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Thermal state of the general time-dependent harmonic oscillator

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Abstract. Taking advantage of dynamical invariant operator, we derived quantum mechanical solution of general time-dependent harmonic oscillator. The uncertainty relation of the system is always larger than $\hbar/2$ not only in number but also in the thermal state as expected. We used the diagonal elements of density operator satisfying Leouville–von Neumann equation to calculate various expectation values in the thermal state. We applied our theory to a special case which is the forced Caldirola–Kanai oscillator.

Keywords. Time-dependent harmonic oscillator; thermal state; density operator.

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1. Introduction

Vibration may be one of the most dominant physical aspect that we come upon in everyday life [1]. For small oscillation, it can be approximated to the motion of harmonic oscillator. Harmonic oscillator that has time-dependent mass or frequency may be a good example of time-dependent Hamiltonian systems. Although a large number of dynamical systems have been investigated using approximation and perturbation method in the literature [2,3], we confine our concern to the exact quantum solution of the time-dependent system. There are about three kinds of methods to solve the quantum solution of time-dependent harmonic oscillator. These are propagator method [4–6], unitary transformation method [7–9] and invariant operator method [10–16]. We will use invariant operator method and unitary transformation method together to evolve the quantum theory and investigate thermal state of the general time-dependent harmonic oscillator. The time-dependent harmonic oscillator has several applications such as electrical behavior of LC-circuit that has time-dependent parameter [17] and/or RLC-circuit [18], path-integral formulation of real-time finite-temperature field theory [19–21], dissipative quantum tunnelling effect in macroscopic system [22–25] and quantum motion of an ion in a Paul trap [7,26,27].

When a system interacts with environment, its coupling parameters may explicitly depend on time. Even if the system is closed so that it is conserved as a whole, its subsystem may implicitly depend on time through interaction with the remnant of the system. The main purpose of this paper is to evolve the thermal state of the general time-dependent harmonic oscillator. The Liouville–von Neumann equation [28,29] for non-equilibrium dynamics can be applicable to both time-dependent harmonic and unharmonic oscillator. The density operator of the system can be obtained using the wave function satisfying Schrödinger equation and can be used to derive various expectation values of variables in the thermal state.

In §2, we investigate quantum mechanical solution of the general time-dependent harmonic oscillator. The thermal state of the system is discussed in §3 on the basis of Liouville–von Neumann approach. In §4, we will apply our theory for a special case which is the forced Caldirola–Kanai oscillator. Finally, §5 summarizes this paper and concludes about some physical results of the system.

2. Hamiltonian, invariant operator and wave function

The Hamiltonian of general time-dependent harmonic oscillator can be written as

$$\hat{H}(\hat{x},\hat{p},t) = A(t)\hat{p}^2 + B(t)(\hat{x}\hat{p}+\hat{p}\hat{x}) + C(t)\hat{p} + D_2(t)\hat{x}^2 + D_1(t)\hat{x} + D_0(t), \quad (1)$$

where $A(t)-C(t)$ and $D_i(t)$ $(i=0,1,2)$ are time-dependent coefficients. These coefficients are real and differentiable with respect to t and note that $A(t)\neq 0$. The corresponding equation of motion can be derived from Hamilton's equation of motion as

$$\ddot{\hat{x}} - \frac{\dot{A}}{A}\dot{\hat{x}} + \left(\frac{2\dot{A}B}{A} - 2\dot{B} - 4B^2 + 4AD_2\right)\hat{x} + \frac{\dot{A}C}{A} - \dot{C} - 2BC + 2AD_1 = 0. \quad (2)$$

The introduction of invariant operator may save the labor of finding quantum mechanical solution of the system. We let the trial invariant operator as the form

$$\begin{aligned}\hat{I}(t) = {} & \alpha_1(t)[\hat{p}-p_p(t)]^2 + \alpha_2(t)\{[\hat{x}-x_p(t)][\hat{p}-p_p(t)] \\ & +[\hat{p}-p_p(t)][\hat{x}-x_p(t)]\} + \alpha_3(t)[\hat{x}-x_p(t)]^2, \end{aligned} \quad (3)$$

where $\alpha_1(t)-\alpha_3(t)$ are time-variable functions which should be determined afterwards and $x_p(t)$ is a particular solution of the equation of motion in $\hat{x}$ space, (eq. (2)) and $p_p(t)$ is the corresponding particular solution of the equation of motion in $\hat{p}$ space. We can choose the dimension of $\hat{I}(t)$ the same as that of the Hamiltonian. By virtue of its definition, the invariant operator must satisfy the following relation:

$$\frac{\mathrm{d}\hat{I}(t)}{\mathrm{d}t} = \frac{\partial \hat{I}(t)}{\partial t} + \frac{1}{i\hbar}[\hat{I}(t),\ \hat{H}] = 0. \quad (4)$$

