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On the invariance properties of the Klein–Gordon equation with external electromagnetic field

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Abstract. Here we attempt to find the nature of the external electromagnetic field such that the KG equation with external electromagnetic field is invariant. Lie's extended group method is applied to obtain the class of external electromagnetic field which admits the invariance of the KG equation. Though, the field potential only explicitly appears in the equation, the constraints for the invariance are only on the electromagnetic field.

Keywords. Klein–Gordon equation; Lie analysis; prolongation group; Poincaré group.

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1. Introduction

In a previous paper [1], the consequences of the presence of external electromagnetic field on the invariance of the Dirac equation was investigated. This paper is devoted to the same problem for the Klein–Gordon equation. There exists a lot of literature on the problem, but all of these investigations are confined to the case of free particle equation, i.e. in the absence of an external electromagnetic field [2]. We introduce infinitesimal transformation of space-time coordinates as well as the wave function to construct the extended group following Lie [3]. Then we apply the operator of the extended group of the equation to obtain the nature of admissible infinitesimal transformations. Since the KG equation is of second-order we have to develop the first and second prolongations for the Lie's extended group. This leads us to the conditions for the invariance. Then we obtain the explicit nature of the admissible infinitesimal transformations. As in the case of the Dirac equation, here also the maximal symmetry group for the free particle namely, the Poincaré group is constrained by the presence of electromagnetic field. Finally, we obtain the general nature of the field for the invariance of the KG equation.

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*since deceased.

2. The Klein–Gordon equation

The Klein–Gordon (KG) equation in the presence of external electromagnetic field is given by

$$\left\{\left(p_0+\frac{e}{c}A_0\right)^2-\left(\mathbf{P}+\frac{e}{c}\mathbf{A}\right)^2-M^2c^2\right\}\psi=0, \tag{1}$$

explicitly

$$\frac{\partial^2\psi}{\partial x^\mu\partial x^\mu}+i2A_\mu\frac{\partial\psi}{\partial x^\mu}-\left(A_\mu\cdot A_\mu+\frac{M^2c^2}{\hbar^2}\right)\psi=0. \tag{2}$$

(Notations: $x_0=ict=i\tau$, $A_0=i\Phi$, sub and superscripts $\mu,\nu,\ldots,j,k,\ldots$ run through 0,1,2,3. Repeated Greek sub or superscripts (no distinction between them) in a term implies summation. But repeated roman ones are free (no sum).)

2.1 *Infinitesimal transformations*

Let us consider the infinitesimal transformations

$$\bar{x}^\mu=x^\mu+\varepsilon\xi^\mu(x,\psi) \tag{3}$$

and

$$\bar{\psi}(x)=\psi(x)+\varepsilon\Phi(x,\psi). \tag{4}$$

ε is an infinitesimal quantity. The first prolongation is given by

$$\begin{aligned}\delta\left(\frac{\partial\psi}{\partial x^\nu}\right)&=\left(\overline{\frac{\partial\psi}{\partial x^\nu}}\right)-\frac{\partial\psi}{\partial x^\nu}\\&=\left(\frac{\partial}{\partial x^\nu}+\frac{\partial\psi}{\partial x^\nu}\frac{\partial}{\partial\psi}\right)\Phi-\left(\frac{\partial\xi^\mu}{\partial x^\nu}+\frac{\partial\psi}{\partial x^\nu}\frac{\partial\xi^\mu}{\partial\psi}\right)\frac{\partial\psi}{\partial x^\mu}\end{aligned} \tag{5}$$

