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Canonical structure of evolution equations with non-linear dispersive terms

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Abstract. The inverse problem of the variational calculus for evolution equations characterized by non-linear dispersive terms is analysed with a view to clarify why such a system does not follow from Lagrangians. Conditions are derived under which one could construct similar equations which admit a Lagrangian representation. It is shown that the system of equations thus obtained can be Hamiltonized by making use of the Dirac's theory of constraints. The specific results presented refer to the third- and fifth-order equations of the so-called distinguished subclass.

Keywords. Evolution equations; non-linear dispersive terms; Lagrangian systems; Hamiltonian structure.

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1. Introduction

The KdV and higher KdV equations [1] are quasilinear in the sense that the dispersive behaviour of the solution of each equation is governed by a linear term giving the order of the equation. The dispersion produced is compensated by non-linear effects resulting in the formation of exponentially localized solitons. In the recent past, Rosenau and Hyman [2] introduced a family of non-linear partial differential equations with non-linear dispersive terms. For brevity, we refer to these equations as fully non-linear evolution (FNE) equations. It was hoped that these might be useful to study formation of patterns in liquids. The solitary wave solutions of the equation in ref. [2] were found to have compact support. That is, they vanish identically outside a finite range. These solutions were given the name compactons. The compactons are robust within their range of existence. However, as opposed to the interaction of solitons in the KdV-like system, the point at which two compactons collide is marked by the birth of a low-amplitude compacton–anticompacton pair.

An awkward analytical constraint for the equations of Rosenau and Hyman is that they do not follow from a Lagrangian density. A non-Lagrangian system does not allow one to carry out a linear stability check [3] as well as to derive a field theory [4] for particles described by compactons. There is still a more fundamental problem. Forming a non-Lagrangian system, these fully non-linear differential equations tend to invalidate the Darboux theorem [5] which states that for $(1+1)$ dimensional cases the Lagrangian always exists. On a very general ground one knows that Helmholtz version of the inverse problem for the calculus of variations [6] provides a useful tool to examine whether a given Newtonian system is Lagrangian. If the Lagrangian exists then one can make use of a homotopy formula to construct an analytical expression for it [7]. However, the Helmholtz solution is somewhat unsatisfactory to determine which systems of differential equations arise from variational principles.

The objective of the present paper is threefold:

(i) We make use of the Helmholtz condition to see why the evolution equations with non-linear dispersive terms are non-Lagrangian.
(ii) Introduce similar equations which follow from a Lagrangian density.
(iii) Hamiltonize the system of equations obtained in (ii) with a view to examine their canonical structure.

We devote §2 to deal with the points in (i) and (ii) and introduce a Lagrangian system of FNEs. We present all the results with particular emphasis on the third-order equation of Rosenau and Hyman [2], which represents the prototypical differential equation of all other evolution equations with non-linear dispersive terms. In §3 we observe that our equations form a higher-order degenerate Lagrangian system [8]. We reduce the action by introducing a set of constraints. This allows us to Hamiltonize the system and study its canonical structure within the framework of Dirac's theory of constraints as used by Olver [9] and by Nutku [10]. In §4 we introduce a class of partial differential equations for which the present method will be applicable. Finally we make some concluding remarks in §5.

2. Lagrangian system of FNE equations

Let $P[u] = P(x, u^{(n)}) \in \mathscr{A}^r$ be an r-tuple of differential functions. The Frechet derivative of P is the differential operator $D_P : \mathscr{A}^q \to \mathscr{A}^r$ and is defined as

$$D_P(Q) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} P[u+\varepsilon Q[u]] \tag{1}$$

for any $Q \in \mathscr{A}^q$. The Helmholtz condition [7] asserts that P is the Euler–Lagrange expression for some variational problem iff D_P is self-adjoint. In the case when self-adjointness is guaranteed, a Lagrangian for P can be explicitly constructed using the homotopy formula

$$\mathscr{L}[u] = \int_0^1 uP[\lambda u]\mathrm{d}\lambda. \tag{2}$$

We begin this section by examining if the condition of self-adjointness holds good for the third-order equation [2] given by

$$u_t + 3u^2 u_x + 6u_x u_{2x} + 2uu_{3x} = 0, \quad u = u(x,t). \tag{3}$$

It is convenient to work with the integral of u,

$$w(x,t) = \int_x^{\infty} \mathrm{d}y \;\; u(y,t) \tag{4}$$

rather than u itself. In terms of this dependent variable, often called the velocity potential, (3) reads

$$w_{xt} = -3w_x^2 w_{2x} + 6w_{2x}w_{3x} + 2w_x w_{4x}. \tag{5}$$

Equivalently

$$w_t + w_x^3 - 2w_{2x}^2 - 2w_x w_{3x} = 0. \tag{6}$$

For (5) we write the Euler–Lagrange expression as

$$E\mathscr{L}[w] = P[w] = 3w_x^2 w_{2x} - 6w_{2x}w_{3x} - 2w_x w_{4x}. \tag{7}$$

