psi}_i - \dot{\bar{\psi}}_i\ \psi_i) - W''_{q_i}\bar{\psi}_i\psi_i. \tag{16}$$

Here this Lagrangian is constrained which satisfies the following primary constraints

$$\phi_1 = p_{\psi_i} + \frac{1}{2}\bar{\psi}_i, \tag{17}$$

$$\phi_2 = p_{\bar{\psi}_i} + \frac{1}{2}\psi_i, \tag{18}$$

$$\phi_3 = p_{A_i} \tag{19}$$

and also the secondary one is

$$\phi_4 = A_i - W'_{q_i}, \tag{20}$$

where $p_{\psi_i}, p_{\bar{\psi}_i}, p_{A_i}$ are canonically conjugate to $\psi_i, \bar{\psi}_i$ and $A_i$, respectively. The graded Poisson brackets of these constraints are being easily calculated from

$$[F,G] = \sum_i \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial y_i} - (-1)^{|F||G|}\frac{\partial F}{\partial y_i}\frac{\partial G}{\partial x_i}, \tag{21}$$

where $x_i, y_i$ are canonically conjugate and $|F|, |G|$ are degrees associated with the dynamical variables $F$ and $G$, respectively. Here the constraints are second class and we find after replacing the original Poisson brackets by the Dirac brackets [18] defined by

$$[\psi_i, \bar{\psi}_j]^*_+ = \delta_{ij}, \quad [q_i, p_i]^* = \delta_{ij}, \quad [q_i, q_j]^* = 0, \quad [p_i, p_j]^* = 0, \tag{22}$$

using the constraints, we obtain the Hamiltonian as

$$\mathscr{H} = \frac{1}{2}p_i^2 + \frac{1}{2}(W'_{q_i})^2 + W''_{q_i}\bar{\psi}_i\psi_i, \tag{23}$$

and the equations of motion as

$$\dot{p}_i = [p_i, \mathscr{H}]^* = \ddot{q}_i = -\left\{\frac{1}{2}(W'_{q_i})^2 + W''_{q_i}\bar{\psi}_i\psi_i\right\}, \tag{24}$$

$$\dot{q}_i = [q_i, \mathscr{H}]^* = p_i, \tag{25}$$

$$\dot{\bar{\psi}}_i = [\bar{\psi}_i, \mathscr{H}]^* = iW''_{q_i}\bar{\psi}_i, \tag{26}$$

$$\dot{\psi}_i = [\psi_i, \mathscr{H}]^* = -iW''_{q_i}\psi_i. \tag{27}$$

These equations (eqs (24)–(27)) are equivalent to eqs (11)–(14) as they should. Here we see that Lagrangian depends explicitly on $A_i$ variables appearing in eq. (16) only because of the super space formulation.

## 3. Construction of the second constant of motion

Here we write a general treatment of the construction of an invariant in two dimensions. We determine the coefficients involved in the corresponding terms which enable us to find the additional invariant.

Let us consider a dynamical system described by the Hamiltonian

$$\mathscr{H} = \frac{1}{2}\dot{q}_i^2 + \frac{1}{2}(W'_{q_i})^2 + W''_{q_i}\bar{\psi}_i\psi_i, \tag{28}$$

and we assume the invariant $I$ as

$$I = a_0(q_i, \psi_i, \bar{\psi}_i) + a_k(q_i, \psi_i, \bar{\psi}_i)\dot{q}_k + \frac{1}{2}a_{kl}(q_i, \psi_i, \bar{\psi}_i)\dot{q}_k\dot{q}_l \tag{29}$$

and putting

$$\dot{I} = \frac{\mathrm{d}I}{\mathrm{d}t} = 0 = [I, H], \tag{30}$$

where $[,]$ denotes the graded bracket. Substituting eq. (29) in eq. (30) and after accounting for the proper symmetrization of the coefficients of $(\dot{q}_i)^n$ equal to zero for each $n$, we get a set of differential equations satisfied by the coefficients $a_0, a_k, a_{kl}$:

