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Bianchi type I string cosmologies

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Abstract. By making use of Letelier's form of energy–momentum tensor for a cloud of string-dust we present some classes of solutions of general relativistic field equations which describe cosmological string-dust models in Bianchi type I space-time. Some of the classes of models obey Takabayashi's equation of state whereas a class of models exhibits inflation in the initial stage. Two of the classes presented here have Kasner's space-time as past asymptote.

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1. Introduction

Standard cosmology provides us a sound theory of cosmic evolution and the observational inputs seem to be tilted in its favor. However, it is unable to account for, on its own, the presence of galactic and other discrete structures against a smooth background of isotropic and homogeneous microwave radiation. Moreover, it gives rise to problems of horizon, flatness and entropy to name a few. It is argued that we can do away with these problems if a proper initial condition, viz., inflation is imposed on the very early universe. Various types of inflationary cosmologies are being considered and the string cosmology is one. It is generally believed that the very early universe underwent phase transitions, which gave rise to topologically stable structures; of particular interest is a linear structure called geometric string. The classical theory of these strings was developed by Stachel [1]. Letelier [2] presented a model of string-dust in which incoherent matter particles are attached to geometric string along its extension. The presence of strings results into anisotropy in the space-time. Strings are not observed in the present epoch. So for a physically meaningful string model it is desirable that either strings disappear at a certain epoch of cosmic evolution or it has a particle dominated future asymptote with undetectable strings.

Bianchi type I anisotropic and homogeneous models are simple and are generally considered to portray early universe reasonably well. Letelier [2] obtained some particular Bianchi type I string-dust models in which at a certain epoch, strings disappear with a phase transition to an anisotropic fluid. Another Bianchi type I model has been presented by Banerjee *et al* [3] which obeys Takabayashi's equation of state. Recently, one of us [4] has reported a string-dust model whose future asymptote is Einstein–de Sitter dust filled universe.

2. Field equations for a cloud of string-dust

The field equations of general relativity are:

$$G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij}. \tag{1}$$

For a cloud of string-dust the energy–momentum tensor T_{ij} is given by [2]

$$T_{ij} = \rho v_i v_j - \lambda x_i x_j \tag{2}$$

with

$$\rho = \rho_P + \lambda, \tag{3}$$

where ρ is the rest energy density of the cloud of string-dust where each string has dust particles attached along its length, ρ_P the density of particles, λ the tension density of strings, v^i the four-velocity of the cloud and x^i the space-like four-vector representing the direction of strings:

$$v_i v^i = 1 = -x_i x^i. \tag{4}$$

x^i and v^i satisfy normalization condition

$$x_i v^i = 0. \tag{5}$$

The energy conditions are

$$\rho \geq \lambda, \quad \lambda \geq 0. \tag{6}$$

3. String cosmology in a Bianchi type I space-time

Bianchi type I space-time is described by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \alpha^2(t)\mathrm{d}x^2 - \beta^2(t)\mathrm{d}y^2 - \gamma^2(t)\mathrm{d}z^2 \tag{7}$$

for which we have [2]

$$u^i = (1,0,0,0), \tag{8}$$

$$x^i = (0,\alpha^{-1},0,0), \tag{9}$$

the latter implying that strings are parallel to the direction $\alpha\mathrm{d}x$. From (2), (7)–(9) the surviving field equations (1) are

$$G_{11} \equiv \frac{\ddot{\beta}}{\beta} + \frac{\ddot{\gamma}}{\gamma} + \frac{\dot{\beta}\dot{\gamma}}{\beta\gamma} = \lambda, \tag{10}$$

$$G_{22} \equiv \frac{\ddot{\alpha}}{\alpha} + \frac{\ddot{\beta}}{\beta} + \frac{\dot{\alpha}\dot{\beta}}{\alpha\beta} = 0, \tag{11}$$

$$G_{33} \equiv \frac{\ddot{\alpha}}{\alpha} + \frac{\ddot{\gamma}}{\gamma} + \frac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma} = 0, \tag{12}$$

$$G_{44} \equiv \frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} + \frac{\dot{\beta}}{\beta}\frac{\dot{\gamma}}{\gamma} + \frac{\dot{\gamma}}{\gamma}\frac{\dot{\alpha}}{\alpha} = \rho. \tag{13}$$

A look into eqs (10)–(13) reveals that these are invariant under the transformation

$$\beta \Leftrightarrow \gamma \tag{14}$$

which is a consequence of (9). Equation (11) is invariant under the transformation

$$\alpha \Leftrightarrow \beta.$$

Also (12) is invariant under the transformation

$$\alpha \Leftrightarrow \gamma.$$

Lastly, we note that the energy density is invariant under the set of transformations

$$\alpha \to \beta, \quad \beta \to \gamma, \quad \gamma \to \alpha.$$

4. Classes of solutions

In this section we shall explore physically meaningful solutions of the set of eqs (10)–(13) by considering simplifying assumptions related to the field variables α, β and γ.