Substituting eqs (1) and (3) in the above equation gives

$$\alpha_1(t) = c_1\rho_1^2(t) + c_2\rho_1(t)\rho_2(t) + c_3\rho_2^2(t), \quad (5)$$

$$\begin{aligned}\alpha_2(t) = {} & \frac{1}{4A}\{4[c_1\rho_1^2(t) + c_2\rho_1(t)\rho_2(t) + c_3\rho_2^2(t)]B \\ & -[2c_1\rho_1(t)\dot{\rho}_1(t) + c_2\dot{\rho}_1(t)\rho_2(t) + c_2\dot{\rho}_2(t)\rho_1(t) + 2c_3\rho_2(t)\dot{\rho}_2(t)]\}, \end{aligned} \quad (6)$$

$$\alpha_3(t) = \frac{1}{2A^2}\{\tfrac{1}{2}[c_1\dot{\rho}_1^2(t) + c_2\dot{\rho}_1(t)\dot{\rho}_2(t) + c_3\dot{\rho}_2^2(t)] \\ -B[2c_1\rho_1(t)\dot{\rho}_1(t) + c_2\dot{\rho}_1(t)\rho_2(t) + c_2\dot{\rho}_2(t)\rho_1(t) + 2c_3\rho_2(t)\dot{\rho}_2(t)] \\ +2B^2[c_1\rho_1^2(t) + c_2\rho_1(t)\rho_2(t) + c_3\rho_2^2(t)]\}, \quad (7)$$

where c_1–c_3 are constants and $\ddot{\rho}_{1,2}(t)$ are two independent solutions of the following differential equation

$$\ddot{\rho}_{1,2}(t) - \frac{\dot{A}}{A}\dot{\rho}_{1,2}(t) + \left(\frac{2\dot{A}B}{A} - 2\dot{B} - 4B^2 + 4AD_2\right)\rho_{1,2}(t) = 0. \quad (8)$$

To simplify the invariant operator, we introduce the unitary operator defined as

$$\hat{U}_t = \hat{U}''\hat{U}'\hat{U}, \quad (9)$$

$$\hat{U} = \exp\left(\frac{i}{\hbar}x_p\hat{p}\right)\exp\left(-\frac{i}{\hbar}p_p\hat{x}\right), \quad (10)$$

$$\hat{U}' = \exp\left(i\frac{\alpha_2}{2\alpha_1\hbar}\hat{x}^2\right), \quad (11)$$

$$\hat{U}'' = \exp\left[\frac{i}{4\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})\ln(2\alpha_1)\right]. \quad (12)$$

We can transform the invariant operator using the above operator as

$$\hat{I}' = \hat{U}_t\hat{I}\hat{U}_t^\dagger. \quad (13)$$

Then, $\hat{I}'$ reduces to the following simple form

$$\hat{I}' = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2, \quad (14)$$

where

$$\omega^2 = 4(\alpha_1\alpha_3 - \alpha_2^2) = \frac{1}{4A^2}(\rho_1\dot{\rho}_2 - \dot{\rho}_1\rho_2)^2(4c_1c_3 - c_2^2) = \text{constant}. \quad (15)$$

For convenience, we only discuss the system for $\omega^2 > 0$. Since eq. (14) is the same as that of the Hamiltonian of the simple harmonic oscillator with unit mass, we can introduce the ladder operators defined as

$$\hat{b} = \sqrt{\frac{\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2\omega\hbar}}\hat{p}, \quad (16)$$

$$\hat{b}^\dagger = \sqrt{\frac{\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2\omega\hbar}}\hat{p}. \quad (17)$$

These satisfy the boson commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$. In terms of these operators, eq. (14) can be simplified to

$$\hat{I}' = \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right). \tag{18}$$

The eigenvalue equation for $\hat{I}'$ can be written as

$$\hat{I}'|n'(t)\rangle = \lambda_n|n'(t)\rangle. \tag{19}$$

We can easily identify the eigenstate in $\hat{x}$ space as

$$\langle\hat{x}|n'(t)\rangle = \left(\frac{\omega}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{\omega}{\hbar}}\hat{x}\right)\exp\left(-\frac{\omega}{2\hbar}\hat{x}^2\right). \tag{20}$$

The eigenstate of the untransformed invariant operator can be obtained from

$$\langle\hat{x}|n(t)\rangle = \hat{U}_t^\dagger\langle\hat{x}|n'(t)\rangle. \tag{21}$$

Substituting eqs (9) and (20) into the above equation gives

$$\langle\hat{x}|n(t)\rangle = \left(\frac{\omega}{2\alpha_1\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{\omega}{2\alpha_1\hbar}}(\hat{x}-x_p)\right)\exp\left(\frac{i}{\hbar}p_p\hat{x}\right)$$
$$\times\exp\left[-\frac{1}{2\alpha_1\hbar}\left(\frac{\omega}{2}+i\alpha_2\right)(\hat{x}-x_p)^2\right]. \tag{22}$$

The $\hat{x}$ space Schrödinger solution $\langle\hat{x}|\psi_n\rangle$ of the Hamiltonian, eq. (1), is the same as the eigenstate of $\hat{I}$, except for some time-dependent phase factor, $\varepsilon_n(t)$ [30]

$$\langle\hat{x}|\psi_n(t)\rangle = \exp[i\varepsilon_n(t)]\langle\hat{x}|n(t)\rangle. \tag{23}$$

Inserting the above equation into Schrödinger equation, we derive the relation

$$\hbar\dot{\varepsilon}_n(t) = \langle n(t)|\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)|n(t)\rangle. \tag{24}$$