$$\frac{\partial a_{kl}}{\partial q_i} = 0, \tag{31}$$

$$\frac{\partial a_k}{\partial q_l} - i\left(\frac{\partial a_{kl}}{\partial \psi_i}\bar{\psi}_i + \frac{\partial a_{kl}}{\partial \bar{\psi}_i}\psi_i\right) = 0, \tag{32}$$

$$\frac{\partial a_0}{\partial q_k} - 2i\left(\frac{\partial a_k}{\partial \psi_i}\bar{\psi}_i + \frac{\partial a_k}{\partial \bar{\psi}_i}\psi_i\right) - a_{ik}(W'_{q_i}W''^2_{q_i} + W'''_{q_i}\bar{\psi}_i\psi_i) = 0, \tag{33}$$

and

$$a_i(W'_{q_i}W''^2_{q_i} + W'''_{q_i}\bar{\psi}_i\psi_i) + 2i\left(\frac{\partial a_0}{\partial \psi_i}\bar{\psi}_i + \frac{\partial a_0}{\partial \bar{\psi}_i}\psi_i\right) = 0. \tag{34}$$

Now, we present solutions of these equations for determining various coefficients. From eq. (31) we get

$$a_{kl} = a_{lk} = \alpha_{kl}\psi_k\psi_l + \beta_{kl}\bar{\psi}_k\bar{\psi}_l, \tag{35}$$

where $\alpha_{kl}, \beta_{kl}$ are some constants. Let us put

$$\alpha_{11} = \beta_{11} = \frac{1}{2}\alpha, \tag{36}$$

$$\alpha_{22} = \beta_{22} = \frac{1}{2}\beta \tag{37}$$

and

$$\alpha_{12} = -\alpha_{21} = \beta_{12} = -\beta_{21} = \frac{1}{2}\gamma \;\; \text{(say)} \tag{38}$$

in eq. (35) to obtain

$$a_{11} = \frac{1}{2}\alpha(\psi_1^2 + \bar{\psi}_1^2), \tag{39}$$

$$a_{22} = \frac{1}{2}\beta(\psi_2^2 + \bar{\psi}_2^2), \tag{40}$$

$$a_{12} = \frac{1}{2}\gamma(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2), \tag{41}$$

$$a_{21} = -\frac{1}{2}\gamma(\psi_2\psi_1 + \bar{\psi}_2\bar{\psi}_1). \tag{42}$$

The coefficients $a_{11}$ and $a_{22}$ are zero since the variables $\psi_i$ and $\bar{\psi}_j$ are Grassmannian and must satisfy

$$\psi_i\psi_j = -\psi_j\psi_i, \quad \bar{\psi}_i\bar{\psi}_j = -\bar{\psi}_j\bar{\psi}_i$$

and

$$\psi_i^2 = \bar{\psi}_i^2 = 0.$$

Now using eqs (39)–(42) in (32), we get $a_k$ as

$$a_1 = -i\gamma(\psi_1\bar{\psi}_2 - \psi_2\bar{\psi}_1)q_2 + C_1, \tag{43}$$

$$a_2 = -i\gamma(\psi_1\bar{\psi}_2 - \psi_2\bar{\psi}_1)q_1 + C_2. \tag{44}$$

For the sake of simplicity if we take $C_1 = C_2 = 0$ and substituting the values (39)–(42) and (43)–(44) in eq. (33), we get $a_0$ as

$$\begin{aligned} a_0 = {} & 4\gamma(\bar{\psi}_2\bar{\psi}_1 + \psi_2\psi_1)q_1q_2 \\ & -\frac{\gamma}{4}(\bar{\psi}_2\bar{\psi}_1 + \psi_2\psi_1)\left[\int (W'_{q_1}W''^2_{q_1} + W'''_{q_1}\bar{\psi}_1\psi_1)\mathrm{d}q_2\right. \\ & \left. + \int (W'_{q_2}W''^2_{q_2} + W'''_{q_2}\bar{\psi}_2\psi_2)\mathrm{d}q_1\right]. \end{aligned} \tag{45}$$

One can get the invariant after determining these coefficients for certain systems and putting in eq. (29).