Case (i). $\beta = \gamma, \alpha = \beta^m$. Here m is an arbitrary constant. In this case eqs (11) and (12) reduce to

$$\frac{\ddot{\beta}}{\beta} + \left(\frac{m^2}{1+m}\right)\frac{\dot{\beta}^2}{\beta^2} = 0. \tag{15}$$

A class of solutions is given by Banerjee *et al* [3].

$$\alpha = t^{[m(m+1)]/(m^2+m+1)}, \quad \beta = \gamma = t^{(m+1)/(m^2+m+1)},$$
$$\lambda = \frac{(2m+1)(1-m^2)}{(m^2+m+1)^2}\frac{1}{t^2}, \quad \rho = \frac{(2m+1)(1+m)^2}{(m^2+m+1)^2}\frac{1}{t^2},$$
$$\rho_P = \frac{2m(m+1)(2m+1)}{(m^2+m+1)^2}\frac{1}{t^2}. \tag{16}$$

In the case $0 \leq m \leq 1$, ρ, ρ_P, $\lambda \geq 0$ and $(\rho/\lambda) = [(1+m)/(1-m)] > 1$. This implies that the energy conditions (6) hold true and that the solution obeys Takabayashi's equation of state $\rho = (1+W)\lambda$ where W is a non-negative constant. In the case $m = 0, \rho_P = 0$ and $\rho = \lambda = 1/t^2$ which describes a cloud of geometric strings; for $m = 1$, we rediscover Einstein–de Sitter flat dust model.

The cosmological parameters are given by [5]

$$\theta = \frac{(m+1)(m+2)}{(m^2+m+1)}\frac{1}{t}, \quad \sigma^2 = \frac{(m^2-1)^2}{3(m^2+m+1)^2}\frac{1}{t^2},$$
$$R^3 = t^{[(m+1)(m+2)]/(m^2+m+1)}, \quad H = \frac{(m+1)(m+2)}{3(m^2+m+1)}\frac{1}{t},$$
$$\Omega = \frac{3(2m+1)}{(m+2)^2}, \quad q = \frac{(2m^2+1)}{(m+1)(m+2)},$$
$$l_\alpha = \left[t^{m(m+1)/(m^2+m+1)}, \quad t^{(m+1)/(m^2+m+1)}, \quad t^{(m+1)/(m^2+m+1)}\right]. \tag{17}$$

As pointed out by Banerjee *et al* [3] the model has a point singularity at $t = 0$. Also $q > 0$, for $0 \leq m \leq 1$, which implies the absence of inflation.

Case (ii). $\beta = \gamma = t^m$. In this case, eqs (11) and (12) reduce to the following:

$$\frac{\ddot{\alpha}}{\alpha} + \frac{m}{t}\frac{\dot{\alpha}}{\alpha} + \frac{m(m-1)}{t^2} = 0. \tag{18}$$

We obtain a one parameter class of solutions:

$$\alpha = t^{(1-m+k)/2}\left[1 - \left(\frac{t_1}{t}\right)^k\right], \quad \beta = \gamma = t^m, \quad \lambda = \frac{m(3m-2)}{t^2},$$
$$\rho = \frac{m}{t^2}\left[\frac{(1+k)t^k + (k-1)t_1^k}{t^k - t_1^k}\right],$$
$$\rho_P = \frac{m}{t^2}\left[\frac{(3-3m+k)t^k - (3-3m-k)t_1^k}{t^k - t_1^k}\right]. \tag{19}$$

Here and in what follows t_1 stands for a non-negative arbitrary constant and

$$k = \sqrt{(1+3m)(1-m)} \quad \text{with } -\frac{1}{3} \leq m \leq 1. \tag{20}$$

The cosmological parameters are given by

$$\theta = \frac{1}{2t}\left[\frac{(1+3m+k)t^k - (1+3m-k)t_1^k}{t^k - t_1^k}\right],$$
$$\sigma^2 = \frac{1}{12t^2}\left[\frac{(1-3m+k)t^k - (1-3m-k)t_1^k}{t^k - t_1^k}\right]^2,$$
$$R^3 = t^{(1+3m+k)/2}\left[1 - \left(\frac{t_1}{t}\right)^k\right], \quad H = \frac{1}{6t}\left[\frac{(1+3m+k)t^k - (1+3m-k)t_1^k}{t^k - t_1^k}\right],$$

$$\Omega = \frac{12m(t^k - t_1^k)\{(1+k)t^k + (k-1)t_1^k\}}{\{(1+3m+k)t^k - (1+3m-k)t_1^k\}^2},$$

$$l_\alpha = \left[t^{(1-m+k)/2}\left\{1 - \left(\frac{t_1}{t}\right)^k\right\}, t^m, t^m\right],$$

$$q = \frac{\begin{matrix}[(1+3m+k)(5-k-3m)t^{2k} \\ -2\{(1+3m)(5-3m)-5k^2\}(tt_1)^k + (1+3m-k)(k+5-3m)t_1^{2k}]\end{matrix}}{\{(1+3m+k)t^k - (1+3m-k)t_1^k\}^2} \quad (21)$$