and the second prolongation is thus

$$\begin{aligned}\delta\left(\frac{\partial^2\psi}{\partial x^k\partial x^k}\right)&=\left(\overline{\frac{\partial^2\psi}{\partial x^k\partial x^k}}\right)-\frac{\partial^2\psi}{\partial x^k\partial x^k}\\&=\frac{\partial^2\Phi}{\partial x^k\partial x^k}+2\frac{\partial\psi}{\partial x^k}\frac{\partial^2\Phi}{\partial\psi\partial x^k}+\left(\frac{\partial\psi}{\partial x^k}\right)^2\frac{\partial^2\Phi}{\partial\psi^2}\\&\quad-\left[\frac{\partial^2\xi^\nu}{\partial x^k\partial x^k}\frac{\partial\psi}{\partial x^\nu}+2\frac{\partial\xi^\nu}{\partial x^k}\frac{\partial^2\psi}{\partial x^k\partial x^\nu}+\frac{\partial\xi^\mu}{\partial\psi}\frac{\partial^2\psi}{\partial x^k\partial x^k}\frac{\partial\psi}{\partial x^\mu}\right.\\&\quad\left.+2\frac{\partial\xi^\nu}{\partial\psi}\frac{\partial\psi}{\partial x^k}\frac{\partial^2\psi}{\partial x^k\partial x^\nu}+\frac{\partial^2\xi^\mu}{\partial x^k\partial\psi}\frac{\partial\psi}{\partial x^k}\frac{\partial\psi}{\partial x^\mu}\right].\end{aligned} \tag{6}$$

In these derivations x^k and ψ are considered as independent variables.

3. Invariance of the equation

In order that the KG equation is invariant under the infinitesimal transformation eqs (3) and (4),

$$\left\{\delta\left(\frac{\partial^2\psi}{\partial x^{\mu^2}}\right)\frac{\partial}{\partial\left(\frac{\partial^2\psi}{\partial x^{\mu^2}}\right)}+\delta\left(\frac{\partial\psi}{\partial x^{\mu}}\right)\frac{\partial}{\partial\left(\frac{\partial\psi}{\partial x^{\mu}}\right)}+\delta\psi\frac{\partial}{\partial\psi}+\delta x^{\mu}\frac{\partial}{\partial x^{\mu}}\right\}(\mathrm{KG})=0. \tag{7}$$

Here, $\partial^2\psi/\partial x^{j^2}$, $\partial\psi/\partial x^k$, ψ and x are considered as independent variables. From eq. (2),

$$\begin{aligned}
&\frac{\partial(\mathrm{KG})}{\partial\frac{\partial^2\psi}{\partial x^{\mu^2}}}=1,\quad \frac{\partial(\mathrm{KG})}{\partial\left(\frac{\partial\psi}{\partial x^{\nu}}\right)}=i2A_{\nu},\\
&\frac{\partial(\mathrm{KG})}{\partial x^{\mu}}=i2\frac{\partial A_{\nu}}{\partial x^{\mu}}\frac{\partial\psi}{\partial x^{\nu}}-2\frac{\partial A_{\nu}}{\partial x^{\mu}}A_{\nu}\psi,\\
&\frac{\partial(\mathrm{KG})}{\partial\psi}=-\left(A_{\mu}A_{\mu}+\frac{M^2c^2}{\hbar^2}\right),\\
&\delta\psi=\Phi,\quad \delta x^{\mu}=\xi^{\mu}.
\end{aligned} \tag{8}$$

Equation (7) should be satisfied identically in ψ and its derivatives along with the original eq. (2) with an indeterminate multiplier, say P, i.e., with the equation

$$P\left\{\frac{\partial^2\psi}{\partial x^{\mu}\partial x^{\mu}}+i2A_{\mu}\frac{\partial\psi}{\partial x^{\mu}}-\left(A_{\nu}A_{\nu}+\frac{M^2c^2}{\hbar^2}\right)\right\}\psi=0. \tag{9}$$