## 4. Examples

In this section, we illustrate some of the interesting potentials in their super form and their respective invariant.

### 4.1 *Harmonic oscillator*

Let us write the harmonic oscillator as

$$\frac{1}{2}W'^2(q) = \frac{1}{2}\lambda^2 q^2 \quad \text{or} \quad W'(q) = \lambda q. \tag{46}$$

Integrating eq. (46), we get

$$W(q) = \frac{1}{2}\lambda q^2. \tag{47}$$

One can write eq. (47) as

$$W(q) = (1/2)\lambda q^2 = (1/2)\lambda(q_1^2 + q_2^2), \tag{48}$$

where

$$q^2 = q_1^2 + q_2^2.$$

Hence

$$W(q_1, q_2) = \frac{1}{2}\lambda(q_1^2 + q_2^2). \tag{49}$$

Differentiating eq. (49) thrice in succession with respect to $q_1$ and $q_2$, we get

$$W'_{q_1} = \lambda q_1, \quad W''_{q_1} = \lambda, \quad W'''_{q_1} = 0, \tag{50}$$

$$W'_{q_2} = \lambda q_2, \quad W''_{q_2} = \lambda, \quad W'''_{q_2} = 0. \tag{51}$$

Now substituting eq. (50) and (51) in eq. (45), we get

$$a_0 = \frac{\gamma}{2}(8 - \lambda^3)\left\{(\bar{\psi}_2\bar{\psi}_1 + \psi_2\psi_1)q_2q_1\right\}. \tag{52}$$

Further, from (39)–(44) we get

$$a_1 = i\gamma(\psi_1\bar{\psi}_2 - \psi_2\bar{\psi}_1)q_2, \tag{53}$$

$$a_2 = -i\gamma(\psi_1\bar{\psi}_2 - \psi_2\bar{\psi}_1)q_1, \tag{54}$$

$$a_{11} = \frac{1}{2}\alpha(\psi_1^2 + \bar{\psi}_1^2), \tag{55}$$

$$a_{22} = \frac{1}{2}\beta(\psi_2^2 + \bar{\psi}_2^2), \tag{56}$$

$$a_{12} = a_{21} = \frac{1}{2}\gamma(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2). \tag{57}$$

Substituting eq. (52) and eqs (53)–(57) in eq. (29), we get the invariant as

$$\begin{aligned} I = {} & \frac{1}{2}\gamma(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2)[\dot{q}_1\dot{q}_2 + (\lambda^3 - 8)q_1q_2] \\ & - i\gamma(\psi_1\bar{\psi}_2 - \psi_2\bar{\psi}_1)(q_2\dot{q}_1 + q_1\dot{q}_2). \end{aligned} \tag{58}$$

Correspondingly, we have the potential for the classical super system as

$$V(q_1, q_2) = \frac{1}{2}\lambda^2(q_1^2 + q_2^2) + \lambda(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2), \tag{59}$$

and the superpotential as

$$\begin{aligned} W = (Z_i, \psi_i, \bar{\psi}_i) = {} & \frac{1}{2}\lambda(q_1^2 + q_2^2) + i\lambda q_i(\theta_i\bar{\psi}_i + \bar{\theta}_i\psi_i) \\ & + \lambda^2 q_i^2\theta_i\bar{\theta}_i + \lambda\theta_i\bar{\theta}_i\bar{\psi}_i\psi_i. \end{aligned} \tag{60}$$

## 4.2 *Kepler potential*

Let us write the Kepler potential as

$$W'^2(q) = -\frac{\lambda^2}{q} \quad \text{or} \quad W'(q) = \frac{i\lambda}{\sqrt{q}}. \tag{61}$$

