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Stability of extended scalar diquark stars vis-à-vis soliton stars

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Abstract. Motivated by relatively large values of the maximum mass $(M = 8.92 M_{\odot})$ and radius ($R = 50.7$ km) obtained for an extended scalar diquark star within the framework of an effective ϕ^4-theory (S K Karn *et al* [1]) some interesting observations are made with regard to the stability of stellar objects describable in general in terms of the polynomial field theories.

Keywords. Stability of stars; diquark gluon plasma; scalar diquark; diquark stars; soliton stars; ϕ^4-theory.

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1. Introduction

It is well known that the stellar objects all through their history of evolution undergo different types of changes. As a result, the duration of their existence in a particular phase vis-a-vis its stability in terms of mass–radius configuration has been the subject of great interest in astrophysics [1a,2]. During this evolution process of stellar objects, the study of the stage of formation of diquark gluon phase has become of particular interest mainly because this phase, like quark gluon phase, can also add to the understanding of stellar objects. This diquark phase, while lies between the extremes of quark and hadron phases with a possibility of its co-existence with the quark phase (i.e., mixed phase), is normally described in terms of scalar diquarks and expected to provide physics in the high density and low temperature regime compared to that required for the deconfined phase (the quark gluon phase). Several authors [3–8] have attempted to study this diquark phase with reference to the properties of diquark gas and diquark stars.

Recently, after accounting for the extended character of scalar diquarks we have derived [7] an equation of state and investigated [8] various properties of a diquark star (including its stability) within the framework of an effective ϕ^4-theory described [4,5] by the Lagrangian density

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi^+ \partial^\mu \phi - m_D^2 \phi^+ \phi) - \lambda (\phi^+ \phi)^2, \tag{1}$$

where $m_D = 615.6$ MeV and $\lambda = 27.8$ are used. Using the Tolman–Oppenheimer–Volkoff (TOV) equations, maximum mass and radius obtained for a diquark star in this model are $M = 8.92M_\odot, R = 50.7$ km corresponding to the central density $\rho_c = 1.488 \times 10^{14}$ g/cm^3, which in fact are much larger than the values expected for a normal star [1]. On the other hand, these figures for M and R cannot be ruled out in the light of the results known for the soliton stars. In fact, Cottingham and Mau [9] for a maximum mass soliton star have predicted these values as $20M_\odot$ and 60 km, respectively by introducing temperature dependence in the Lee–Wick model [10]. These observations have motivated us to look into the stability aspects of a star. It could be either a soliton star in general or an extended scalar diquark (ESD) star (studied in the spirit of eq. (1)) in particular.

2. Aspects of the stability of a star

The stability of stellar objects against their constituents and the corresponding interactions have been the subject of study for a long time [2,11]. A theory of stability accommodating these features *a priori* can be examined either in terms of oscillations of the radial coordinate [2,12] and/or be assumed to be due to the occurrence of the phase transition(s) (in the present case it may be either from the quark gluon phase to the diquark gluon phase or from the diquark gluon phase to the hadronic phase). While the former situation provides a necessary condition for the stability of a stellar object in terms of its gravitational mass M and central density ρ_c through

$$(\mathrm{d}M/\mathrm{d}\rho_c) > 0 \tag{2}$$

the latter, on the other hand, can possibly offer a sufficiency condition in this regard. Condition (2) suggests the value of the adiabatic index $\bar{\Gamma}_1 > 4/3$, which after accounting for the post-Newtonian approximation becomes [2]

$$\bar{\Gamma}_1 > \frac{4}{3} + \frac{2GM\kappa}{Rc^2}, \tag{3}$$

where $\kappa \sim 1$ and depends on the structure of the star [3]. Further, $\bar{\Gamma}_1$, also computed from [2]

$$\bar{\Gamma}_1 = \left[\int_0^R \Gamma_1(r)p(r)r^2\mathrm{d}r\right] \Big/ \left[\int_0^R p(r)r^2\mathrm{d}r\right] \tag{4}$$

for the present ESD model, is compared with the results [4] obtained from (3) and they conform to the stability of a ESD star for certain values of ρ_c. In eq. (4), $p(r)$ and $\Gamma_1(r)$, respectively are the pressure and the adiabatic index computed at the radial distance r from the center using the numerical form [8] of the equation of state.