3.1 *Determination of ξ and Φ*

Writing eq. (7) in full, with the help of eqs (5), (6) and (8), one observes that there is only one term each as $(\partial\psi/\partial x^k)^2$ with the coefficient $\partial^2\phi/\partial\psi^2$ and as $(\partial\psi/\partial x^k)(\partial^2\psi/\partial x^k\partial x^{\nu})$ with coefficient $\partial\xi^{\nu}/\partial\psi$. Hence,

$$\frac{\partial^2\Phi}{\partial\psi^2}=0\quad\text{and}\quad\frac{\partial\xi^{\mu}}{\partial\psi}=0 \tag{10}$$

so that

$$\Phi=b(x)+B(x)\psi \tag{11}$$

and

$$\xi^{\mu}=\xi^{\mu}(x). \tag{12}$$

Next collecting the coefficients of terms $\partial^2\psi/\partial x^{k2}$ and $\partial^2\psi/\partial x^k\partial x^j(k\neq j)$, one obtains

$$B+P=2\frac{\partial\xi^{p}}{\partial x^{p}}=2\frac{\partial\xi^{q}}{\partial x^{q}} \tag{13}$$

and

$$\frac{\partial \xi^k}{\partial x^j} + \frac{\partial \xi^j}{\partial x^k} = 0 \ (k \neq j). \tag{14}$$

It is important to note that these relations are independent of the potential A, i.e. of E and H. Since B and P carry no index, the right-hand side of eq. (13) are the same for all the four indices. Equations (13) and (14) are sufficient to determine ξ^p uniquely. Since the result is well-known, we omit the proof and quote the result as

$$\xi^p = a_\mu x_\mu x^p - \frac{1}{2} a_p x_\nu x_\nu + \Gamma^p_\nu x^\nu + \beta x^p + T^p, \tag{15}$$

a_μ, Γ^p_q, β and T^p being independent constants, excluding the relations

$$\Gamma^p_q + \Gamma^q_p = 0. \tag{16}$$

There are 15 independent parameters. a_μ induces the well-known conformal transformation. We will see later that Γ^p_q is associated with rotation and Lorentz transformation, T^μ generates translation and β leads to dilatation.

3.2 *Equations for B, P and b*

Next, collecting the coefficients of $\partial\psi/\partial x^k$, ψ and terms independent of ψ one gets

$$2\frac{\partial B}{\partial x^k} - \frac{\partial^2 \xi^k}{\partial x^\mu \partial x^\mu} + i2A_k(B+P) - i2A_\mu \frac{\partial \xi^k}{\partial x^\mu} + i2\xi^\mu \frac{\partial A_k}{\partial x^\mu} = 0, \tag{17}$$

$$\frac{\partial^2 B}{\partial x^\mu \partial x^\mu} + i2A_\mu \frac{\partial B}{\partial x^\mu} - 2\xi^\mu \frac{\partial A_\nu}{\partial x^\mu} A_\nu - \left(A_\mu A_\mu + \frac{M^2c^2}{\hbar^2} \right)(B+P) = 0, \tag{18}$$

$$\frac{\partial^2 b}{\partial x^\mu \partial x^\mu} + i2A_\mu \frac{\partial b}{\partial x^\mu} - \left(A_\nu, A_\nu + \frac{M^2c^2}{\hbar^2} \right) b = 0. \tag{19}$$

The above equation (19) for b is exactly the same as that of ψ (eq. (2)), but from eq. (11) $b(x)$ is a function of x alone and independent of ψ. Thus,

$$b(x) = 0. \tag{20}$$

From (17), on further differentiating, one gets

$$\frac{\partial^2 B}{\partial x^\mu \partial x^\mu} + i2A_\nu a_\nu = 0. \tag{21}$$

Eliminating $\partial B/\partial x^k$, between eqs (17) and (18), and taking note of eqs (13) and (14) one obtains

$$\frac{M^2c^2}{\hbar^2}(B+P) = 2\frac{M^2c^2}{\hbar^2}(a_\mu x_\mu + \beta) = 0, \tag{22}$$

since $M \neq 0$, $a_\mu = 0$ and $\beta = 0$, i.e., the KG equation with mass term does not admit conformal invariance. It needs to be pointed out that our results up to this stage are independent of $\underset{\sim}{A}$ i.e., the presence of the external field.