Integrating eq. (61) we get

$$W(q) = 2i\lambda\sqrt{q}. \tag{62}$$

Equation (62) can be written as

$$W(q_1, q_2) = 2i\lambda(q_1^2 + q_2^2)^{1/4}. \tag{63}$$

Again three successive differentiation of (63) with respect to $q_1$ and $q_2$ gives

$$W'_{q_i} = \frac{i\lambda q_i}{\sqrt{q^3}}, \tag{64}$$

$$W''_{q_1} = \frac{i\lambda}{2\sqrt{q^7}}(2q_2^2 - q_1^2), \tag{65}$$

$$W''_{q_2} = \frac{i\lambda}{2\sqrt{q^7}}(2q_1^2 - q_2^2) \tag{66}$$

and

$$W'''_{q_1} = \frac{3}{4}\frac{i\lambda q_1}{q^{11/2}}(q_1^2 - 6q_2^2), \tag{67}$$

$$W'''_{q_2} = \frac{3}{4}\frac{i\lambda q_2}{q^{11/2}}(q_2^2 - 6q_1^2). \tag{68}$$

Substituting eqs (64)–(68) in eq. (45), we get the expression for $a_0$ and putting back $a_0, a_k, a_{kl}$ in eq. (29) we get the invariant as

$$\begin{aligned}
I = 4\gamma(\bar{\psi}_2\bar{\psi}_1 + \psi_2\psi_1)\Bigg[ & q_1q_2 + \frac{i\lambda^3}{16}\left\{\frac{q_1q_2}{2q^2}\left(\frac{1}{q_1^{5/2}} + \frac{1}{q_2^{5/2}}\right)\right. \\
& \left. + \frac{1}{2}\left(\frac{\tan^{-1}(q_2/q_1)}{q_1^{5/2}} + \frac{\tan^{-1}(q_1/q_2)}{q_2^{5/2}}\right)\right\} \\
& - \frac{6i\lambda^3}{11q}\left\{\frac{q_1q_2}{q^{10}} + \frac{5}{9q^8}\left(\frac{q_2}{q_1} + \frac{q_1}{q_2}\right) + \frac{40}{63q^6}\left(\frac{q_2}{q_1^3} + \frac{q_1}{q_2^3}\right) + \frac{80}{105q^4}\left(\frac{q_2}{q_1^5} + \frac{q_1}{q_2^5}\right)\right. \\
& \left. + \frac{320}{315q^2}\left(\frac{q_2}{q_1^7} + \frac{q_1}{q_2^7}\right) + \frac{640}{315}\left(\frac{q_2}{q_1^9} + \frac{q_1}{q_2^9}\right)\right\} + \frac{9}{4}\frac{i\lambda^3}{q}\left\{\frac{41q_1q_2}{195q^{12}} + \frac{56}{715q^{10}}\left(\frac{q_2}{q_1} + \frac{q_1}{q_2}\right)\right. \\
& + \frac{336}{2861q^8}\left(\frac{q_2}{q_1^3} + \frac{q_1}{q_2^3}\right) + \frac{2688}{20027q^6}\left(\frac{q_2}{q_1^5} + \frac{q_1}{q_2^5}\right) + \frac{16124}{100135q^4}\left(\frac{q_2}{q_1^7} + \frac{q_1}{q_2^7}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{64532}{300405q^2}\left(\frac{q_2}{q_1^9}+\frac{q_1}{q_2^9}\right)+\frac{129064}{300405}\left(\frac{q_2}{q_1^{11}}+\frac{q_1}{q_2^{11}}\right)\bigg\}\\
&-\frac{3}{4}\frac{i\lambda}{q}\left\{q_1^{3/2}\left(\frac{q_1^2q_2}{q^2}+2\right)\bar{\psi}_1\psi_1+q_2^{3/2}\left(\frac{q_1q_2^2}{q^2}+2\right)\bar{\psi}_2\psi_2\right\}\\
&+\frac{i\lambda}{2q}\bigg\{-\frac{q_1q_2}{q^8}(\bar{\psi}_1\psi_1+\bar{\psi}_2\psi_2)+\frac{1}{q^6}\left(\frac{q_2}{q_1}\bar{\psi}_1\psi_1+\frac{q_1}{q_2}\bar{\psi}_2\psi_2\right)\\
&+\frac{6}{35q^4}\left(\frac{q_2}{q_1^3}\bar{\psi}_1\psi_1+\frac{q_1}{q_2^3}\bar{\psi}_2\psi_2\right)+\frac{8}{35q^2}\bigg(\frac{q_2}{q_1^5}\bar{\psi}_1\psi_1\\
&+\frac{q_1}{q_2^5}\bar{\psi}_2\psi_2\bigg)+\frac{16}{35}\left(\frac{q_2}{q_1^7}\bar{\psi}_1\psi_1+\frac{q_1}{q_2^7}\bar{\psi}_2\psi_2\right)\bigg\}+\frac{\dot{q}_1\dot{q}_2}{8}\bigg]\\
&-i\gamma(\psi_1\bar{\psi}_2-\psi_2\bar{\psi}_1)(q_2\dot{q}_1+q_1\dot{q}_2).
\end{aligned}
\tag{69}
$$