Another aspect through which the stability of a star (against the microscopic collapse) is much discussed in the literature [13], is based on a direct manifestation of the equation of state through the inequalities

$$p \geq 0 \quad \text{and} \quad (\mathrm{d}p/\mathrm{d}\rho) \geq 0, \tag{5}$$

where $(\mathrm{d}p/\mathrm{d}\rho)^{1/2}$ is a measure of hydrostatic phase velocity of sound waves in the stellar matter. Again for certain values of ρ_c, the inequalities (5) are found [8] to be marginally satisfied for the present ESD model. The concept of causality, appearing through the inequality $c_s^2 = (\mathrm{d}p/\mathrm{d}\rho) < c^2$, where c_s is the sound velocity, further restricts [13,14] the maximum mass configuration to $3M_\odot$ in this aspect of the stability of a star. This figure of $3M_\odot$ is also smaller than the one obtained for the ESD star. Clearly, the results obtained for the diquark star in ref. [8] violate causality in certain density ranges (cf. tables 2 and 3 of ref. [8], and other references cited therein). These results, in fact, have motivated the present investigation.

From this brief survey it appears that in addition to these so-called 'physical' factors responsible for the stability of a star, there exists yet another but not much highlighted 'mathematical' factor and that is the one which is already built-in the theoretical framework used for the purpose. It is this mathematical factor which dominates the theories [10,15–17] of soliton stars and contributes to the existence of a particular phase. In the next section we continue exploring this last aspect of the stability of a star.

3. Mathematical aspect of the stability of a star

Scalar field theories with a polynomial (in ϕ) form of the potential term in the Lagrangian have played an important role in explaining varieties of phenomena in different branches of theoretical physics mostly on the basis of structural analogy. While the term quartic in ϕ has provided a satisfactory explanation for several phenomena like phase transition in magnetism, structural phase transition in solids, Higg's mechanism in particle physics etc., necessity of ϕ^6- and ϕ^8-terms are also felt in some cases mainly for a fine tuning of the results. In fact physical common sense suggests that any system with analogous features in the structure of the Lagrangian can exhibit similar phenomena under suitable circumstances. In this spirit these theories have been applied to study the evolution and subsequently the stability of stellar objects.

Lee and Wick [10] discussed the stability of a system in which the potential $U(\phi)$ in $\mathscr{L} = -(1/2)(\partial_\mu \phi)^2 - U(\phi) +$ counter terms, is of the form (note the choice of different metric in this L from eq. (1))

$$U(\phi) = \frac{1}{2}a\phi^2 + \frac{1}{6}b\phi^3 + \frac{1}{24}c\phi^4, \quad (a, b, c \text{ real}). \tag{6}$$

Here, after ignoring the effect of counter terms, the extrema of the energy density functional $E(\phi)$ can be obtained [10] from the Hamiltonian density $\mathscr{H} = (1/2)\pi^2 + (1/2)(\nabla\phi)^2 + U(\phi)$ and through $(\partial U(\phi)/\partial\phi) = 0$ one defines the stability of the system. Here π is the conjugate momentum of ϕ. For the form (6), $\phi = 0$ and

$$\phi = \phi_\pm = (3/2c)[-b \pm (b^2 - (8/3)ac)^{1/2}]$$

are the extrema of $U(\phi)$. Note that for certain choices of the parameters a, b, c, while $\phi = 0$ could be an absolute maximum, $\phi = \phi_+$ and $\phi = \phi_-$ may turn out to be the local minima as in the case of a symmetric double well potential (cf. figure 2a). Lee and Wick [10] further study the behavior of energy functional $E(\underline{\phi}) = \lambda_J - J\phi$, where $\underline{\phi} = (\partial\lambda_J/\partial J)$ as a function of ϕ and the Lagrange multiplier J defined by $J = -(\partial E(\underline{\phi})/\partial\underline{\phi})$. As J increases, the two local minima will move and correspondingly $U(\phi)$ will also change. For certain critical value of $J = J_c$ (say) these two minima become degenerate. These features of potential (6) are further pursued by Lee and his coworkers [15,16] in a series of papers to develop a theory of soliton stars. For this purpose they introduce an additive quantum number N (like the baryon number) carried by a spin-(1/2) field ψ, or a spin-0 complex field ϕ with its elementary field quantum having $N = \pm 1$. In addition, they also introduce a scalar field σ whose self-interactions again are described by a potential like (6), namely

$$U(\sigma) = \frac{1}{2}m^2\sigma^2\left(1 - \frac{\sigma}{\sigma_0}\right)^2 \tag{7}$$

and describe similar degeneracy features as (6). The significance of the cubic term in $U(\phi)$ is also found [18] important in nuclear matter theory in terms of 3-body forces.

Theories up to quartic terms in $U(\phi)$ (but without linear and cubic terms) have found relatively more importance in the literature in different contexts (a case used in the present ESD model), viz.,

$$U(\phi) = d + \frac{1}{2}a\phi^2 + \frac{1}{24}c\phi^4. \tag{8}$$

In (8) again, while the opposite algebraic signs of a and c give rise to one type of features with regard to the degeneracy of vacuum/stability, the same signs (as is the case with the present ESD model, cf. eq. (1)) give rise to the only minimum of the system. The case of double well (particularly the symmetric ones) features of (8) (when a and c are of opposite sign) has been of tremendous interest in the literature. In what follows we consider this example to demonstrate the mathematical aspect of the stability of a star.