Using eqs (1), (22) and (24), $\varepsilon_n(t)$ can be obtained as

$$\varepsilon_n(t) = -\omega\left(n+\frac{1}{2}\right)\int_0^t\frac{A(t')}{\alpha_1}\mathrm{d}t' - \frac{1}{\hbar}\int_0^t H_p(x_p(t'),p_p(t'),t')\mathrm{d}t', \tag{25}$$

where $H_p(x_p(t),p_p(t),t)$ is defined as

$$H_p(x_p(t),p_p(t),t) = A(t)\;p_p^2(t) + C(t)p_p(t) - D_2(t)x_p^2(t) + D_0(t). \tag{26}$$

Substituting eq. (25) into (23), we can obtain the exact wave function as

$$\langle\hat{x}|\psi_n(t)\rangle = \langle\hat{x}|n(t)\rangle\exp\left[-i\omega\left(n+\frac{1}{2}\right)\int_0^t\frac{A(t')}{\alpha_1}\mathrm{d}t'\right.$$
$$\left.-\frac{i}{\hbar}\int_0^t H_p(x_p(t'),p_p(t'),t')\mathrm{d}t'\right]. \tag{27}$$

The $\hat{p}$ space wave function is related to the $\hat{x}$ space by the Fourier transformation

$$\langle \hat{p}|\psi_n(t)\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\langle \hat{x}|\psi_n(t)\rangle \exp\left(-i\frac{\hat{p}\hat{x}}{\hbar}\right)\mathrm{d}\hat{x}. \tag{28}$$

Using (28), the above equation can be calculated as

$$\begin{aligned}\langle \hat{p}|\psi_n(t)\rangle = (-i)^n\left(\frac{2\omega\alpha_1}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\left[\frac{(\omega-2i\alpha_2)^n}{(\omega+2i\alpha_2)^{n+1}}\right]^{1/2} \\ \times H_n\left[\sqrt{\frac{2\alpha_1\omega}{\hbar(\omega^2+4\alpha_2^2)}}(\hat{p}-p_p)\right] \\ \times\exp\left[-\frac{i}{\hbar}x_p(\hat{p}-p_p)-\frac{\alpha_1(\hat{p}-p_p)^2}{\hbar(\omega+2i\alpha_2)}\right] \\ \times\exp\left[-i\omega\left(n+\frac{1}{2}\right)\int_0^t\frac{A(t')}{\alpha_1}\mathrm{d}t' \right. \\ \left. -\frac{i}{\hbar}\int_0^t H_p(x_p(t'),p_p(t'),t')\mathrm{d}t'\right].\end{aligned} \tag{29}$$

To express $\hat{I}(t)$ in a simple form, we introduce another ladder operator as

$$\hat{a}(t) = \frac{1}{\sqrt{\omega\hbar\alpha_1}}\left[\left(\frac{\omega}{2}+i\alpha_2\right)(\hat{x}-x_p)+i\alpha_1(\hat{p}-p_p)\right], \tag{30}$$

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{\omega\hbar\alpha_1}}\left[\left(\frac{\omega}{2}-i\alpha_2\right)(\hat{x}-x_p)-i\alpha_1(\hat{p}-p_p)\right]. \tag{31}$$

These operators also satisfy $[\hat{a},\ \hat{a}^\dagger]=1$. In terms of eqs (30) and (31), (3) can be expressed as

$$\hat{I}(t) = \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right). \tag{32}$$

From eqs (30) and (31), we can confirm that the coordinate and the momentum can be expressed as

$$\hat{x} = \sqrt{\frac{\hbar\alpha_1}{\omega}}(\hat{a}+\hat{a}^\dagger)+x_p, \tag{33}$$

$$\hat{p} = i\sqrt{\frac{\hbar}{\alpha_1\omega}}\left[\left(\frac{\omega}{2}+i\alpha_2\right)\hat{a}^\dagger-\left(\frac{\omega}{2}-i\alpha_2\right)\hat{a}\right]+p_p. \tag{34}$$

Using eqs (27), (33) and (34), we can calculate the following expectation values

$$\langle\psi_n|\hat{x}|\psi_n\rangle = x_p, \tag{35}$$

$$\langle\psi_n|\hat{p}|\psi_n\rangle = p_p, \tag{36}$$

$$\langle\psi_n|\hat{x}^2|\psi_n\rangle = \frac{\hbar\alpha_1}{\omega}(2n+1)+x_p^2, \tag{37}$$

$$\langle\psi_n|\hat{p}^2|\psi_n\rangle = \frac{\hbar}{\alpha_1\omega}\left(\frac{\omega^2}{4}+\alpha_2^2\right)(2n+1)+p_p^2, \tag{38}$$

$$\langle\psi_n|(\hat{x}\hat{p}+\hat{p}\hat{x})|\psi_n\rangle = -\frac{2\alpha_2\hbar}{\omega}(2n+1)+2x_pp_p. \tag{39}$$

Then, we can easily identify the uncertainty relation as

$$\begin{aligned}\Delta\hat{x}\Delta\hat{p} &= [\langle\psi_n|\hat{x}^2|\psi_n\rangle-(\langle\psi_n|\hat{x}|\psi_n\rangle)^2]^{1/2}[\langle\psi_n|\hat{p}^2|\psi_n\rangle-(\langle\psi_n|\hat{p}|\psi_n\rangle)^2]^{1/2}\\ &= \frac{\hbar}{\omega}\sqrt{\omega^2+4\alpha_2^2}\left(n+\frac{1}{2}\right).\end{aligned} \tag{40}$$

This is always larger than $\hbar/2$ as expected. The uncertainty relation of q-deformed harmonic oscillator also differs from the uncertainty relation for the simple harmonic oscillator [31].