Correspondingly, the potential for the classical super system is

$$
V(q_1,q_2)=-\frac{\lambda^2}{2q}+\frac{i\lambda}{q^{7/2}}\left[(2q_2^2-q_1^2)\bar{\psi}_1\psi_1+(2q_1^2-q_2^2)\bar{\psi}_2\psi_2\right] \tag{70}
$$

and the superpotential is

$$
\begin{aligned}
W(Z_i,\psi_i,\bar{\psi}_i)&=2i\lambda\sqrt{q}-\frac{\lambda q_i}{q^{3/2}}[\theta_i\bar{\psi}_i+\bar{\theta}_i\psi_i]\\
&\quad+\frac{i\lambda}{q^{3/2}}\theta_i\bar{\theta}_i\left[\frac{i\lambda q_i}{q^{3/2}}+\frac{1}{2q^2}\{(2q_2^2-q_1^2)\bar{\psi}_1\psi_1+(2q_1^2-q_2^2)\bar{\psi}_2\psi_2\}\right].
\end{aligned}
\tag{71}
$$

### 4.3 *Potential* $q^{-2/3}$

We write the potential as

$$
W'^2(q)=\lambda^2q^{-2/3}\quad\text{or}\quad W'=\lambda q^{-1/3}. \tag{72}
$$

Integrating eq. (72) we get

$$
W(q)=\frac{3}{2}\lambda q^{2/3}. \tag{73}
$$

Writing eq. (73) in terms of $q_1, q_2$, we have

$$
W(q_1,q_2)=\frac{3}{2}\lambda(q_1^2+q_2^2)^{1/3}. \tag{74}
$$

Differentiating eq. (74) with respect to $q_1$ and $q_2$ successively thrice, we get

$$
W'_{q_i}=\frac{\lambda q_i}{q^{4/3}},\quad W''_{q_i}=\frac{\lambda}{q^{4/3}}\left[1-\frac{4q_i^2}{3q^2}\right] \tag{75}
$$

and

$$W'''_{q_i} = 4\lambda q_i (q_1^2 + q_2^2)^{-5/3} \left[ \frac{10 q_i^2}{9q^2} - 1 \right]. \tag{76}$$

