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Matrix factorization method for the Hamiltonian structure of integrable systems

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Abstract. We demonstrate that the process of matrix factorization provides a systematic mathematical method to investigate the Hamiltonian structure of non-linear evolution equations characterized by hereditary operators with Nijenhuis property.

Keywords. Matrix factorization; Hamiltonian structure; derivative non-linear Schrödinger equation.

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1. Introduction

The factorization process for matrices has hardly been discussed in the literature for useful application in mathematical physics. However, it has occasionally been noted in the context of numerical analysis that Choleski's method for matrix factorization [1] plays a role in the solution of simultaneous equations as well as calculation of eigenvalues and eigenvectors of matrices. This method can also be used to calculate the inverse of square matrices. Any matrix A can be factorized as

$$A = LU, \tag{1}$$

where L and U are matrices of same dimension as that of A. The matrix L is lower triangular while U is upper triangular. Specializing to a 3×3 square matrix, we write (1) in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}. \tag{2}$$

From (2), it is clear that the elements u_{ij} and l_{km} can be obtained in terms of a_{ap}s. The process is, however, not unique. For example, the non-uniqueness could be demonstrated by simply making different choices for the diagonal elements of L. The object of the present note is to point out the rationale of the matrix factorization method in the study of canonical and/or Hamiltonian structure of integrable non-linear evolution equations.

The Hamiltonian structure of non-linear evolution equations solvable by the inverse spectral method was discovered in 1971 by Zakharov and Faddeev [2] and by Gardner [3] who interpreted the Kortweg-de Vries (KdV) equation as a completely integrable Hamiltonian system in an infinite dimensional phase space with $\partial_x = \partial/\partial x$ as the relevant Hamiltonian operator. Almost simultaneously, hierarchies of infinitely many commuting vector fields and constants of the motion in evolution for the KdV equation were constructed by Lax [4] and by Gel'fand and Dikii [5] using the equation for squared eigenfunction of the Schrödinger operator. Similar hierarchies were also obtained for other important non-linear evolution equations by dealing with squared eigenfunction equation of the Dirac operator [6]. In any case, the squared eigenfunction operator could be interpreted as an operator generating higher symmetries [7] so as to have the name hereditary recursion operator. Discovery of the recursion operator initiated further important development in the Hamiltonian theory. For example, Magri [8] realized that integrable Hamiltonian systems have an additional structure, namely, they are bi-Hamiltonian systems. This implies that they are Hamiltonian with respect to two different compatible Hamiltonian operators. More precisely, if a hereditary operator Φ can be factorized in terms of the Hamiltonian operators J_1 and J_0 as

$$\Phi = J_1 J_0^{-1}, \tag{3}$$

then the associated evolution equation is Hamiltonian with respect to both J_1 and J_0. The operators may possess Nijenhuis property [9]. The Nijenhuis operators are non-local and non-locality of Φ often poses problem to factorize it in the form (3).

In two recent papers, Ma [10] and Zhou [11] considered the Hamiltonian formulation of the coupled KdV and Kaup–Newell systems of derivative non-linear Schrödinger (DNLS) equations. The KdV systems have been extensively discussed in the literature and the KdV-like equations appear in a wide variety of physical context. The DNLS equation was found by Kaup and Newell [12] by slightly modifying the scattering problem of Zakharov and Shabat [13] and that of [6]. This equation could account for the propagation of circularly polarized Alfven waves in plasma. Looking closely into the works in [10] and [11], we observe that the authors do not use any systematic mathematical method for their development, rather they proceed by making use of their personal experience in dealing with similar problems. To bring some order into the situation we demonstrate below that the process of matrix factorization plays a central role both in constructing recursion operators and in deriving the corresponding Hamiltonian hierarchies [10,11].

2. Matrix factorization and Hamiltonian structure of DNLS equations

To construct the recursion operator in [10], Ma proceeds by assuming two specific forms for the matrix differential operators J and M such that $\Phi = MJ^{-1}$. We note that both J and M are lower triangular matrices and MJ^{-1} has the form of (2) giving the standard formula for matrix factorization. Further, since the elements of both matrices are simple differential operators rather than integro-differential ones, there were no problems to determine the coefficients multiplying the elements. The bi-Hamiltonian structure could also be derived easily. The situation was slightly more complicated for the recursion operator in [11] where the authors had chosen to work with the DNLS system. In this case, Φ is a matrix operator whose elements are integro-differential operators. The specific form of Φ is given by

$$\Phi = \begin{pmatrix} \frac{1}{2}\partial - \frac{1}{2}\partial q\partial^{-1}r & -\frac{1}{2}\partial q\partial^{-1}q \\ -\frac{1}{2}\partial r\partial^{-1}r & -\frac{1}{2}\partial - \frac{1}{2}\partial r\partial^{-1}q \end{pmatrix}. \tag{4}$$