By performing a similar procedure, we obtain the expectation value of Hamiltonian as

$$\langle\psi_n|\hat{H}|\psi_n\rangle = \frac{\hbar}{\omega}[\alpha_1D_2-2\alpha_2B+\alpha_3A](2n+1)+H_p(x_p(t),p_p(t),t). \tag{41}$$

3. Thermal state

We consider an ensemble of particles that satisfies the given general time-dependent harmonic oscillator motion. Let us assume that these particles conform to the Bose–Einstein distribution function.

Density operator of the system may satisfy Liouville–von Neumann equation as

$$\frac{\partial\hat{\rho}(t)}{\partial t}+\frac{1}{i\hbar}[\hat{\rho}(t),\ \hat{H}]=0. \tag{42}$$

Then, we can express the density operator in $\hat{x}$ space as

$$\hat{\rho}(\hat{x},\hat{x}',t) = \frac{1}{Z(t)}\sum_{n=0}^{\infty}\langle\hat{x}|\psi_n(t)\rangle\exp\left[-\frac{\hbar\omega}{kT}\left(n+\frac{1}{2}\right)\right]\langle\psi_n(t)|\hat{x}'\rangle, \tag{43}$$

where k is the Boltzmann constant and T the temperature of the system at initial time.

The partition function of the system can be given by

$$Z(t) = \sum_{n=0}^{\infty}\langle\psi_n(t)|\mathrm{e}^{-\hat{I}(t)/(kT)}|\psi_n(t)\rangle. \tag{44}$$

Using eq. (27), the partition function, eq. (44) and the density operator, eq. (43) can be calculated as

$$Z(t) = \frac{1}{2\sinh[\hbar\omega/(2kT)]}, \tag{45}$$

$$\hat{\rho}(\hat{x},\hat{x}',t) = \left[\frac{\omega}{2\pi\hbar\alpha_1}\tanh\left(\frac{\hbar\omega}{2kT}\right)\right]^{1/2}\exp\left\{-\frac{i\alpha_2}{2\hbar\alpha_1}[(\hat{x}-x_p)^2-(\hat{x}'-x_p)^2] - \frac{\omega}{8\hbar\alpha_1}\left[(\hat{x}+\hat{x}'-2x_p)^2\tanh\left(\frac{\hbar\omega}{2kT}\right)+(\hat{x}-\hat{x}')^2\coth\left(\frac{\hbar\omega}{2kT}\right)\right]\right\} \times\exp\left[\frac{i}{\hbar}(\hat{x}-\hat{x}')p_p\right]. \tag{46}$$

At high temperature, eq. (46) reduces to

$$\hat{\rho}(\hat{x},\hat{x}',t) \simeq \frac{\omega}{2}\frac{1}{(\alpha_1\pi kT)^{1/2}}\exp\left[\frac{i}{\hbar}(\hat{x}-\hat{x}')p_p\right] \times\exp\left[-\frac{i\alpha_2}{2\hbar\alpha_1}[\hat{x}^2-\hat{x}'^2-2(\hat{x}-\hat{x}')x_p]-\frac{kT}{4\hbar^2\alpha_1}(\hat{x}-\hat{x}')^2\right]. \tag{47}$$

If the difference between $\hat{x}$ and $\hat{x}'$ are sufficiently small compared to $(2\hbar\alpha_1/\alpha_2)^{1/2}$ and $\hbar/p_p$, and $1/(kT)$ approaches zero, the density operator may be simply represented as

$$\hat{\rho}(\hat{x},\hat{x}',t) \simeq \frac{\hbar\omega}{kT}\,\delta(\hat{x}-\hat{x}'). \tag{48}$$

On the other hand, at low temperature, it becomes

$$\hat{\rho}(\hat{x},\hat{x}',t) \simeq \left(\frac{\omega}{2\pi\hbar\alpha_1}\right)^{1/2}\exp\left[\frac{i}{\hbar}p_p(\hat{x}-\hat{x}')\right] \times\exp\left\{-\frac{i\alpha_2}{2\hbar\alpha_1}[\hat{x}^2-\hat{x}'^2-2(\hat{x}-\hat{x}')x_p] -\frac{\omega}{4\hbar\alpha_1}(\hat{x}^2+\hat{x}'^2+2x_p^2-2\hat{x}x_p-2\hat{x}'x_p)\right\}. \tag{49}$$

The diagonal element of the density operator, eq. (46), can be written as

$$f(\hat{x}) = \left[\frac{\omega}{2\pi\hbar\alpha_1}\tanh\left(\frac{\hbar\omega}{2kT}\right)\right]^{1/2}\exp\left[-\frac{\omega}{2\hbar\alpha_1}\tanh\left(\frac{\hbar\omega}{2kT}\right)(\hat{x}-x_p)^2\right]. \tag{50}$$

The above equation represents the probability that the mass of the oscillator reside at $\hat{x}$. As temperature increases, it becomes

$$f(\hat{x}) \simeq \frac{\omega}{2}\frac{1}{(\pi\alpha_1 kT)^{1/2}}\exp\left(-\frac{\omega^2}{4kT\alpha_1}(\hat{x}-x_p)^2\right). \tag{51}$$