Using eqs (75) and (76) we obtain $a_0$ and again putting the coefficients $a_0, a_k$ and $a_{kl}$ in eq. (29) we write the invariant as

$$\begin{aligned} I = 4\gamma(\bar{\psi}_2\bar{\psi}_1 + \psi_2\psi_1) \Bigg[ q_1q_2 - \frac{\lambda^3}{54} \Bigg\{ -\frac{q_1q_2}{q^4} - \frac{3}{4q_1q_2} \\ + \frac{3}{16}\left( \frac{\tan^{-1}(q_2/q_1)}{q_1^2} + \frac{\tan^{-1}(q_1/q_2)}{q_2^2} \right) \Bigg\} \\ - \frac{5\lambda}{27q} \Bigg\{ \frac{59}{9} \left( \frac{q_2}{q_1^{4/3}} \bar{\psi}_1\psi_1 + \frac{q_1}{q_2^{4/3}} \bar{\psi}_2\psi_2 \right) \\ - \frac{56}{27q^2} \left( \frac{q_2^3}{q_1^{4/3}} \bar{\psi}_1\psi_1 + \frac{q_1^3}{q_2^{4/3}} \bar{\psi}_2\psi_2 \right) \\ - \frac{4}{3} \left( \frac{\tan^{-1}(q_2/q_1)}{q_1^{1/3}} \bar{\psi}_1\psi_1 + \frac{\tan^{-1}(q_1/q_2)}{q_2^{1/3}} \bar{\psi}_2\psi_2 \right) \\ - \left( \frac{q_2}{q_1^{4/3}} (\tan^{-1}(q_2/q_1))^2 \bar{\psi}_1\psi_1 + \frac{q_1}{q_2^{4/3}} (\tan^{-1}(q_1/q_2))^2 \bar{\psi}_2\psi_2 \right) \\ + \frac{1}{3q^2} \left( \frac{q_2^3}{q_1^{4/3}} (\tan^{-1}(q_2/q_1))^3 \bar{\psi}_1\psi_1 + \frac{q_1^3}{q_2^{4/3}} (\tan^{-1}(q_1/q_2))^3 \bar{\psi}_2\psi_2 \right) \\ - \frac{2}{9q^2} (q_1^{5/3} \tan^{-1}(q_2/q_1) \bar{\psi}_1\psi_1 + q_2^{5/3} \tan^{-1}(q_1/q_2) \bar{\psi}_2\psi_2) \Bigg\} - \frac{\dot{q}_1\dot{q}_2}{8} \Bigg] \\ - i\gamma(\psi_1\bar{\psi}_2 - \psi_2\bar{\psi}_1)(q_2\dot{q}_1 + q_1\dot{q}_2). \end{aligned} \tag{77}$$

Correspondingly, the potential for the classical super system is

$$V(q_1,q_2) = \frac{\lambda}{q^{2/3}} \left[ \frac{\lambda}{2} + q^{-2/3} \left\{ \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 - \frac{4}{3q^2}(q_1^2\bar{\psi}_1\psi_1 + q_2^2\bar{\psi}_2\psi_2) \right\} \right] \tag{78}$$

and the superpotential is

$$\begin{aligned} W(Z_i, \psi_i, \bar{\psi}_i) = \frac{3}{2}\lambda q^{2/3} + \frac{i\lambda q_i}{q^{4/3}} [\theta_i \bar{\psi}_i + \bar{\theta}_i \psi_i] \\ + \theta_i\bar{\theta}_i \frac{\lambda}{q^{4/3}} \left[ \frac{\lambda q_i^2}{q^{4/3}} + \left( 1 - \frac{4}{3}\frac{q_i^2}{q^2} \right) \bar{\psi}_i\psi_i \right]. \end{aligned} \tag{79}$$

## 5. Summary and conclusions

In this paper, we have demonstrated a method which furnishes the integrable systems in super space with spatial dimension $N \geq 2$. Invoking Dirac's constraints the dynamical system under consideration reduces to a time independent one. Three examples have been considered to discuss its integrability. As we know, the harmonic oscillator is an integrable system. However, the potential has been investigated as an integrable system in its bosonic and fermionic part using the method mentioned in this paper. The fermionic part which effectively involve the fermionic operator, implies that the super symmetry formalism could be envisaged as dealing with scalar spinors. We obtain the Kepler potential as integrable system in its bosonic and fermionic parts. In the same spirit, we find that the potential $q^{-2/3}$ [8] is also integrable. One can see that the last two cases, eqs (69)–(71) and eqs (77)–(79), however have yielded extra degeneracies in the corresponding quantum systems.