From (1) and (7)

$$D_P = 6w_x w_{2x} D_x + 3w_x^2 D_x^2 - 6w_{2x} D_x^3 - 6w_{3x} D_x^2 - 2w_x D_x^4 - 2w_{4x} D_x. \tag{8}$$

To construct the adjoint D_P^* of the above Frechet derivative we rewrite (8) as

$$D_P = \sum_{j=1}^{4} P_j(w) D_j \tag{9}$$

and make use of the definition

$$D_P^* = \sum_{j=1}^{4} (-D)_j P_j(w). \tag{10}$$

This gives

$$D_P^* = 6w_x w_{2x} D_x + 3w_x^2 D_x^2 - 2w_{2x} D_x^3 - 2w_x D_x^4 \;\; . \tag{11}$$

Here D_x stands for $\partial/\partial x$. Clearly $D_P \neq D_P^*$ verifying that the Frechet derivative of $P(w)$ is non-self-adjoint. Thus eq. (3) does not have an analytic representation [11] to follow from a Lagrangian density and the variational structure of the system will remain undiscovered. Understandably, use of the homotopy formula does not have any relevance in the present context and we cannot proceed further to construct Lagrangian system of FNE equations. We now show that the usual Hamilton's variational principle supplemented by a straightforward dimensional analysis can be used to construct such equations.

Multiplying (6) by δw_x and integrating over t and x we write

$$\int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{d}x \left[w_t + w_x^3 - 2w_{2x}^2 - 2w_x w_{3x}\right] \delta w_x = 0. \tag{12}$$

The first and second terms in the square bracket of (12) can be put in the variational form as

$$\frac{1}{2}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \; w_t w_x \quad \text{and} \quad \frac{1}{4}\delta \int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \; w_x^4 \tag{13}$$

respectively, while the others cannot be put in the variational form. Equation (3) belongs to a subclass [12] for which w has a dimension [L^{-1}]. Thus it is evident from (13) that each term of the Lagrangian density for (6) has the dimension [L^{-8}]. We venture further to suggest that a linear combination of the form $\sum_i \alpha_i w_x^p w_{2x}^q w_{3x}^r$ will take care of the terms which could not be expressed in the variational form. Here the summation is taken over all possible combinations of p, q and r such that

$$2p+3q+4r=8, \quad p,q,r \geq 0 \quad \text{and integers.} \tag{14}$$

The integer programming problem [13] implied in (14) leads to the Lagrangian density

$$\mathscr{L} = \frac{1}{2}w_t w_x + \frac{1}{4}w_x^4 + \alpha_1 w_{3x}^2 + \alpha_2 w_x^2 w_{3x} + \alpha_3 w_x w_{2x}^2. \tag{15}$$

If we now demand that $\mathscr{L}$ in (15) when substituted in the higher-order Euler–Lagrange equation [14]

$$\frac{\delta \mathscr{L}}{\delta w} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial w_t}\right) = 0 \tag{16}$$

with the variational derivative

$$\frac{\delta}{\delta w} = \frac{\partial}{\partial w} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial w_x} + \frac{d^2}{dx^2}\frac{\partial}{\partial w_{2x}} - \cdots \tag{17}$$

should reproduce eq. (5), we immediately arrive at $\alpha_1 = 0$ and a set of inconsistent equations

$$2\alpha_2 - \alpha_3 = -1 \text{ and } 2\alpha_2 - \alpha_3 = -\frac{3}{2}. \tag{18}$$

This implies that the fully non-linear equation (3) or (5) does not form a Lagrangian system. Interestingly, the non-self-adjointness of (3) or (5) has now been reflected in the process of determining the values of Lagrange's multipliers α_is. We postulate that a slightly modified form of (6), namely,

$$w_t + w_x^3 - q w_{2x}^2 - 3p w_x w_{3x} = 0 \tag{19}$$

for arbitrary values of p and q will follow from a Lagrangian provided

$$2\alpha_2 - \alpha_3 = -q \quad \text{and} \quad 2\alpha_2 - \alpha_3 = -\frac{3p}{2}. \tag{20}$$