4. Nature of $\underset{\sim}{A}$ for the invariance

Since $B+P=0$ (eq. (22)), B can be eliminated from eq. (17) by differentiating further to obtain $\partial^2 B/\partial x^k \partial x^j (k \neq j)$ and subtracting it from $\partial^2 B/\partial x^j \partial x^k$. We finally obtain the relation between ξ^μ and A_ν as

$$\frac{\partial \xi^k}{\partial x^\mu} F_{\mu j} - \frac{\partial \xi^j}{\partial x^\mu} F_{\mu k} = \xi^\mu \frac{\partial}{\partial x^\mu} F_{kj}, \tag{23}$$

where

$$F_{\mu j} = \frac{\partial A_\mu}{\partial x^j} - \frac{\partial A_j}{\partial x^\mu} \tag{24}$$

are the electric and magnetic field intensities. Equation (23) is the master equation which leads to inter-relation between the field intensities for the invariance. It is important to note that the final constraints are only on the electric (E) and magnetic (H) field intensities.

5. The relations between electric and magnetic fields

Since in the expressions for ξ^k the constants Γ^k_j and T^k are all independent parameters, we can investigate their roles individually. It may be pointed out that eq. (23) is linear and homogeneous in ξ^js. Hence the relations are independent of the parameters of the transformation.

5.1 *The constraints on E and H due to* Γ^p_q

We first take one of the $\Gamma^p_q \neq 0$ and then fix suitable pairs of (x^k, x^j).

5.1.1 Let both indices be spatial ones. $(\alpha)\Gamma^x_y \neq 0$. With the choice of pairs of (x^k, x^j) as stated below, eq. (23) gives

$$(x,y): \frac{\partial H_z}{\partial \varphi_z} = 0, \tag{25a}$$

$$(y,z): H_y - \frac{\partial H_x}{\partial \varphi_z} = 0, \tag{25b}$$

$$(z,x): H_x + \frac{\partial H_y}{\partial \varphi_z} = 0. \tag{25c}$$

Here and in the sequel φ_x, φ_y and φ_z are azimuthal angles about x, y and z-axis respectively.

$$(x,x_0): E_y - \frac{\partial E_x}{\partial \varphi_z} = 0, \tag{26a}$$

$$(y, x_0) : E_x + \frac{\partial E_y}{\partial \varphi_z} = 0, \tag{26b}$$

$$(z, x_0) : \frac{\partial E_z}{\partial \varphi_z} = 0. \tag{26c}$$

These equations show the relations between different components of magnetic field and those of electric field among themselves under rotation about the *z*-axis (see Appendix).

For $\Gamma_z^y \neq 0$ and $\Gamma_x^z \neq 0$, one obtains similarly six relations each with cyclic permutations of *x*, *y* and *z*.

5.1.2 Let one of the indices be time and the other, space. $(\beta)\Gamma_\tau^x \neq 0$, $(\tau = ct)$.

With the choice of pairs of (x^k, x^j), eq. (23) gives

$$(x, \tau) : \frac{\partial E_x}{\partial \chi_x} = 0, \tag{27a}$$

$$(y, \tau) : H_z + \frac{\partial E_y}{\partial \chi_x} = 0, \tag{27b}$$

$$(z, \tau) : H_y - \frac{\partial E_z}{\partial \chi_x} = 0, \tag{27c}$$

$(\tanh \chi_q = x^q / et,\ x^q = (x, y, z))$

$$(x, y) : E_y + \frac{\partial H_z}{\partial \chi_x} = 0, \tag{28a}$$

$$(y, z) : E_z - \frac{\partial H_y}{\partial \chi_x} = 0, \tag{28b}$$

$$(y, z) : \frac{\partial H_x}{\partial \chi_x} = 0. \tag{28c}$$

These equations show the inter-relation between the components of the electric and magnetic fields, under Lorentz transformation (see Appendix II) with velocity along the *x*-direction.

For $\Gamma_\tau^y \neq 0$ and $\Gamma_\tau^z \neq 0$, one obtains similarly six relations each with cyclic permutations of x, y and z.