The Nijenhuis property of (4) tends to pose a problem to factorize it in the form (3). However, it is of interest to note that the Φ in (4) can be factorized in the form

$$\Phi = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \left(-\frac{1}{2}\right) \begin{pmatrix} r\partial^{-1}r & 1+r\partial^{-1}q \\ -1+q\partial^{-1}r & q\partial^{-1}q \end{pmatrix} \tag{5}$$

giving

$$J_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{6}$$

and

$$J_0^{-1} = -\frac{1}{2} \begin{pmatrix} r\partial^{-1}r & 1+r\partial^{-1}q \\ -1+q\partial^{-1}r & q\partial^{-1}q \end{pmatrix}. \tag{7}$$

The operator can also be factorized as

$$\Phi = J_2 J_1^{-1}, \tag{8}$$

where J_2 is given by

$$J_2 = \begin{pmatrix} -\frac{1}{2}\partial q\partial_{-1}q\partial & \frac{1}{2}\partial^2 - \frac{1}{2}\partial q\partial^{-1}r\partial \\ \frac{1}{2}\partial^2 - \frac{1}{2}\partial r\partial^{-1}q\partial & -\frac{1}{2}\partial r\partial^{-1}r\partial \end{pmatrix}. \tag{9}$$

The non-local nature of elements in Φ does not allow one to write it in any other factorizable form. Thus J_0, J_1 and J_2 form a Hamiltonian triplet for the DNLS equations giving a tri-Hamiltonian–Lax hierarchy.

In the above context, it remains a problem to compute J_0 from J_0^{-1}. This can be done by noting the identity $J_0 J_0^{-1} = I$ and assuming $J_0 = C_{ij}$. We thus write

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} r\partial^{-1}r & 1+r\partial^{-1}q \\ -1+q\partial^{-1}r & q\partial^{-1}q \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{10}$$

From (10) we get

$$C_{11}r\partial^{-1}r + C_{12}(-1+q\partial^{-1}r) = -2, \tag{11}$$

$$C_{11}(1+r\partial^{-1}q) + C_{12}q\partial^{-1}q = 0, \tag{12}$$

$$C_{21}r\partial^{-1}r + C_{22}(-1+q\partial^{-1}r) = 0, \tag{13}$$

and

$$C_{21}(1+r\partial^{-1}q) + C_{22}q\partial^{-1}q = -2. \tag{14}$$

It is important to note that C_{ij}s are partially decoupled in the set of equation from (11)–(14). For example, C_{11} and C_{12} can be obtained from (11) and (12) only. However, in solving them one must take proper care for the non-locality involved. To that end we re-arrange (11) and (12) to write

$$(C_{11}r+C_{12}q)\partial^{-1}r-C_{12}=-2, \tag{15}$$

and

$$(C_{11}r+C_{12}q)\partial^{-1}q+C_{11}=0. \tag{16}$$

Multiplying (15) and (16) by q and r respectively from the right and subtracting we find

$$C_{11}r+C_{12}q=2q. \tag{17}$$

From (15) and (17) we get

$$C_{12}=2+2q\partial^{-1}r. \tag{18}$$

Again using (17) in (16) we have $C_{11}=-2q\partial^{-1}q$. Similar considerations also apply for (13) and (14), and we find all the elements of J_0. Thus construction of the inverse operator for the DNLS equations as carried out by Ma and Zhou [11] is now in order.

3. Conclusion

We conclude by noting that in deriving the bi-Hamiltonian formulation for a coupled KdV system, Ma [10] has chosen to work with two specific forms for the matrix differential operators J and M such that MJ^{-1} appears in the form LU. The process of matrix factorization is, therefore, implicit in the canonical structure of coupled systems. Referring to the work of Koup–Newell hierarchy [11] we observe that the hereditary recursion operator in (4) was found by Ma *et al* [14] only a few years ago. We have rederived the tri-Hamiltonian structure and reconstructed the inverse operator Φ^{-1} for the system using a strict mathematical procedure involved in the matrix factorization method. The merit of the present approach is that it does not rely on additional intuitive assumptions for the Hamiltonian operators.

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