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A charged spherically symmetric solution

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Abstract. We find a solution of the Einstein–Maxwell system of field equations for a class of accelerating, expanding and shearing spherically symmetric metrics. This solution depends on a particular *ansatz* for the line element. The radial behaviour of the solution is fully specified while the temporal behaviour is given in terms of a quadrature. By setting the charge contribution to zero we regain an (uncharged) perfect fluid solution found previously with the equation of state $p = \mu +$ constant, which is a generalisation of a stiff equation of state. Our class of charged shearing solutions is characterised geometrically by a conformal Killing vector.

Keywords. Einstein–Maxwell equations; charged cosmological models; inhomogeneous space-times.

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1. Introduction

Spherically symmetric solutions of the Einstein field equations are important models in cosmological and astrophysical processes. Most of the exact solutions found have vanishing shear as this condition substantially simplifies the field equations [1]. We seek to extend these solutions to the more general class of non-zero shear and to include the effects of charge. Consequently we are investigating the Einstein–Maxwell system of field equations for the general spherically symmetric line element. The aim of finding solutions to this system is to obtain a deeper physical understanding of the electromagnetic field in a curved space-time. Despite the global observed neutrality of the universe, it is important to study the Einstein–Maxwell field equations for a number of reasons [2]. For example, the role of charge in gravitational collapse has been pointed out in a number of treatments. Also, the electromagnetic field may play a role in preventing a Big Bang singularity.

The Einstein–Maxwell field equations describe the coupling between curvature of space-time and the matter content which now includes the electromagnetic field. The analysis with charge is more complex because of the additional Maxwell equations and the presence of the electromagnetic gauge potential. In §2, the Einstein–Maxwell field equations for a comoving frame of reference are given. We simplify the field equations in §3 by making an *ansatz* for the line element which was used in earlier investigations. This enables us to integrate the equations and an Einstein–Maxwell solution is exhibited. This solution reduces to an uncharged perfect fluid model found previously which is characterised by the barotropic equation of state $p = \mu +$ constant. In §4, we briefly consider some physical aspects of the solution and point out the geometric characterisation of the Einstein–Maxwell model by a conformal Killing symmetry.

2. Spherically symmetric space-times

With the condition of spherical symmetry and using the comoving coordinates $(x^i) = (t, r, \theta, \phi)$ we can show that the line-element takes the form [1,2]

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)\left[d\theta^2 + \sin^2\theta d\phi^2\right]. \qquad (1)$$

A consequence of spherical symmetry is that the vorticity of the space-time vanishes. From this result it follows that we can choose the comoving 4-velocity $u^a = (e^{-\nu}, 0, 0, 0)$ which is hypersurface orthogonal. The remaining kinematical quantities, namely the acceleration, expansion and shear, are given by

$$\dot{u}^a = \left(0, \nu', 0, 0\right), \qquad (2)$$

$$\Theta = e^{-\nu}\left(\dot{\lambda} + \frac{2\dot{Y}}{Y}\right), \qquad (3)$$

$$\sigma^1_1 = \sigma^2_2 = -\tfrac{1}{2}\sigma^3_3 = \tfrac{1}{3}e^{-\nu}\left(\frac{\dot{Y}}{Y} - \dot{\lambda}\right), \qquad (4)$$

respectively. For non-zero shear we require $\dot{Y}/Y \neq \dot{\lambda}$.

The Einstein–Maxwell field equations, for a charged perfect fluid matter distribution, are

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = (\mu + p)u_a u_b + pg_{ab} + F_{ac}F^c_b - \tfrac{1}{4}g_{ab}F_{cd}F^{cd}, \qquad (5)$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \qquad (6)$$

$$F^{ab}{}_{;b} = J^a, \qquad (7)$$

where $J^a = \kappa u^a$ is the 4-current density and κ the proper charge density. The electromagnetic field tensor is $F_{ab} = A_{b;a} - A_{a;b}$. We utilise the gauge freedom in choosing the 4-potential. We assume that $A_a = (\phi(t,r), 0, 0, 0)$ for the 4-potential which is consistent with spherical symmetry and has been extensively utilised in inhomogeneous cosmological models by Sussman [3–5]. We take the electromagnetic gauge potential ϕ to be a function of both the radial and time coordinates analogous to the dependence of the metric potentials ν, λ and Y. Then the nonzero components of the electromagnetic field tensor are