5.1.3 $T^p \neq 0$, eq. (23) leads to

$$\frac{\partial}{\partial x^p} F_{kj} = 0. \tag{29}$$

They express translation invariance of the electric and magnetic fields in the respective directions.

6. Discussion

The above analysis shows that the KG equation with the introduction of the electromagnetic field may admit the same invariance properties as that of the field free equation namely the Poincaré group consisting of translation, rotation and Lorentz transformation. This result is quite expected from the structure of the equation. But an important point to emphasize is that exhaustive investigation of infinitesimal transformation shows that there are no other groups. Further, it is quite natural that the invariances are restricted by the exact nature of the electromagnetic field which has been worked out in detail.

Appendix I

With the non-vanishing $\Gamma^x_y \equiv \gamma$ (say) from eq. (3) for infinitesimal transformation

$$\delta_x = \varepsilon\gamma y \quad \text{and} \quad \delta_y = -\varepsilon\gamma x. \tag{A1}$$

Again from the operator on the right-hand side of eq. (23)

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) x = y, \tag{A2a}$$

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) x = -x \tag{A2b}$$

leads to the same infinitesimal changes as above. Introducing a finite parameter θ from the infinitesimal one $\varepsilon\gamma$ as defined by

$$\theta = \varepsilon\gamma n,$$

where n is a very large positive integer, we get

$$\bar{x}_1 = \left(x + \frac{\theta}{n} y \right), \quad \bar{y}_1 = \left(y - \frac{\theta}{n} x \right) \tag{A3}$$

so that

$$(\bar{x} + i\bar{y})_1 = \left(1 - \frac{i\theta}{n} \right)(x + iy). \tag{A4}$$

The subscript indicates transformed x and y after the first stage. Hence after n stages,

$$(\bar{x}+i\bar{y})_n = \left(1-\frac{i\theta}{n}\right)^n (x+iy). \tag{A5}$$

The finite change is obtained as $n \to \infty$. Thus

$$\bar{x}+i\bar{y} = \mathrm{e}^{-i\theta}(x+iy) \tag{A6}$$

leading to

$$\bar{x} = x\cos\theta + y\sin\theta,$$

$$\bar{y} = -x\sin\theta + y\cos\theta \tag{A7}$$

which is the expression for rotation about the *z*-axis.

Appendix II

With the non-vanishing $\Gamma^x_\tau = \gamma'$ (say) from eq. (3)

$$\delta x = \varepsilon\gamma'\tau \quad \text{and} \quad \delta\tau = \varepsilon\gamma' x. \tag{A8}$$

As before from the right-hand side of eq. (23),

$$\left(\tau\frac{\partial}{\partial x} + x\frac{\partial}{\partial\tau}\right)x = \tau \tag{A9a}$$

and

$$\left(\tau\frac{\partial}{\partial x} + x\frac{\partial}{\partial\tau}\right)\tau = x \tag{A9b}$$

leads to the same infinitesimal changes. Introducing a finite parameter χ from the infinitesimal one $\varepsilon\gamma'$ as defined by

$$\chi = \varepsilon\gamma' m,$$

where m is a large positive integer, one gets

$$\bar{x}_1 = x + \frac{\chi}{m}\tau \quad \text{and} \quad \bar{\tau}_1 = \tau + \frac{\chi}{m}x. \tag{A10}$$

Hence,

$$(\bar{x}\pm\bar{\tau})_1 = \left(1\pm\frac{\chi}{m}\right)(x\pm\tau) \tag{A11}$$

so that for finite change

$$\bar{x}\pm\bar{\tau} = \left(1\pm\frac{\chi}{m}\right)^m (x\pm\tau)(m\to\infty) = \mathrm{e}^{\pm\chi}(x\pm\tau). \tag{A12}$$

Thus

$$\bar{\chi} = x \cosh\chi + \tau \sinh\chi,$$
$$\bar{\tau} = x \sinh\chi + \tau \cosh\chi. \tag{A13}$$

Introducing a new parameter $(v/c) = \tanh\chi$, one gets the usual form of Lorentz transformation with velocity v along the x-direction.

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