PRAMANA — journal of physics

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Vol. 60, No. 6
June 2003
pp. 1153–1159

Bianchi-IX string cosmological model in Lyra geometry

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MS received 3 June 2002; revised 16 October 2002; accepted 20 December 2002

Abstract. A class of cosmological solutions of massive strings for the Bianchi-IX space-time are obtained within the framework of Lyra geometry. Various physical and kinematical properties of the models are discussed.

Keywords. Bianchi-IX space-time; Lyra geometry; string cosmology.

PACS Nos 04.20.Jb; 98.80.cq; 04.50.+h

1. Introduction

The origin of structure in the Universe is one of the greatest cosmological mysteries even today. The exact physical situation at very early stages of the formation of our Universe is still unknown. The concept of string theory was developed to describe events at the early stages of the evolution of the Universe. The present day configurations of the Universe are not contradicted by large-scale network of the strings in the early Universe. They may be one of the sources of density perturbations that are required for formations of large-scale structures in the Universe [1–5] .

The general relativistic treatment of strings was initiated by Stachel and Letelier [1,2]. According to Letelier [2] the massive strings are nothing but geometric strings (massless) with particles attached along its extension. So, the total energy momentum tensor for a cloud of massive strings can be written as [1,2]

$$T_a^b = \rho V_a V^b - \lambda X_a X^b. \tag{1}$$

Here ρ is the rest energy for a cloud of strings with particles attached to them (P-strings).

So one can write

$$\rho = \rho_p + \lambda, \tag{2}$$

ρ_p being the particle energy density and λ being the tension density of the string. The four velocity V^a for the cloud of particle and the four vector X^a, the direction of string will satisfy

$$V_a V^a = -1 = X_a X^a \quad \text{and} \quad V_a X^b = 0. \tag{3}$$

Several authors have worked on it [1–5].

In the last few decades, there has been considerable interest in alternative theories of gravitation. The most important among them are scalar–tensor theories proposed by Lyra [6] and Brans–Dicke [6]. Lyra [6] proposed a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance to Weyl's geometry. In general relativity, Einstein succeeded in geometrising gravitation by identifying the metric tensor with the gravitational potentials.

In the scalar–tensor theory of Brans–Dicke, on the other hand, scalar field remains alien to the geometry. Lyra's geometry is more in keeping with the spirit of Einstein's principle of geometrisation, since both the scalar and tensor fields have more or less intrinsic geometrical significance. In the consecutive investigations, Sen [7] and Sen and Dunn [7] proposed a new scalar–tensor theory of gravitation and constructed an analog of the Einstein field equation based on Lyra's geometry which in normal gauge may be written as

$$R_{ik} - \tfrac{1}{2} g_{ik} R + (3/2)\phi_i \phi_k - \tfrac{3}{4} g_{ik} \phi_m \phi^m = -T_{ik}, \tag{4}$$

where ϕ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford [8] has pointed out that the constant displacement field ϕ_i in Lyra's geometry play the role of cosmological constant Λ in the normal general relativistic treatment. According to Halford the present theory predicts the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearised form of field equations. Soleng [9] has pointed out that the constant displacement field in Lyra's geometry will either include a creation field and be equal to Hoyle's creation field cosmology or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term.

Subsequent investigations were done by several authors in scalar–tensor theory and cosmology within the framework of Lyra geometry [10].

So far the isotropic cosmological models have been studied extensively. But there are only few works on anisotropic models, essentially on Bianchi-IX cosmological model, due to the complicated nature of field equations. Actually the study of anistropic models were started after the discovery of microwave background radiation in 1965. It was found that the radiation was isotropic to one part of 10^4 apart from a dipole anisotropy, which was attributed to the peculiar motion of our galaxy [11].

In this paper, we would like to study string cosmology for Bianchi IX space-time within the framework of Lyra geometry.

2. Field equations

The time-like displacement vector ϕ_i in (1) is given by

$$\phi_i = (\beta(t), 0, 0, 0). \tag{5}$$

The Bianchi type-IX metric is taken as [11]

$$ds^2 = -dt^2 + a^2 dx^2 + b^2 dy^2 + (b^2 \sin^2 y + a^2 \cos^2 y) dz^2 - 2a^2 \cos y \, dx \, dz, \tag{6}$$

where a, b are functions of time co-ordinate only. The field equation (1) for the metric (6) are

$$2a'b'/ab + b'^2/b^2 + 1/b^2 - \tfrac{1}{4}a^2/b^4 - \tfrac{3}{4}\beta^2 = \rho, \tag{7}$$

$$2b''/b + b'^2/b^2 + 1/b^2 - \tfrac{3}{4}a^2/b^4 + \tfrac{3}{4}\beta^2 = \lambda, \tag{8}$$

$$a''/a + b''/b + a'b'/ab + \tfrac{1}{4}a^2/b^4 + \tfrac{3}{4}\beta^2 = 0 \tag{9}$$

(prime denotes differentiation with respect to t).