Equation (50) can be used to calculate the expectation value in coordinate space as

$$\langle\hat{x}^l\rangle_{\mathrm{T}} = \int_{-\infty}^{\infty}\hat{x}^l f(\hat{x})\mathrm{d}\hat{x}. \tag{52}$$

For example, we can obtain for $l = 1, 2$ as

$$\langle \hat{x} \rangle_{\mathrm{T}} = x_p, \tag{53}$$

$$\langle \hat{x}^2 \rangle_{\mathrm{T}} = \frac{\hbar \alpha_1}{\omega} \coth\left(\frac{\hbar\omega}{2kT}\right) + x_p^2. \tag{54}$$

When considering eq. (44), the expectation value of $\hat{I}$ in the thermal state can be derived from

$$\langle \hat{I} \rangle_{\mathrm{T}} = kT^2 \frac{\partial}{\partial T} \ln Z(t). \tag{55}$$

Making use of eq. (45), the above equation becomes

$$\langle \hat{I} \rangle_{\mathrm{T}} = \frac{1}{2} \hbar\omega \coth\left(\frac{\hbar\omega}{2kT}\right). \tag{56}$$

Using the same procedure in the $\hat{x}$ space, the $\hat{p}$ space representation of the density operator can be obtained as

$$\begin{aligned} \rho(\hat{p}, \hat{p}', t) = {} & \left[\frac{2\alpha_1\omega}{\hbar\pi(\omega^2 + 4\alpha_2^2)} \tanh\left(\frac{\hbar\omega}{2kT}\right)\right]^{1/2} \exp\left[-\frac{i}{\hbar} x_p (\hat{p} - \hat{p}')\right] \\ & \times \exp\left\{\frac{\alpha_1\omega}{\hbar(\omega^2 + 4\alpha_2^2)} \left\{\frac{2i\alpha_2}{\omega} [\hat{p}^2 - \hat{p}'^2 - 2p_p(\hat{p} - \hat{p}')]\right.\right. \\ & \left.\left. - \frac{1}{2}\left[(\hat{p} + \hat{p}' - 2p_p)^2 \tanh\left(\frac{\hbar\omega}{2kT}\right) + (\hat{p} - \hat{p}')^2 \coth\left(\frac{\hbar\omega}{2kT}\right)\right]\right\}\right\}. \end{aligned} \tag{57}$$

At high temperature, the above equation becomes

$$\begin{aligned} \rho(\hat{p}, \hat{p}', t) \simeq {} & \left(\frac{\alpha_1\omega^2}{\pi(\omega^2 + 4\alpha_2^2)kT}\right)^{1/2} \exp\left[-\frac{i}{\hbar} x_p (\hat{p} - \hat{p}')\right] \\ & \times \exp\left\{\frac{\alpha_1}{\hbar(\omega^2 + 4\alpha_2^2)} \left\{2i\alpha_2 [\hat{p}^2 - \hat{p}'^2 - 2p_p(\hat{p} - \hat{p}')]\right.\right. \\ & \left.\left. - \frac{kT}{\hbar} (\hat{p} - \hat{p}')^2\right\}\right\}. \end{aligned} \tag{58}$$

On the other hand, at low temperature it can be expressed as

$$\begin{aligned} \rho(\hat{p}, \hat{p}', t) \simeq {} & \left(\frac{2\alpha_1\omega}{\hbar\pi(\omega^2 + 4\alpha_2^2)}\right)^{1/2} \exp\left[-\frac{i}{\hbar} x_p (\hat{p} - \hat{p}')\right] \\ & \times \exp\left\{\frac{\alpha_1\omega}{\hbar(\omega^2 + 4\alpha_2^2)} \left\{\frac{2i\alpha_2}{\omega} [\hat{p}^2 - \hat{p}'^2 - 2p_p(\hat{p} - \hat{p}')]\right.\right. \\ & \left.\left. - (\hat{p}^2 + \hat{p}'^2 - 2\hat{p}p_p - 2\hat{p}'p_p + 2p_p^2)\right\}\right\}. \end{aligned} \tag{59}$$

The probability that the mass of the oscillator resides at $\hat{p}$ is obtained taking the diagonal elements of eq. (57) as

$$f(\hat{p}) = \left[\frac{2\alpha_1\omega}{\hbar\pi(\omega^2+4\alpha_2^2)}\tanh\left(\frac{\hbar\omega}{2kT}\right)\right]^{1/2} \times \exp\left[-\frac{2\alpha_1\omega}{\hbar(\omega^2+4\alpha_2^2)}\tanh\left(\frac{\hbar\omega}{2kT}\right)(\hat{p}-p_p)^2\right]. \quad (60)$$

Using eq. (60), we can calculate the expectation value in $\hat{p}$ space as

$$\langle\hat{p}\rangle_{\mathrm{T}} = p_p, \quad (61)$$

$$\langle\hat{p}^2\rangle_{\mathrm{T}} = \frac{\hbar(\omega^2+4\alpha_2^2)}{4\alpha_1\omega}\coth\left(\frac{\hbar\omega}{2kT}\right)+p_p^2. \quad (62)$$

