PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 61, No. 1
July 2003
pp. 153–159

Higher dimensional homogeneous cosmology in Lyra geometry

F RAHAMAN[1,2], S DAS[1], N BEGUM[1] and M HOSSAIN[1]
[1]Department of Mathematics, Jadavpur University, Kolkata 700 032, India
[2]Khodar Bazar, 24 Parganas (South), Baruipur 743 302, India
Email: farook_ rahaman@yahoo.com

MS received 24 June 2002; revised 16 December 2002; accepted 30 January 2003

Abstract. Assuming a homogeneous perfect fluid with $\rho = \rho(t)$ and $p = p(t)$, we have obtained exact solutions for cosmological models in higher-dimension based on Lyra geometry. Depending on the form of metric chosen, the model is similar to FRW type. The explicit solutions of the scale factor are found via the assumption of an equation of state $p = m\rho$, where m is a constant. Some astrophysical parameters are also calculated.

Keywords. Higher dimensional cosmology; Lyra geometry; homogeneous perfect fluid.

PACS Nos 98.80.Cq; 04.20.Jb; 04.50.+h

1. Introduction

The idea of higher dimensional theory was originated in super string and super gravity theories with the other fundamental forces in nature. Solutions of Einstein field equations in higher dimensional space-times are believed to be of physical relevance possibly at the extremely early times before the Universe underwent compactification transitions [1,2]. It is argued that the extra dimensions are observable at the present time, owing to their size being assumed to be of the order of the Planck length, but may perhaps be relevant for the very early Universe [3].

While attempting to unify gravitation and electromagnetism in a single space-time geometry, Weyl [4] showed how one can introduce a vector field with an intrinsic geometrical significance. But this theory was not accepted as it was based on non-integrability of length transfer. Lyra [5] proposed a new modification of Riemannian geometry by introducing a gauge function which removes the non-integrability condition of a vector under parallel transport.

In consecutive investigations, Sen [5] and Sen and Dunn [5] proposed a new scalar tensor theory of gravitation and constructed an analog of Einstein's field equations based on Lyra's geometry which in normal gauge may be written as

$$R_{ab} - \frac{1}{2} g_{ab} R + \left(\frac{3}{2}\right) \varphi_a \varphi_b - \frac{3}{4} g_{ab} \varphi_c \varphi^c = -\chi T_{ab}, \tag{1}$$

where ϕ_a is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

According to Halford [6], the present theory predicts the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearized form of field equations. Soleng [7] has pointed out that the constant displacement field in Lyra's geometry will either include a creation field and be equal to Hoyle's creation field cosmology or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term.

Subsequent investigations were done by several authors in scalar tensor theory and cosmology within the frame work of Lyra geometry [8].

All the above discussions are the general relativistic four-dimensional models.

It is well-known that our Universe was much small in its early stage than it is today. Indeed the present four-dimensional stage of the Universe could have been preceded by a higher dimensional stage, which at later times becomes effectively four-dimensional in the sense that the extra dimensions become unobservably small due to dynamical contraction [1,2]. Therefore it is interesting to study higher dimensional cosmological model based on Lyra geometry as the higher dimensional concept is important at the early stage of the Universe.

In this paper we shall consider higher dimensional spherically symmetric perfect fluid model based on Lyra geometry.

2. The basic equations

The time-like displacement vector in (1) is taken as

$$\phi_a = (\beta(t), 0, 0, 0, 0). \tag{2}$$

We consider a spherically symmetric in 5-dimensional space-time with topology of 4-space is S^1XS^3 as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{d}r^2 + b^2\mathrm{d}\Omega_3^2, \tag{3}$$

where $a = a(r,t), b = b(r,t)$ and $\mathrm{d}\Omega_3^2$ is the metric on unit three sphere. The field equations (1) for the metric (3) are

$$\begin{aligned} &3a^\bullet b^\bullet/ab + 3b^{\bullet 2}/b^2 + 3/b^2 \\ &\qquad - (3/a^2)(-a'b'/ab + b'^2/b^2 + b''/b) = \chi\rho + \frac{3}{4}\beta^2, \end{aligned} \tag{4}$$

$$3b^{\bullet\bullet}/b + 3b^{\bullet 2}/b^2 + 3/b^2 - (3/a^2)(b'^2/b^2) = -\chi p - \frac{3}{4}\beta^2, \tag{5}$$