$$F_{10} = -F_{01} = \phi'.$$

Thus the Einstein–Maxwell equations (5)–(7) become

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right)$$

$$-\frac{1}{2}\mathrm{e}^{-2(\lambda+\nu)}\phi'^2, \tag{8}$$

$$p = -\frac{1}{Y^2} + \frac{2}{Y}\mathrm{e}^{-2\lambda}\left(\nu' Y' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}\mathrm{e}^{-2\nu}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right)$$

$$+\frac{1}{2}\mathrm{e}^{-2(\lambda+\nu)}\phi'^2, \tag{9}$$

$$p = \mathrm{e}^{-2\lambda}\left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{Y}\left(\nu' Y' - \lambda' Y' + Y''\right)\right]$$

$$-\mathrm{e}^{-2\nu}\left[\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right]$$

$$-\frac{1}{2}\mathrm{e}^{-2(\lambda+\nu)}\phi'^2, \tag{10}$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda}, \tag{11}$$

$$\kappa = \mathrm{e}^{-2\lambda-\nu}\phi'\left(\lambda' + \nu' - \frac{\phi''}{\phi'} - 2\frac{Y'}{Y}\right), \tag{12}$$

$$\dot{\phi}' = \phi'\left(\dot{\lambda} + \dot{\nu} - 2\frac{\dot{Y}}{Y}\right), \tag{13}$$

for the spherically symmetric model (1) with charged matter. With $\phi = \phi(t)$ we regain the field equations for uncharged matter.

3. A charged solution

It is unlikely that much progress is possible with the field equations (8)–(13) without simplifying assumptions. Here we follow the approach of Maharaj *et al* [6] for uncharged perfect fluids in an attempt to generate a solution. Note that a similar form for the line element was chosen by Hajj–Boutros [7]. Both classes of solution contain the Gutman–Bespal'ko model [8] with a stiff equation of state. We make the *ansatz*

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(r)}\mathrm{d}r^2 + r^2T^2(t)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{14}$$

Then eq. (13) can be written as

$$\phi' = A_1(r)\mathrm{e}^{\nu+\lambda}r^{-2}T^{-2}$$

and (12) is a definition for the proper charge density κ. With the simplified form of the line element (14), the remaining field equations (8)–(11) become

$$\mu = \frac{1}{r^2T^2} + \mathrm{e}^{-2\lambda}\left(-\frac{1}{r^2} + \frac{2}{r}\lambda'\right) + \mathrm{e}^{-2\nu}\frac{\dot{T}^2}{T^2} - \frac{A_1^2(r)}{2r^4T^4}, \tag{15}$$

$$p = -\frac{1}{r^2T^2} + \mathrm{e}^{-2\lambda}\left(\frac{1}{r^2} + \frac{2}{r}\nu'\right) - \frac{1}{T^2}(2\ddot{T}T + \dot{T}^2)\mathrm{e}^{-2\nu} + \frac{A_1^2(r)}{2r^4T^4}, \tag{16}$$

$$p = \mathrm{e}^{-2\lambda}\left((\nu'' + \nu'^2 - \nu'\lambda') + \frac{1}{r}(\nu' - \lambda')\right) - \mathrm{e}^{-2\nu}\frac{\ddot{T}}{T} - \frac{A_1^2(r)}{2r^4T^4}, \tag{17}$$

$$0 = 1 - r\nu'. \tag{18}$$

Equation (18) is immediately integrated to yield

$$e^{2\nu} = a^2 r^2,$$

where a is a constant. On equating (16) to (17), and using the above expression for $e^{2\nu}$, we obtain

$$\frac{1}{T^2} + \frac{1}{a^2T^2}(\ddot{T}T + \dot{T}^2) - \frac{A_1^2(r)}{r^2T^4} = 2e^{-2\lambda}(1+\lambda' r),$$

which is often referred to as the pressure isotropy equation. To continue we make the choice