We can also write the expectation value of $\hat{I}$ in the thermal state as

$$\langle\hat{I}\rangle_{\mathrm{T}} = \alpha_1(\langle\hat{p}^2\rangle_{\mathrm{T}}-p_p^2)+2\alpha_2(\langle\hat{x}\hat{p}\rangle_{\mathrm{T}}-x_pp_p)+\alpha_3(\langle\hat{x}^2\rangle_{\mathrm{T}}-x_p^2). \quad (63)$$

Substituting eqs (54), (56) and (62) into the above equation gives

$$\langle\hat{x}\hat{p}\rangle_{\mathrm{T}} = \frac{\hbar\omega}{2\alpha_2}\left[\frac{1}{4}-\frac{1}{\omega^2}(\alpha_2^2+\alpha_1\alpha_3)\right]\coth\left(\frac{\hbar\omega}{2kT}\right)+x_pp_p. \quad (64)$$

Then, the expectation value of the Hamiltonian, eq. (1), can be calculated as

$$\langle\hat{H}\rangle_{\mathrm{T}} = \frac{\hbar}{\omega}(\alpha_3A-2\alpha_2B+\alpha_1D_2)\coth\left(\frac{\hbar\omega}{2kT}\right)+H_p(x_p(t),p_p(t),t) \quad (65)$$

and the uncertainty relation in thermal state can be calculated as

$$(\Delta\hat{x}\Delta\hat{p})_{\mathrm{T}} = [(\langle\hat{x}^2\rangle_{\mathrm{T}}-\langle\hat{x}\rangle_{\mathrm{T}}^2)(\langle\hat{p}^2\rangle_{\mathrm{T}}-\langle\hat{p}\rangle_{\mathrm{T}}^2)]^{1/2} = \frac{\hbar}{2\omega}\sqrt{\omega^2+4\alpha_2^2}\coth\left(\frac{\hbar\omega}{2kT}\right). \quad (66)$$

By comparing the above equation with eq. (40), we can confirm that the uncertainty relation in thermal state varies as time goes by, with the same fashion in number state.

4. Forced Caldirola–Kanai oscillator

We can apply our theory to various kinds of time-dependent Hamiltonian systems. As an example, let us see for the forced Caldirola–Kanai oscillator [32,33]. For this system, the time-dependent coefficients in eq. (1) are given by

$$A(t) = \frac{1}{2m}\mathrm{e}^{-\beta t}, \quad (67)$$

$$D_2(t) = \frac{1}{2}m\omega_0^2\mathrm{e}^{\beta t}, \quad (68)$$

$$D_1(t) = -F(t)\mathrm{e}^{\beta t}, \quad (69)$$

$$B(t) = C(t) = D_0(t) = 0, \quad (70)$$

so that we can rewrite the Hamiltonian as

$$\hat{H} = \mathrm{e}^{-\beta t}\frac{\hat{p}^2}{2m} + \mathrm{e}^{\beta t}\frac{1}{2}m\omega_0^2\hat{x}^2 - \mathrm{e}^{\beta t}F(t)\hat{x}, \tag{71}$$

where m is the mass, β the damping constant and $F(t)$ the arbitrary time-dependent driving force. Equation (8) becomes

$$\ddot{\rho}_{1,2} + \beta\dot{\rho}_{1,2} + \omega_0^2\rho_{1,2} = 0. \tag{72}$$

The two classical solutions of the above equation can be written as

$$\rho_1(t) = \rho_1(0)\mathrm{e}^{-\beta t/2}\mathrm{e}^{i\omega t}, \tag{73}$$

$$\rho_2(t) = \rho_2(0)\mathrm{e}^{-\beta t/2}\mathrm{e}^{-i\omega t}, \tag{74}$$

where ω is given by

$$\omega = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}. \tag{75}$$

We choose c_1–c_3 in eqs (5)–(7) as

$$c_2 = \frac{1}{2m\rho_1(0)\rho_2(0)}, \quad c_1 = c_3 = 0. \tag{76}$$

The particular solutions x_p and p_p satisfy the following relations:

$$\ddot{x}_p + \beta\dot{x}_p + \omega_0^2 x_p = \frac{F(t)}{m}, \tag{77}$$

$$\ddot{p}_p - \beta\dot{p}_p + \omega_0^2 p_p = \mathrm{e}^{\beta t}\dot{F}(t). \tag{78}$$

The solutions of the above equations depend on $F(t)$. If, we choose $F(t)$ as

$$F(t) = F_0 t, \tag{79}$$

the solutions of eqs (77) and (78) will be

$$x_p(t) = \frac{F_0}{m\omega_0^2}t - \frac{\beta F_0}{m\omega_0^4}, \tag{80}$$

$$p_p(t) = \frac{F_0}{\omega_0^2}\mathrm{e}^{\beta t}. \tag{81}$$

We will also investigate the system driven by the exponentially decaying force:

$$F(t) = F_0\mathrm{e}^{-\gamma t}, \tag{82}$$

where γ is an arbitrary real constant. In this case, the particular solutions are given by

$$x_p(t) = \frac{F_0/m}{\gamma^2 - \beta\gamma + \omega_0^2}\mathrm{e}^{-\gamma t}, \tag{83}$$

$$p_p(t) = -\frac{\gamma F_0}{\gamma^2 - \beta\gamma + \omega_0^2}\mathrm{e}^{(\beta-\gamma)t}. \tag{84}$$