$$\begin{aligned} &2a^\bullet b^\bullet/ab + b^{\bullet 2}/b^2 + 1/b^2 + 2b^{\bullet\bullet}/b + a^{\bullet\bullet}/a \\ &\qquad - (1/a^2)(-2a'b'/ab + b'^2/b^2 + 2b''/b) = -\chi p - \frac{3}{4}\beta^2, \end{aligned} \tag{6}$$

$$2b^{\bullet\prime}/b - 2a^\bullet b'/ab = 0. \tag{7}$$

The energy conservation equation is

$$\chi\rho^\bullet + (3/2)\beta^\bullet\beta + [\chi(\rho+p) + (3/2)\beta^2][a^\bullet/a + 3b^\bullet/b] = 0 \tag{8}$$

(dot and prime denote the differentiation with respect to t and r respectively). We assume the equation of state as

$$p = m\rho \qquad (0 \le m \le 1). \tag{9}$$

3. Solutions to the field equations

Equation (7) can easily be integrated to obtain

$$a(r,t) = f(r)b', \tag{10}$$

where $f(r)$ is an arbitrary function of r alone. Now subtracting (6) from (5), we get

$$\begin{aligned} &-a^\bullet b^\bullet/ab + 2b^{\bullet 2}/b^2 + 2/b^2 + b^{\bullet\bullet}/b \\ &\qquad - a^{\bullet\bullet}/a = (1/a^2)(2a'b'/ab + 2b'^2/b^2 - 2b''/b). \end{aligned} \tag{11}$$

Multiplying eq. (4) by m and adding with eq. (5), we get

$$\begin{aligned} &[a^\bullet b^\bullet/(m+1)ab] + b^{\bullet 2}/b^2 + 1/b^2 + [b^{\bullet\bullet}/(m+1)b] = (1/a^2) \\ &\quad \times[-\{ma'b'/(m+1)ab\} + b'^2/b^2 + \{mb''/(m+1)b\}] + \frac{3}{4}(m-1)\beta^2. \end{aligned} \tag{12}$$

From the above two equations, we get

$$\begin{aligned} &-[\{(2m+1)/((m+1)\}(a^\bullet b^\bullet/ab)] - a^{\bullet\bullet}/2a + [\{(m-1)/2(m+1)\}b^{\bullet\bullet}/b] \\ &\qquad + \frac{3}{4}\{(m-1)/(m+1)\}\beta^2 = (1/a^2)[\{(2m+1)/(m+1)\}] \\ &\qquad \times[a'b'/ab - b''/b]. \end{aligned} \tag{13}$$

This partial differential equation can only be solved using separation of variables.

So we write,

$$b(r,t) = b_1(r)b_2(t). \tag{14}$$

And consequently

$$a(r,t) = f(r)b_1'b_2 = a_1(r)b_2(t), \tag{15}$$

where

$$a_1(r) = f(r)b_1'. \tag{16}$$

Now from eq. (13), we obtain (using eqs (14) and (15))

$$-[\{1/(m+1)\}b_2^{\bullet\bullet}b_2] + \frac{3}{4}\{(m-1)/(m+1)\}\beta^2 b_2^2 - \{(2m+1)/(m+1)\}b_2^{\bullet 2}$$
$$= (1/a_1^2)[\{(2m+1)/(m+1)\}][a_1' b_1'/a_1 b_1 - b_1''/b_1] = k_1. \tag{17}$$

(k_1 is a separation constant).
From eq. (17), we get

$$a_1' b_1'/a_1 b_1 - b_1''/b_1 = \{(m+1)/(2m+1)\}a_1^2 k_1. \tag{18}$$

This implies

$$z' + 2[b_1''/b_1]z = \{(m+1)/(2m+1)\}k_1, \tag{19}$$

where

$$z = -(1/a_1^2). \tag{20}$$

Hence we get

$$a_1^2 = b_1'^2[F - \{(m+1)/(2m+1)\}b_1^2 k_1]^{-1/2}. \tag{21}$$

(F is an integration constant).
Using eq. (21) in eq. (16), we get

$$f^2 = 1/[F\{1 - kb_1^2\}]. \tag{22}$$

(where $k = \{k_1(m+1)/F(2m+1)\}$).
If we introduce a new radial variable $\rho = b_1$, then the metric (3) can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (b_2^2/F)[(1-k\rho^2)^{-1}\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_3^2]. \tag{23}$$

Thus we have the five-dimensional well-known FRW type metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)[(1-kr^2)^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2]. \tag{24}$$

4. Cosmological implications of our model

In this section we shall discuss the cosmological dynamics of our model in Lyra geometry. Using the metric (24), we get only two independent field equations:

$$6(R^{\bullet 2} + k)R^{-2} = \chi\rho + \frac{3}{4}\beta^2, \tag{25}$$

$$-R^{\bullet\bullet}/R - 3(R^{\bullet 2} + k)R^{-2} = \chi p - \frac{3}{4}\beta^2. \tag{26}$$

Now we shall consider the dynamical behavior of different models with constant displacement vector i.e. $\beta =$ constant.