The expansion scalar θ and shear σ^2 are given by

$$\theta = (a'/a + 2b'/b), \tag{10}$$

$$\sigma^2 = \tfrac{2}{3}(a'/a - b'/b)^2. \tag{11}$$

The different equations of state for string model are [1–5]

(a) $\rho = \rho(\lambda)$(barotropic equation of state),

(b) $\rho = \lambda$(geometric string),

(c) $\rho = (1+w)\lambda$ (Takabayasi string, i.e., P-string). (12)

In the following section, we shall determine the exact solution of field equations using the above equations of state for string model in Lyra geometry.

3. Solutions

Case I : Barotropic equation of state

In this case we take displacement vector as constant, i.e., $\beta =$ constant. To solve the field equations one notes that there are three field equations connecting 4-unknowns. So one more relation connecting these variables is needed.

Here we assume the relation

$$a = \mu b^n \quad (\mu, n \text{ are constants}) \tag{13}$$

between the scale factors for unique solutions.

Using this relation, we get from eq. (9) as

$$b'/b + kb'^2/b^2 = -\{\mu^2/4(n+1)\}b^{2n-4} - [3/\{4(n+1)\}]\beta^2, \tag{14}$$

where $k = \{n^2/(n+1)\}$. Solving this differential equation, we get

$$\int \{[\mu^2/4(n+1)(1-n-k)]b^{2n-4}[3/\{4(n+1)(k+1)\}]b^2\beta^2 - Db^{-2k}\}^{-1/2}\mathrm{d}b = \pm(t-t_0). \tag{15}$$

(t_0 and D are integration constants).

The above integral can be solved only when $n = 0$. Hence we get the analytic form of b as

$$b^2 = (2/3\beta^2)A(t), \tag{16}$$

where $A(t) = [-D + \surd(D^2 + \frac{3}{4}\beta^2\mu^2)\sin\surd(3\beta^2)(t - t_0)]$. Here we get

$$a = \mu. \tag{17}$$

The other physical parameters have the following expressions:

$$\theta = \surd(D^2 + \tfrac{3}{4}\beta^2\mu^2)\surd(3\beta^2)\cos\surd(3\beta^2)(t - t_0)[A(t)]^{-1}, \tag{18}$$

$$\sigma^2 = (1/6)\theta^2, \tag{19}$$

$$\rho = \tfrac{1}{4}\theta^2 + [(2/3\beta^2)A(t)]^{-1} - \tfrac{1}{4}\mu^2[(2/3\beta^2)A(t)]^{-2} - \tfrac{3}{4}\beta^2, \tag{20}$$

$$\lambda = \tfrac{1}{4}\theta^2 + [(2/3\beta^2)A(t)]^{-1} - (3/8)\mu^2[(2/3\beta^2)A(t)]^{-2} - \tfrac{3}{4}\beta^2, \tag{21}$$

$$\rho_p = \rho - \lambda = \tfrac{1}{2}\mu^2[(2/3\beta^2)A(t)]^{-2}. \tag{22}$$

Case II: Geometric string ($\rho = \lambda$)

Here we also assume the same relation between the metric coefficients, i.e., $a = \mu b^n$, but the displacement vector is not constant.

Following the relation (7)–(8) + 2.(9) and using the above polynomial relation between the metric coefficients, we get

$$b''/b + (n+1)b'^2/b^2 = -\{\mu^2/2n\}b^{2n-4}. \tag{23}$$

This differential equation can be written in the integral form as

$$\int[-(\mu^2/8n^2)b^{2n-4} + Db^{-2n-2}]^{-1/2}\mathrm{d}b = \pm(t - t_0). \tag{24}$$

(t_0 and D are integration constants).

One can obtain b in closed form evaluating the above integral only for $n = -2, 2$.

Sub Case I: $n = -2$. We obtain from eq. (24) as

$$b^4 = (\mu^2/32D)\cosh 4\surd(D)(t - t_0). \tag{25}$$

So the metric coefficient and other physical parameters have the following expressions:

$$a = \mu(\mu^2/32D)^{-1/4}[\cosh 4\surd(D)(t - t_0)]^{-1/2}, \tag{26}$$

$$\theta = 0, \tag{27}$$

$$\sigma^2 = 6D[\tanh 4\surd(D)(t - t_0)]^2, \tag{28}$$

$$\rho = \lambda = (\mu^2/32D)^{-1/4}[\cosh 4\surd(D)(t-t_0)]^{-1/2}$$
$$+[8D(\mu^2-2)/\mu^2][\text{sech } 4\surd(D)(t-t_0)]^2, \tag{29}$$
$$\tfrac{3}{4}\beta^2 = [8D(1-\mu^2)/\mu^2][\text{sech } 4\surd(D)(t-t_0)]^2$$
$$-3D[\tanh 4\surd(D)(t-t_0)]^2. \tag{30}$$