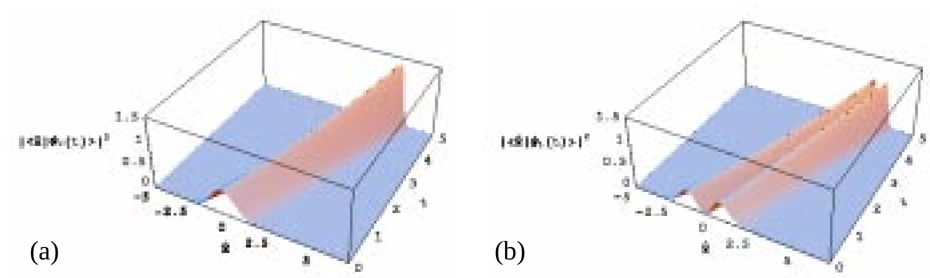

Figure 1. Ground state (**a**) and first excited state (**b**) probability density in number state for the forced Caldirola–Kanai oscillator that the driving force is given by eq. (79), as a function of position $\hat{x}$ and time t. We used $F_0 = 1$, $\omega_0 = 1$, $\beta = 0.4$, $\rho_1(0) = \rho_2(0) = 1$, $m = 1$ and $\hbar = 1$.

The system, we finally consider is the one driven by the periodic force

$$F(t) = F_0 \cos(\omega_1 t + \phi), \tag{85}$$

where ω_1 is a real driving force and ϕ an arbitrary phase. Equation (85) has the classical particular solutions which are given by

$$x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + \beta^2 \omega_1^2}} \cos(\omega_1 t + \phi - \delta), \tag{86}$$

$$p_p(t) = -\frac{F_0 \omega_1}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + \beta^2 \omega_1^2}} \mathrm{e}^{\beta t} \sin(\omega_1 t + \phi - \delta), \tag{87}$$

where

$$\delta = \tan^{-1} \frac{\beta \omega_1}{\omega_0^2 - \omega_1^2}. \tag{88}$$

We depicted ground state and first excited state probability densities in figures 1–3 for eqs (79), (82) and (85). The center of probability densities shifted from zero point along $\hat{x}$-axis with time according to the magnitude of the driving force.

We will investigate the system driven by the periodic force, eq. (85), in more detail. For this case, the expectation value of the Hamiltonian, eq. (65), in thermal state can be evaluated as

$$\begin{aligned} \langle \hat{H} \rangle_{\mathrm{T}} &= \frac{\hbar \omega_0^2}{2\omega} \coth \frac{\hbar \omega}{2kT} + \frac{1}{2} \mathrm{e}^{\beta t} m \dot{x}_p^2 - \frac{1}{2} m \omega_0^2 \mathrm{e}^{\beta t} x_p^2, \\ &= \frac{\hbar \omega_0^2}{2\omega} \coth \frac{\hbar \omega}{2kT} + \mathrm{e}^{\beta t} \frac{F_0^2}{m[(\omega_0^2 - \omega_1^2)^2 + \beta^2 \omega_1^2]} \Theta(t), \end{aligned} \tag{89}$$

where

$$\Theta(t) = \frac{1}{2} [\omega_1^2 \sin^2(\omega_1 t + \phi - \delta) - \omega_0^2 \cos^2(\omega_1 t + \phi - \delta)]. \tag{90}$$

We can confirm that eq. (89) oscillate with time.

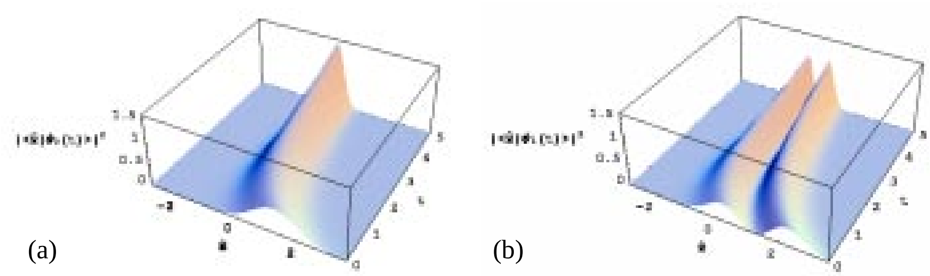

Figure 2. Ground state (**a**) and first excited state (**b**) probability density in number state for the forced Caldirola–Kanai oscillator that the driving force is given by eq. (82), as a function of position $\hat{x}$ and time t. We used $F_0 = 2$, $\omega_0 = 1$, $\beta = 0.4$, $\rho_1(0) = \rho_2(0) = 1$, $m = 1$, $\gamma = 1$ and $\hbar = 1$.

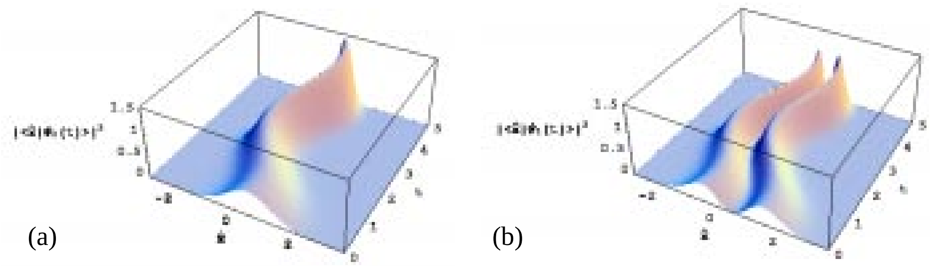

Figure 3. Ground state (**a**) and first excited state (**b**) probability density in number state for the forced Caldirola–Kanai oscillator that the driving force is given by eq. (85), as a function of position $\hat{x}$ and time t. We used $F_0 = 1$, $\omega_0 = 1$, $\omega_1 = 1.5$, $\beta = 0.4$, $\rho_1(0) = \rho_2(0) = 1$, $m = 1$, $\phi = 0$ and $\hbar = 1$.