The above study would play a vital role in the domain of a variety of fields namely molecular physics, accelerator physics, plasma physics, laser physics etc. If the invariant is constructed for a system, it may help in reducing some non-linear dynamical problems to a quadrature [1,2,4,11]. These studies are capable of facilitating the reduction of the problem of the solution of equation of motion [9,10]. No doubt, in some cases the system is found to be integrable just by accident. These studies are also important in time dependent Kepler problem, alpha decay, time dependent gravitational constant and time varying mass for accelerating dynamical systems.

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## References

[1] E T Whittaker, *Analytical dynamics of particle and rigid bodies* (Cambridge University Press, London, 1960) p. 332

[2] S C Mishra, R S Kaushal and K C Tripathy, *J. Math. Phys.* **25**, 2217 (1984)
R S Kaushal, S C Mishra and K C Tripathy, *Phys. Lett.* **A102**, 7 (1984)
R S Kaushal, S C Mishra and K C Tripathy, *J. Math. Phys.* **26**, 420 (1985)
C R Holt, *J. Math. Phys.* **23**, 1037 (1982)

[3] E D'Hoker and L Vinet, *Lett. Math. Phys.* **8**, 439 (1989)
L Bouguiax, P Dauby and V Hussain, *J. Math. Phys.* **28**, 447 (1987)
K C Tripathy, L K Tripathy, *ICTP*, Trieste, preprint No. IC/89/233 (1988)
S C Mishra and Veena Mishra, *Sing. J. Phys.* **8**, 7 (1991)

[4] M Lutzky, *J. Phys.* **A12**, 973 (1979)

[5] M Lakshmanan and P Kaliappan, *J. Math. Phys.* **24**, 795 (1983)

[6] P G L Leach, *J. Math. Phys.* **26**, 77 (1985)

[7] W Sarlet, F M Mahomed and P G L Leach, *J. Phys.* **A20**, 277 (1987)

[8] V I Inozemtsev, *Phys. Lett.* **A96**, 447 (1983)

[9] R S Kaushal and S C Mishra, *Pramana – J. Phys.* **26**, 109 (1986)

[10] Y F Chang, M Tabor and J Weiss, *J. Math. Phys.* **23**, 531 (1982)

[11] B Dorizzi, B Grammaticos and A Ramani, *J. Math. Phys.* **24**, 2282 (1983)
A Ramani, B Dorizzi and B Grammaticos, *Phys. Rev. Lett.* **49**, 1539 (1982)
[12] J R Ray and J L Reid, *J. Math. Phys.* **20**, 2054 (1979)
[13] P J Olver, *Applications of Lie groups to differential equations* (Springer, New York, 1986)
[14] Yu A Gol'fand and E P Likhtman, *JETP Lett.* **13**, 323 (1971)
P Ramond, *Phys. Rev.* **D3**, 2415 (1971)
[15] J L Gervais and B Sakita, *Nucl. Phys.* **B34**, 633 (1971)
A Neveu and J Schwarz, *Nucl. Phys.* **B31**, 86 (1971)
[16] J Wess and B Zumino, *Nucl. Phys.* **B70**, 39 (1974)
[17] P Fayet and S Ferrara, *Phys. Rep.* **C32**, 249 (1977)
[18] E C G Sudarshan and N Mukunda, *Classical dynamics – a modern prospective* (Wiley, New York, 1974)
[19] F Ravndal, *Proceeding of the CERN School of Physics* (CERN, Geneva, 1984) p. 300
[20] H R Lewis and P G L Leach, *J. Math. Phys.* **23**, 2371 (1982)