$$A_1(r) = a_1 r,$$

where a_1 is a constant. Then the variables r and t separate in the pressure isotropy equation. We obtain the result

$$\frac{1}{T^2} + \frac{1}{a^2T^2}(\ddot{T}T + \dot{T}^2) - \frac{a_1^2}{T^4} = 2e^{-2\lambda}(1+\lambda' r)$$

which is equivalent to

$$\frac{1}{T^2} + \frac{1}{a^2T^2}(\ddot{T}T + \dot{T}^2) - \frac{a_1^2}{T^4} = 2k, \tag{19}$$

$$e^{-2\lambda}(1+\lambda' r) = k, \tag{20}$$

where k is a constant. The differential equation (20) has the general solution

$$e^{2\lambda} = \frac{1}{k+br^2}.$$

Note that Maharaj *et al* [6] obtained a similar form for $e^{2\lambda}$; however their analysis was valid only for uncharged matter.

Equation (19) is transformed via

$$T = Z^{1/2}$$

to the simpler differential equation

$$\ddot{Z} + 2a^2 - 4ka^2 Z - \frac{2a^2a_1^2}{Z} = 0 \tag{21}$$

which is easily reduced to the form

$$\int \frac{dZ}{\sqrt{C_1 - 4a^2Z + 4ka^2Z^2 + 4a^2a_1^2 \log Z}} = t - t_0,$$

where C_1 and t_0 are constant. Thus we have reduced the solution of the last field equation (19) to a quadrature. On evaluating this quadrature we obtain the form of T via the transformation $T = Z^{1/2}$.

The matter variables μ and p are then obtained from (15) and (16) respectively as

$$\mu = -3b + \frac{k}{r^2} - \frac{1}{a^2 r^2}\frac{\ddot{T}}{T} - \frac{a_1^2}{2r^2 T^4},$$
$$p = 3b + \frac{k}{r^2} - \frac{1}{a^2 r^2}\frac{\ddot{T}}{T} + \frac{a_1^2}{2r^2 T^4}.$$

We note that p and μ are related by

$$p - \mu = 6b + \frac{a_1^2(r)}{r^2 T^4}. \tag{22}$$

We regain the stiff equation of state $p = \mu$ if we set $a_1 = b = 0$. Therefore we have generated an exact solution to the Einstein–Maxwell field equations with the equation of state (22). This exact solution has non-vanishing acceleration, expansion and shear from (2)–(4).

The choice $a_1 = 0$ implies that $\phi' = 0$ which corresponds to the uncharged case. With this assumption the differential equation (21) for Z becomes

$$\ddot{Z} + 2a^2 - 4ka^2 Z = 0.$$

Depending on the sign of k this equation can be integrated to yield the following forms of the line element

$$k = 0:$$
$$\mathrm{d}s^2 = -a^2 r^2 \mathrm{d}t^2 + \frac{1}{br^2}\mathrm{d}r^2 + r^2(-a^2t^2 + ct + d)\mathrm{d}\Omega^2,$$
$$k = -n^2 < 0:$$
$$\mathrm{d}s^2 = -a^2 r^2 \mathrm{d}t^2 + \frac{1}{-n^2 + br^2}\mathrm{d}r^2 + r^2\left(c\sin(2ant) + d\cos(2ant) - \frac{1}{2n^2}\right)\mathrm{d}\Omega^2,$$
$$k = n^2 > 0:$$
$$\mathrm{d}s^2 = -a^2 r^2 \mathrm{d}t^2 + \frac{1}{n^2 + br^2}\mathrm{d}r^2 + r^2\left(c\mathrm{e}^{2ant} + d\mathrm{e}^{-2ant} + \frac{1}{2n^2}\right)\mathrm{d}\Omega^2,$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ and c and d are constants of integration. For the case $k \geq 0$ we require that $b > 0$ while for $k < 0$ we must have that $r \geq \sqrt{-k/b}$. These results were obtained by Maharaj *et al* [6]. Thus we have recovered their solutions when the charge vanishes. The equation of state for this uncharged class of models is

$$p - \mu = 6b.$$

This is a generalisation of the stiff equation of state $p = \mu$. If we set $k = 1, a = \frac{1}{2}$ and $b = 0$ we obtain the line element

$$\mathrm{d}s^2 = -\frac{r^2}{4}\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(c\mathrm{e}^{t} + d\mathrm{e}^{-t} + \frac{1}{2}\right)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

which is the original Gutman–Bespal'ko [8] solution. Note that the models (Maharaj *et al* [6]) contain this particular case.