In general, the mechanical energy is somewhat different from the Hamiltonian for the time-dependent system such as Caldirola–Kanai oscillator [34]. For the case driven by eq. (85), the mechanical energy expectation values in the thermal state can be calculated as [34]

$$\begin{aligned}\langle \hat{E} \rangle_{\mathrm{T}} &= \mathrm{e}^{-2\beta t}\frac{1}{2m}\langle \hat{p}^2 \rangle_{\mathrm{T}} + \frac{1}{2}m\omega_0^2 \langle \hat{x}^2 \rangle_{\mathrm{T}} \\ &= \mathrm{e}^{-\beta t}\frac{\hbar\omega_0^2}{2\omega}\coth\frac{\hbar\omega}{2kT} + \mathrm{e}^{-2\beta t}\frac{p_p^2}{2m} + \frac{1}{2}m\omega_0^2 x_p^2. \\ &= \mathrm{e}^{-\beta t}\frac{\hbar\omega_0^2}{2\omega}\coth\frac{\hbar\omega}{2kT} + \frac{F_0^2}{2m[(\omega_0^2-\omega_1^2)^2+\beta^2\omega_1^2]} \\ &\quad \times [\omega_1^2 \sin^2(\omega_1 t + \phi - \delta) + \omega_0^2\cos^2(\omega_1 t + \phi - \delta)]. \end{aligned} \tag{91}$$

Thus, we see that eq. (91) also oscillates with time since the system exchanges the energy with the surroundings. However, at $\omega_1 = \omega_0$, the energy does not oscillate and becomes (in the limit $t \to \infty$):

$$\langle \hat{E} \rangle_{\mathrm{T}} = \frac{F_0^2}{2m[(\omega_0^2-\omega_1^2)^2+\beta^2\omega_1^2]}\omega_0^2. \tag{92}$$

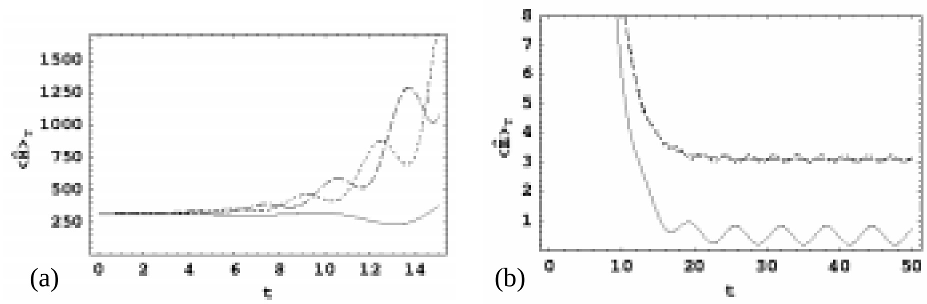

Figure 4. Hamiltonian (**a**) and quantum-mechanical energy (**b**) expectation values for the forced Caldirola–Kanai oscillator in the thermal state as a function of time t. The solid line is for $\omega_1 = 0.5$, the long dotted-line for $\omega_1 = \omega_0 = 1$ and the short dotted line for $\omega_1 = \sqrt{\omega_0^2 - \beta^2/2} \simeq 0.959$. We used $F_0 = 1$, $\omega_0 = 1$, $\beta = 0.4$, $\rho_1(0) = \rho_2(0) = 1$, $m = 1$, $\phi = 0$ $k = 1$, $T = 300$ and $\hbar = 1$.

There are two resonant frequencies for this system [35]. One is the velocity resonant frequency which is same as the natural frequency ω_0 and the other is the amplitude resonant frequency which is given by $\sqrt{\omega_0^2 - \beta^2/2}$. In figure 4, we depicted the Hamiltonian and the mechanical energy expectation values for these two resonant frequency in the thermal state. Even if the mechanical energy far from the resonance points in frequency gradually disappears with time, the mechanical energy near the resonance points remained the same.

5. Summary

Taking advantage of the invariant operator, we obtained the solution of the Schrödinger equation for the general time-dependent harmonic oscillator. We assumed that an ensemble of particles that satisfies the general time-dependent harmonic oscillator motion conform to the Bose–Einstein distribution function at equilibrium temperature. We investigated uncertainty relation in number and thermal states. Comparing eqs (40) and (66), we can confirm that the uncertainty relation in the thermal state varies in the same manner as in the number state.

The uncertainty relation is always larger than $\hbar/2$ in both number and thermal states. We determined density operators satisfying the Leouville–von Neumann equation and used it to calculate various expectation values of the variables in the thermal state.

We applied our theory to a special case which is the forced Caldirola–Kanai oscillator. The center of probability densities in number state shifted from zero point along the $\hat{x}$-axis with time according to the magnitude of the driving force. Even if the mechanical energy far from the resonance points in frequency gradually disappears with time, the mechanical energy near the resonance points remained the same.

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