Case I: *Empty universe*

In this case

$$\rho = p = 0. \tag{27}$$

Here we get,

$$R = a\cosh(1/2\sqrt{2})\beta t, \tag{28}$$

where $a^2 = 8k/\beta^2$. We see that this is a singularity free model. Here the deceleration parameter is

$$q = -RR^{\bullet\bullet}/R^{\bullet 2} = -\cosh^2(1/2\sqrt{2})\beta t. \tag{29}$$

The Hubble function is

$$H = R^{\bullet}/R = (1/2\sqrt{2})\beta\tanh(1/2\sqrt{2})\beta. \tag{30}$$

Case II: *Matter-filled universe*

From eqs (25), (26) and (9) we obtain, after some mathematical manipulation, a first integral

$$R^{\bullet 2} = (1/8)\beta^2 R^2 - k + DR^{-(4m+2)}. \tag{31}$$

(D is an integration constant).

Hence the integral form of R is

$$\int [(1/8)\beta^2 R^2 - k + DR^{-(4m+2)}]^{-1/2}\mathrm{d}R = \pm(t - t_0), \tag{32}$$

where t_0 is another integration constant.

(a) *Dust case* $(m = 0)$: In this case, R takes the following form

$$R^2 = (4/\beta^2)A(t), \tag{33}$$

where

$$A(t) = [k + \sqrt{((1/2)D\beta^2 - k^2)}\sinh(\beta/\sqrt{2})(t - t_0)].$$

Also we get

$$\begin{aligned}\chi\rho = {} & (6\beta^2/8)((1/2)D\beta^2 - k^2)\cosh(\beta/\sqrt{2})(t - t_0)[A(t)]^{-2} \\ & + (6k\beta^2/4)[A(t)]^{-1} - (3/4)\beta^2,\end{aligned} \tag{34}$$

$$H = (1/2\sqrt{2})\beta\sqrt{((1/2)D\beta^2 - k^2)}\cosh(\beta/\sqrt{2})(t - t_0)[A(t)]^{-1}, \tag{35}$$

$$q = 1 - [2/\beta\sqrt{((1/2)D\beta^2 - k^2)}]A(t)\tanh(\beta/\sqrt{2})(t-t_0)\mathrm{sech}(\beta/\sqrt{2})(t-t_0). \tag{36}$$

(b) *Zeldovich fluid*: In this case

$$m = -1 \quad \text{and} \quad p = -\rho. \tag{37}$$

Here we get

$$R = a\cosh(\sqrt{k}/a)(t-t_0), \tag{38}$$

where $a^2 = [k/\{D + (1/8)\beta^2\}]$,

$$\chi\rho = -\chi\rho = (6k/a^2) - (3/4)\beta^2. \tag{39}$$

Here we see that ρ and p should remain separately constant. Here we also see that the space-time is singularity free. Here

$$q = -\coth^2\sqrt{\{D + (1/8)\beta^2\}}(t-t_0), \tag{40}$$

$$H = \sqrt{\{D + (1/8)\beta^2\}}\tanh\sqrt{\{D + (1/8)\beta^2\}}(t-t_0). \tag{41}$$

(c) *Stiff fluid*: In this case $m = -1$ and $p = -\rho$. From eq. (32), we get (here we assume $k = 0$)

$$R^2 = (8/\beta^2)B(t), \tag{42}$$

where $B(t) = [2\beta^2(t-t_0)^2 - D]$. Also we get

$$\chi\rho = \chi p = 6\beta^2(t-t_0)^2[B(t)]^{-2} + (3\beta/\sqrt{2})[B(t)]^{-1/2} - (3/4)\beta^2, \tag{43}$$

$$H = \beta^2(t-t_0)[B(t)]^{-1}, \tag{44}$$

$$q = -3\beta^2(t-t_0) + 4[B(t)](t-t_0)^{-1}. \tag{45}$$

From the above solutions we observe that at the initial epoch $(t-t_0) = \sqrt{(D/2\beta^2)}$. The model starts with an initial singularity $\sqrt{(-g) \to 0}$ while ρ, H diverge. But the deceleration parameter was constant at the initial epoch.