The choice $p = 2n$ and $q = 3n$ with $n > 0$ and an integer makes equations in (20) consistent and reduces them to a single equation

$$2\alpha_2 - \alpha_3 = -3n. \tag{21}$$

From (15), (19) and (21) we find that the equation

$$w_t + w_x^3 - 3n w_{2x}^2 - 6n w_x w_{3x} = 0 \tag{22}$$

follows from the gauge equivalent Lagrangian densities

$$\mathscr{L} = \frac{1}{2}w_t w_x + \frac{1}{4}w_x^4 + \frac{1}{2}(\alpha_3 - 3n)w_x^2 w_{3x} + \alpha_3 w_x w_{2x}^2 \tag{23}$$

for arbitrary values of α_3. The equation for $u(x,t)$ is obtained from

$$w_{xt} + 3w_x^2 w_{2x} - 12nw_{2x}w_{3x} - 6nw_x w_{4x} = 0. \tag{24}$$

One can easily verify that the Frechet derivative for the Euler–Lagrange expression in (24) is self-adjoint.

3. Hamiltonian formulation

We shall construct the Hamiltonian density corresponding to $\mathscr{L}$ in (23) and examine the Poisson structure of (24). For clarity of presentation we take $\alpha_3 = -3n$ and write $\mathscr{L}$ in the form

$$\mathscr{L} = \frac{1}{2}w_t w_x + \frac{1}{4}w_x^4 - 3nw_x^2 v_x - 3nw_x v^2. \tag{25}$$

This is, however, no loss of generalization since α_3 is still arbitrary. For (25) the constraint used is given by

$$v - w_{2x} = 0. \tag{26}$$

The form of $\mathscr{L}$ in (25) resembles the result of Whitham [15] for the KdV equation. Considering independent variations with respect to w and v one recovers the constraint and also arrives at (22). Thus by introducing an extra variable, namely v, we have converted the original variational problem to a new one. An important virtue of the expression for $\mathscr{L}$ in (25) is that it depends only on the first-order derivatives of the velocity potentials w and v and thereby provides the simplest basis to examine the Hamiltonian structure of our fully non-linear equation. It is important to note that the original $\mathscr{L}$ in (23) involves higher-order space derivatives.

To consider the canonical formulation for eq. (24) we find that the momentum densities

$$\pi_v = 0 \text{ and } \pi_w = \frac{1}{2}w_x \tag{27}$$

corresponding to w and v fields cannot be inverted for the velocities v_t and w_t. Thus our Lagrangian density is degenerate such that we require Dirac's theory of constraints [16] to discuss the complete Hamiltonian structure of our equation. Henceforth we shall work with $\mathscr{L}$ in (25) for $n = 1$ only. Following Dirac, we define the constraints by the weak equations

$$c_1 \approx \pi_v \text{ and } c_2 \approx \pi_w - \frac{1}{2}w_x. \tag{28}$$

The Poisson brackets of the constraints can easily be found as

$$\left[c_1(x), c_1(x')\right] = 0,$$
$$\left[c_1(x), c_2(x')\right] = 0$$

and

$$\left[c_2(x), c_2(x')\right] = -\delta_x(x - x'), \tag{29}$$

using the canonical Poisson brackets of the w and v fields. The results in (29) indicate that constraints are second class. In this theory the total Hamiltonian is given by

$$H = \int \mathscr{H}\, \mathrm{d}x \tag{30}$$

with the Hamiltonian density defined as

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1. \tag{31}$$

Here the free part $\mathscr{H}_0$ is determined by

$$\mathscr{H}_0 = \pi_v v_t + \pi_w w_t - \mathscr{L} \tag{32}$$

while $\mathscr{H}_1$ is a linear combination of the constraints

$$\mathscr{H}_1 = \sum_{i=1}^{2} \beta^i c_i \tag{33}$$

with β^1 and β^2 as undetermined multipliers. From (25) and (32) we have

$$\mathscr{H}_0 = 3w_x^2 v_x + 3w_x v^2 - \frac{1}{4} w_x^4. \tag{34}$$

It is of interest to note that $\mathscr{H}_0$ alone when substituted in the Zakharov–Faddeev–Gardner equation [17] reproduces the non-linear equation (22). This might be one of the reasons why Olver [9] refers to non-linear evolution equations as Hamiltonian systems in non-canonical coordinates.