Example: The time evolution of the system $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, describing a symmetrical double well potential, is given by

$$\ddot{x} = x - x^3. \tag{9}$$

Defining $\dot{x} = y$, we have $\dot{y} = x - x^3$. The critical points (x,y) of the system are (0,0), ($\pm$1,0). To determine the nature of the critical points, we compute the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$ at $x = 0$ and ± 1. At $x = 0$, the eigenvalues are $\mu_1 = +1, \mu_2 = -1$ implying [19] a saddle (unstable) point S. On the other hand at $x = \pm 1$, the eigenvalues are $\mu_1 = \sqrt{2}i, \mu_2 = -\sqrt{2}i$ implying [19] centers (stable) C_1 and C_2 (cf. figure 1) with closed trajectories (circles) of constant energy.

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

The phase portraits of system (9) are shown in figure 1. The trajectory which separates the small and big closed contours has a special feature (called 'separatrix'). Along this trajectory, the system takes forever to reach the critical point.

A connection between the phase portrait and the motion of an undamped particle in a double well can be understood from figures 2a and 2b as a function of energy. Large orbits represent more energetic oscillations about the critical points taking the particle back and forth. In figure 2b, the graphs of the energy function $E(x,y)$ are sketched. The energy plots above each point (x,y) of the phase plane represent constant energy surface.

Thus, in a nut shell, the mathematical structure of $U(\phi)$ is also responsible for the stability of the system described by $\mathscr{L}$ in (1) in general. In the above example when the system passes through the saddle S it is highly unstable. On the other hand, when it passes through the centers C_1 and C_2 it is highly stable.

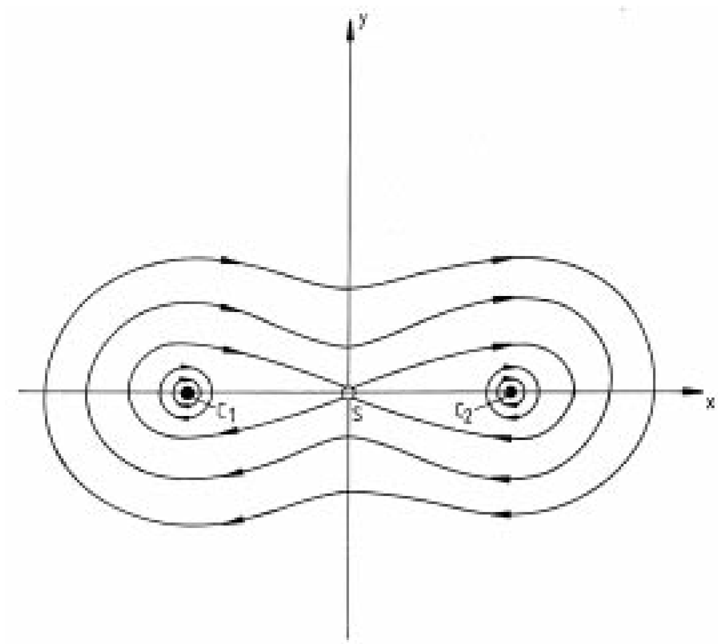

Figure 1. Phase portraits for system (9).

4. Stability of ESD stars: Discussion and conclusions

In the present ESD model, in fact, the effective nature of the ϕ^4-theory (cf. eq. (1)) does not allow any discussion of the individual terms as such. The value of m_D, derived using the $\Delta - N$ mass difference in the process of accounting for the spin–spin interactions of quarks and that of λ, obtained using *P*-matrix method of Jaffe and Low [20], in the work Donogue and Sateesh [4] or of Kastor and Traschen [5] brings in enormous physics content in the theory. The existence of the only minimum in $U(\phi)$, in this case, further confirms the stability of ESD stars as far as the mathematical aspect of stability is concerned. In a way such a setting of $U(\phi)$ does not require higher powers of ϕ in it. The only assumption in the ESD model, which can be questioned in the work of Karn *et al* [7,8] pertains to using the same diquark structure inside the nucleon as that when it is free. But in the absence of any other theory for the diquark structure these ingredients of the model are unavoidable.

In summary, besides the importance of 'physical' factors in the study of the stability of a star, the mathematical aspect originating within the framework of the underlying theory itself should not be ignored. Sometimes even the methodology of computing various physical quantities also matters. For example, in the semi-quantal approach the system (9) also exhibits [21] the chaotic behavior, which otherwise is absent in the pure classical or quantum approaches.