Sub Case II : $n = 2$. If we solve eq. (24), we have the following expressions for the metric coefficient and other physical parameters have the following expressions:

$$b^4 = \surd(8Dn^2/\mu^2)\sin\surd(2\mu^2/n^2)(t-t_0), \tag{31}$$

$$a^2 = \mu^2\surd(8Dn^2/\mu^2)\sin\surd(2\mu^2/n^2)(t-t_0), \tag{32}$$

$$\theta = (\surd(2\mu^2/n^2)\tan\surd(2\mu^2/n^2)(t-t_0), \tag{33}$$

$$\sigma^2 = (\mu^2/12n^2)[\tan\surd(2\mu^2/n^2)(t-t_0)]^2, \tag{34}$$

$$\rho = \lambda = (8Dn^2/\mu^2)^{-1/4}[\sin\surd(2\mu^2/n^2)(t-t_0)]^{-1/2} - (5\mu^2/4), \tag{35}$$

$$\tfrac{3}{4}\beta^2 = (5\mu^2/8n^2)[\tan\surd(2\mu^2/n^2)(t-t_0)]^2 + \tfrac{1}{2}\mu^2. \tag{36}$$

Case III: Takabayasi string (i.e. P-string)

Here the equation of state $\rho = (1+w)\lambda$, where $w > 0$, a constant and it is small for string dominant era and large for particle dominant era. Further using the polynomial relation $a = \mu b^n$, from the field equations, we get

$$Ab''/b + Bb'^2/b^2 + Cb^{2n-4} = wb^{-2}, \tag{37}$$

where $A = (w+2)n - w; B = (2n^2 + wn^2 + 2n + w); C = (1+w)\mu^2$. This equation can be solved only when $A = 0$, i.e., $w = 2n/(1-n)$. As $w > 0$, therefore n must be restricted as $0 < n < 1$. We also note that eq. (37) is solvable only for $n = \frac{1}{2}$. Here we get,

$$\surd(4B/w)[\tfrac{1}{2}\surd(b)\surd\{b-(C/w)\} + \tfrac{1}{2}(C/w)\cosh^{-1}\surd(bw/C)] = \pm(t-t_0). \tag{38}$$

In this case, we cannot get explicit form of b in terms of t and consequently all physical parameters cannot be determined in terms of t. Therefore no physical conclusion can be drawn from the solution.

4. Properties of the solutions

Case I

In this model, the Universe starts at an initial epoch $t = t_0 + [1/\surd(3\beta^2)]\sin^{-1}[D/\surd(D^2 + \frac{3}{4}\beta^2\mu^2)]$, which is a line singularity. At this epoch all the physical quantities $\theta, \rho, \lambda, \sigma^2 \to$

∞. At a later instant when $t = t_0 + [\pi/2\surd(3\beta^2)]$, we have $\lambda = 0$ and $\rho = \rho_p$ provided $[2/(3\beta^2)\{-D+\surd(D^2+\frac{3}{4}\beta^2\mu^2)\}]^{-1} - \{(3/8)\mu^2\}[2/(3\beta^2)\{-D+\surd(D^2+\frac{3}{4}\beta^2\mu^2)\}]^{-2} = \frac{3}{4}\beta^2$. Therefore, at this instant strings vanish and we are left with a dust-filled Universe. At this stage,

$$\rho = \{\tfrac{1}{2}\mu^2\}[2/(3\beta^2)\{-D+\surd(D^2+\tfrac{3}{4}\beta^2\mu^2)\}]^{-2}, \quad \theta = \sigma^2 = 0, \quad a = \mu,$$
$$b = \{2/(3\beta^2)\}\{-D+\surd(D^2+\tfrac{3}{4}\beta^2\mu^2)\}.$$

So all the physical parameters are of finite magnitude. Moreover, we note that anisotropy and expansion rate vanish at this moment.

Case II

For $n = -2$. We note that the solutions in (25) and (26) describe a nonsingular space-time. It is easy to verify that all the physical quantities in (27)–(30) remain finite and regular for the entire range of variables: $-\infty < t < \infty$. This clearly indicates that the model is free of singularity.

Since ρ and $\beta^2 > 0$ for all time, so at the instant $t = t_0$, the positivity conditions hold only when $\mu^2 < 1$. If $t \to \infty, \rho \to 0$ and β becomes imaginary, then $\sigma^2 \to$ nonzero finite value. Thus string concept and concept of Lyra geometry will not linger for infinite time. Interestingly, we see that our model retains anisotropy as nonzero shear necessarily implies anisotropy.

For $n = 2$. For this solution, the Universe starts at an initial epoch $t = t_0$, which is a point singularity. At this epoch the physical quantities $\theta, \sigma^2 \to 0$ but ρ diverges. At the later instant when $t = t_0 + [\pi/\mu^2]$, the scale factors are constants and consequently the volume and density of the Universe is constant.

Acknowledgements

We are thankful to the members of Relativity and Cosmology Center, Jadavpur University. FR is thankful to UGC for financial support and IUCAA for sending preprints and papers. Finally we wish to thank the anonymous referee for pointing out an oversight which has led to stronger results than the one in the earlier version.

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