The Poisson bracket of c_1 and c_2 with H must vanish since consistency demands the constraints to be preserved in time such that their time derivatives are weakly zero. This requirement will determine the values of the multipliers β^1 and β^2 if there are no further constraints in the problems. But we find that

$$\left[c_1, H\right] = -6w_x(w_{2x} - v). \tag{35}$$

The Poisson bracket in (35) cannot be made to vanish by any choice of β^1 and β^2. In view of this, we introduce a secondary constraint

$$c_3 \approx w_{2x} - v. \tag{36}$$

To include the effect of the secondary constraint c_3 in the total Hamiltonian H we modify (33) as

$$\mathscr{H}_1 = \sum_{i=1}^{3} \beta^i c_i \tag{37}$$

with β^3, a new multiplier. The arbitrary linear combination of constraints in (37) is now expected to bring about the correct equations of motion with usual Poisson brackets. We have found that the results for $[c_1,H]$, $[c_2,H]$ and $[c_3,H]$ can now be set equal to zero by choosing the following values for the Lagrange's multipliers

$$\beta^3 = 6w_x(v - w_{2x}),$$

$$\beta^2 = 6w_{2x}^2 + 6w_x w_{3x} - 6w_{2x}v - w_x^3 + 3v^2,$$

and

$$\beta^1 = (6w_{2x}^2 + 6w_x w_{3x} - 6w_{2x}v - w_x^3 + 3v^2)_{xx}. \tag{38}$$

Equations in (38) indicate that c_1, c_2 and c_3 exhaust all constraints of the problem. Thus from (28), (31), (34), (36), (37) and (38) we write the total Hamiltonian density in the form

$$\begin{aligned}\mathscr{H} &= 3w_x^2 v_x - \frac{9}{2} w_x v^2 + \frac{1}{4} w_x^4 + 15 w_x w_{2x} v - 9 w_x w_{2x}^2 - 3w_x^2 w_{3x} \\ &\quad + (6w_{2x}^2 + 6w_x w_{3x} - 6w_{2x}v - w_x^3 + 3v^2)_{xx}\pi_v \\ &\quad + (6w_{2x}^2 + 6w_x w_{3x} - 6w_{2x}v - w_x^3 + 3v^2)\pi_w.\end{aligned} \tag{39}$$

It is straightforward to verify that each Poisson bracket $[\pi_w,H]$ or $[\pi_v,H]$ reproduces the fully non-linear equation (22).

4. Higher-order equations

The analysis presented in §§2 and 3 can easily be extended to deal with non-linear evolution equations of arbitrary order. For example, the class of partial differential equations is given by

$$u_t + \sum_{i=0}^{m} (u^{q+p-i})_{(2i+1)x} = 0 \tag{40}$$

with $q > 1$, $p \geq 1$ can be Hamiltonized using our approach. For (40) to represent an evolution equation with non-linear dispersive term we must have $m < q + p - 1$ with $u^{q+p-m-1}$ as the allowed power of non-linearity. As for the fifth-order non-linear dispersive equation considered by Dey and Khare [3] we found

$$w_t - w_x^4 + 6l w_x w_{2x}^2 - 6n w_{3x}^2 - 8n w_{2x} w_{4x} + 6l w_x^2 w_{3x} - 4n w_x w_{5x} = 0 \tag{41}$$

to follow from a Lagrangian density given by

$$\begin{aligned}\mathscr{L} &= \frac{1}{2} w_t w_x - \frac{1}{5} w_x^5 + \alpha_1 w_x w_{2x} w_{4x} + \alpha_3 w_x^3 w_{3x} + (\alpha_1 - 2\alpha_2 - 2n) w_x w_{3x}^2 \\ &\quad + 3(\alpha_3 - l) w_x^2 w_{2x}^2 + \alpha_2 w_x^2 w_{5x}\end{aligned} \tag{42}$$

with l and $n > 0$ and integers. The coefficients αs can be chosen arbitrarily to generate a family of gauge equivalent Lagrangians. An interesting situation arises for $\alpha_1 = \alpha_2 = \alpha_3 = 0$. In this case, a third-order Lagrangian

$$\mathscr{L} = \frac{1}{2} w_t w_x - \frac{1}{5} w_x^5 - 3l w_x^2 w_{2x}^2 - 2n w_x w_{3x}^2 \tag{43}$$

reproduces the non-linear equation (41) giving an example of order reduction of $\mathscr{L}$ from the choice of Lagrange's multipliers used in our dimensional analysis.

5. Conclusion

In this work we derived an approach to the inverse problem of variational calculus to construct Lagrangian system of fully non-linear evolution equations and studied their canonical structure using the Dirac's theory of constraints. Relatively recently, Faddeev and Jackiw [18] sought a geometrical realization of constrained systems. The novelty of their approach is that it treats all constraints on equal basis such that one does not require to distinguish between primary and secondary and, first and second class constraints. Further, contrary to Dirac's procedure, constraints involving canonical momenta p_t conjugate to q_t (also their continuum analogs) for which velocities q_t occur linearly in the Lagrangian are absent in the Faddeev–Jackiw approach. However the two methods are formally equivalent and lead to the same canonical structure [19].

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