5. Summary

In the present work, we have derived some exact solutions for cosmological models in five dimensions within the framework of Lyra geometry under various symmetry conditions postulating homogeneity in matter distribution.

We have assumed the space-time to be spherically symmetric in 5-dimensions and obtain a unique FRW-like solution. Thus our spherically symmetric cosmological model in Lyra geometry agree with the standard result in general relativity that D-dimensional spherically symmetric metric whose source is a homogeneous matter energy distribution must be FRW type [3]. Later we have utilized the metric form to investigate the cosmological dynamics of the model. In contrast to the general relativity case the 5-D vacuum energy model in Lyra geometry, the space-time is singularity free. For dust case, the solution may be interpreted as representing a bounded distribution of matter.

An interesting situation occurs in the case of $\rho = -p$. Here space-time is non-singular and all the components of the stress tensor become constants conforming to the usual inflationary model.

For stiff fluid model, we see that at the initial epoch $(t - t_0) = (D/2\beta^2)$, the model starts with an initial singularity $\sqrt{(-g) \rightarrow 0}$. Further the scale factors increase, starting from an initial singularity. On the other hand, the pressure has an infinite large value initially and then gradually decreases. Hence the solutions represent an expanding model of the Universe. This result is in agreement with the general relativity case. As our models lie in the assumption of constant displacement vector, so for a future exercise, one should take into account the time-dependent displacement vector and current work is in progress in this direction.

Acknowledgement

One of the authors (FR) is grateful to IUCAA for sending preprints and papers.

References

[1] T Appelquist, A Chodas and P G O Freund, *Modern Kaluza Klein theories* (Addison Wesley, MA, 1987)
S Chatterji, *Ann. Phys.* **218**, 121 (1992) and references therein
[2] A Chodos and S Detweller, *Phys. Rev.* **D21**, 2167 (1980)
[3] W J Marciano, *Phys. Rev. Lett.* **52** 489 (1984)
S Chatterji, *Astro. Astrophys.* **179**, 1 (1987); **230**, 1 (1990)
S Chatterji and B Bhui, *Mon. Not. R. Astron. Soc.* **247**, 57 (1990)
S Chatterji, *Ann. Phys.* **218**, 121 (1992) and references therein
[4] H Weyl and S Ber, *Preuss. Akad. Wiss.* 465 (1918)
[5] G Lyra, *Math. Z.* **54**, 52 (1951)
D K Sen, *Z. Phys.* **149**, 311 (1957)
D K Sen and K A Dunn, *J. Math. Phys.* **12**, 578 (1971)
For brief notes on Lyra's geometry, see also A Beesham, *Aust. J. Phys.* **41**, 833 (1988)
T Singh and G P Singh, *Int. J. Theor. Phys.* **31**, 1433 (1992)
[6] W D Halford, *Aust. J. Phys.* **23**, 863 (1970); *J. Math. Phys.* **13**, 1399 (1972)
[7] H H Soleng, *Gen. Relativ. Gravit.* **19**, 1213 (1987)
[8] K S Bharma, *Aust. J. Phys.* **27**, 541 (1972)
T M Karade and S M Borker, *Gen. Relativ. Gravit.* **9**, 413 (1978)
Shetti and B B Waghmode, *Gen. Relativ. Gravit.* **14**, 823 (1982)
D Reddy and Innaiah, *Ast. Space Sci*. **123**, 49 (1986)
A Beesham, *Ast. Space Sci.* **127**, 189 (1986)
D Reddy and R Venkatteswarlu, *Ast. Space Sci.* **136**, 191 (1987)
T Singh and G P Singh, *J. Math. Phys.* **32**, 2456 (1991); *Il. Nuovo Cimento* **B106**, 617 (1991); *Fortschr. Phys.* **41**, 737 (1993)
G P Singh and K Dwsikan *Pramana – J. Phys.* **49**, 20 (1997)
F Rahaman and Bera, *J. Int. J. Mod. Phys.* **D10**, 729 (2001)
F Rahaman, S Chakraborty and Bera, *J. Int. Mod. Phys.* **D11**, 1501 (2002); *Astrophys. Space Sci.* **281**, 595 (2002)