4. Discussion

We have generated a charged solution for the Einstein–Maxwell system of field equations. This solution arises as a result of the *ansatz* (14) for the line element. It is clear from the kinematical quantities that our solutions are accelerating, expanding and shearing. There are few solutions with non-zero shear and our approach may help to isolate other classes of solution. When the charge vanishes we regain the solutions of Maharaj *et al* [6]. The singularity in μ and p at $r = 0$ mean that these charged solutions are not very suitable as cosmological models. However, the models found are solutions of the Einstein–Maxwell system and have a simple form. This could assist in future investigations for more realistic models for cosmological processes. In our future work we would need to choose a more general form than the line element (14), with a corresponding increase in the complexity of the field equations.

We should note two points relating to the physical aspects of our solution. Firstly, the equation of state (22) necessarily follows from our *ansatz* on the metric functions. This equation is time-dependent which is not usually the case in the phenomenological modelling of a perfect fluid matter distribution where the pressure is a function of the density. However we observe that for large values of t for the special case $k = 0$, the term $a_1^2(r)/r^2T^4$ in (22) becomes increasingly small, and we do in fact generate the equation of state $p = \mu + 6b$, which is a barotropic equation of state for realistic matter, in the limit of infinitely large t. In future work on more realistic models of non-stationary spherically symmetric charged models we intend to generate solutions in which the dependence of time in the equation of state is avoided for all t. Secondly, we point out that our model has a non-zero proper charge density κ. This serves as additional sources for gravity since charges have mass. However these additional sources are not included in the energy-momentum (stress) tensor which suggests that the charge density (but not the electromagnetic field) could be set to zero. Unfortunately with this assumption in our model we find that the separability of the field equations is lost, and we cannot obtain the equivalent of equations (19) and (20). Consequently, it is for this technical difficulty that we do not consider the case of vanishing κ.

It was pointed out by Maharaj and Maharaj [9] that the form of the line element (14) admits a conformal symmetry for uncharged fluids in spherically symmetric models. As the general form of the line element remains the same (however the behaviour of $T(t)$ in this analysis is different) for the class of charged Einstein–Maxwell solutions presented here, we conclude that the proper conformal Killing vector

$$\boldsymbol{\xi} = \xi^1 \frac{\partial}{\partial r}$$

is admitted by the charged spherically symmetric fluid. This suggests the possibility of finding other solutions by imposing the requirement of a conformal symmetry on spacetime. For an example of a conformal Killing vector in the t–r plane corresponding to an exact solution in shear–free spacetimes, see Maharaj *et al* [10].

References

[1] D Kramer, H Stephani, M A H MacCullum and E Herlt, *Exact solutions of Einstein's field equations* (Cambridge University Press, Cambridge, 1980)

[2] A Krasinski, *Inhomogeneous cosmological models* (Cambridge University Press, Cambridge, 1997)
[3] R A Sussman, *J. Math. Phys.* **28**, 1118 (1987)
[4] R A Sussman, *J. Math. Phys.* **29**, 945 (1988)
[5] R A Sussman, *J. Math. Phys.* **29**, 1177 (1988)
[6] S D Maharaj, R Maartens and M S Maharaj, *Il. Nuovo Cimento* **B108**, 75 (1993)
[7] J Hajj-Boutros, *J. Math. Phys.* **26**, 771 (1985)
[8] I I Gutman and R M Bespal'ko, *Sbornik Sovrem. Probl. Grav. Tbilissi* **201** (1967) (in Russian)
[9] S D Maharaj and M S Maharaj, *Il. Nuovo Cimento* **B109**, 983 (1994)
[10] S D Maharaj, P G L Leach, and R Maartens, *Gen. Relativ. Gravit.* **23**, 261 (1991)