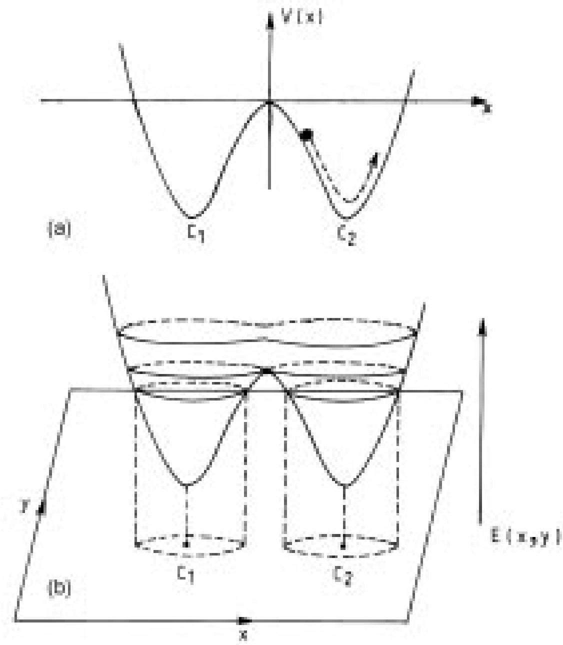

Figure 2. (**a**) Motion of an undamped particle in a symmetric double well potential. (**b**) Graphs of the energy function $E(x,y)$ in the *x*–*y* plane for the potential of (a).

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References

[1] S K Karn, R S Kaushal and Y K Mathur, *Euro. Phys. J.* **C14**, 487 (2000)
[1a] S Chandrasekhar, *Principles of stellar dynamics* (Dover Publication, New York, 1942)
[2] S L Shapiro and S A Teukolsky, *Black holes, white dwarfs and neutron stars* (Wiley, New York, 1983)
[3] S Ekelin and S Fredriksson, *Phys. Lett.* **B162**, 373 (1985); *Phys. Rev.* **D30**, 2310 (1984)
[4] J F Donoghue and K S Sateesh, *Phys. Rev.* **D38**, 360 (1988)
[5] D Kastor and J Traschen, *Phys. Rev.* **D44**, 3791 (1991)
K S Sateesh, *Phys. Rev.* **D45**, 866 (1992)
[6] J E Horvath *et al*, *Phys. Rev.* **D46**, 4754 (1992)
[7] S K Karn, R S Kaushal and Y K Mathur, *Z. Phys.* **C72**, 297 (1996)
[8] S K Karn, R S Kaushal and Y K Mathur, *Euro. Phys. J.* **C14**, 487 (2000)
S K Karn, Ph.D. Thesis (unpublished) (Delhi University, 1998)

[9] W N Cottingham and R V Mau, *Phys. Lett.* **B261**, 93 (1991); *Phys. Rev.* **D44**, 1652 (1991)
[10] T D Lee and G C Wick, *Phys. Rev.* **D9**, 2291 (1974)
[11] S Chandrasekhar, *Rev. Mod. Phys.* **56**, 137 (1984)
S Weinberg, *Gravitation and cosmology* (Wiley, New York, 1972)
J B Hartle, *Phys. Rep.* **46**, 201 (1978)
[12] Ch Kettner, F Weber, M K Weigel and N K Glendenning, *Phys. Rev.* **D51**, 1440 (1995)
[13] V Kalogera and G Baym, *Astrophys. J.* **470**, L61 (1996)
J B Hartle in ref. [11] above
[14] P J Ellis, J M Lattimer and M Prakash, *Comm. Nucl. Part. Phys.* **22**, 63 (1996)
[15] R Friedberg and T D Lee, *Phys. Rev.* **D15**, 1694 (1977)
S Coleman, *Aspects of symmetry* (Univ. Press, Cambridge, 1985)
[16] T D Lee and Y Pang, *Phys. Rev.* **D35**, 3678 (1987) and the reference therein
[17] A Goyal, *Proc. Natl. Acad. Sci. India* **A66**, 93 (1996)
[18] A D Jackson, *Ann. Rev. Nucl. Part. Sci.* **33**, 105 (1983)
[19] R S Kaushal and D Parashar, *Advanced methods of mathematical physics* (Published jointly by Narosa, New Delhi and C.R.C. Press, USA) Ch. 6
S H Strogatz, *Nonlinear dynamics and chaos* (Addison Wesley, Reading, 1994)
[20] R L Jaffe and F E Low, *Phys. Rev.* **D19**, 2105 (1979)
[21] Analabha Roy and J K Bhattacharjee, *Phys. Lett.* **A288**, 1 (2001)