The solution has two singularities, viz., at $t = 0$ and $t = t_1$. The singularity at $t = 0$ is cigar type and at $t = t_1$ the singularity is pancake type. In the case $2/3 \leq m \leq 1$, $\lambda \geq 0$ for $0 \leq t < \infty$, whereas ρ, $\rho_P \geq 0$ for $t_1 \leq t < \infty$. At $t = t_1, \rho, \rho_P$ diverge while λ is finite and $q = 2$. The energy condition $\rho \geq \lambda$ is violated in the interval $0 \leq t \leq t_1$. Also $q > 0$ for $0 \leq t < \infty$ showing the absence of inflation. In the case $m = 2/3$, $\alpha = t^{2/3}[1 - (t_1/t)]$, $\beta = \gamma = t^{2/3}$, $\lambda = 0$, $\rho = \rho_P = 4/3t(t - t_1)$, which corresponds to a dust filled universe without strings whose future asymptote is Einstein–de Sitter universe. In the case $m = 1$, $\rho_P = 0$, $\lambda = \rho = t^{-2}$ and we are left with a universe filled with geometric strings.

Case (iii). $\beta = \gamma, \alpha = t^m$. Field equations (11) and (12) reduce to

$$\frac{\ddot{\beta}}{\beta} + \frac{m}{t}\frac{\dot{\beta}}{\beta} + \frac{m(m-1)}{t^2} = 0. \quad (22)$$

The resulting class of solutions is given by

$$\alpha = t^m, \quad \beta = \gamma = t^{(1-m+k)/2}\left[1 - \left(\frac{t_1}{t}\right)^k\right],$$

$$\lambda = \frac{k}{2t^2}\left[\frac{(1-3m+k)t^{2k} - (1-3m-k)t_1^{2k}}{(t^k - t_1^k)^2}\right],$$

$$\rho = \frac{k}{2t^2}\left[\frac{(1+m+k)t^{2k} - (1+m-k)t_1^{2k}}{(t^k - t_1^k)^2}\right], \quad \rho_P = \frac{2mk}{t^2}\left[\frac{t^k + t_1^k}{(t^k - t_1^k)}\right],$$

$$\frac{\rho_P}{\lambda} = 4m\left[\frac{t^{2k} - t_1^{2k}}{(1-3m+k)t^{2k} - (1-3m-k)t_1^{2k}}\right]. \quad (23)$$

The cosmological parameters are as follows:

$$\theta = \frac{1}{t}\left[\frac{(1+k)t^k + (k-1)t_1^k}{t^k - t_1^k}\right],$$

$$\sigma^2 = \frac{1}{12t^2}\left[\frac{(3m-1-k)t^k - (3m-1+k)t_1^k}{t^k - t_1^k}\right]^2,$$

$$R^3 = t^{(1+k)}\left[1 - \left(\frac{t_1}{t}\right)^k\right]^2, \quad H = \frac{1}{3t}\left[\frac{(1+k)t^k + (k-1)t_1^k}{t^k - t_1^k}\right],$$

$$\Omega = \frac{3[(1-m+k)(1+3m+k)t^{2k} + (1-m-k)(1+3m-k)t_1^{2k}]}{4\{(1+k)t^k - (1-k)t_1^k\}^2},$$

$$q = \frac{[(2-k)(k+1)t^{2k} + (3k+4)(k-1)(tt_1)^k + (1-k)(2+k)t_1^{2k}]}{\{(1+k)t^k - (1-k)t_1^k\}^2},$$

$$l_\alpha = \left[t^m, t^{(1-m+k)/2}\left\{1-\left(\frac{t_1}{t}\right)^k\right\}, t^{(1-m+k)/2}\left\{1-\left(\frac{t_1}{t}\right)^k\right\}\right]. \tag{24}$$

Each solution has two distinct singularities, viz., $t = 0$ and $t = t_1$. The singularity at $t = t_1$ is barrel type whereas the singularity at $t = 0$ does not fall in any of the types of the sense of Thorne [6].

In the case $0 \le m \le 2/3$, $\lambda > 0$ for $0 \le t < \infty$ and $\rho, \rho_P, q > 0$ for $t_1 < t < \infty$. At $t = t_1$, $(\rho/\lambda) \approx 1$, $(\rho_P/\lambda) \approx 0$, $q = (1+k)/4k > 0$. For $0 < t < t_1$ the energy condition $\rho \ge \lambda$ is violated. At $t = 0$, $q = -[2+k)/(1-k)] < 0$, implying the inflationary character of the model in the initial stage. In the case $m = 0$, $\rho_P = 0$ and $\rho = \lambda = 1/(t-t_1)^2$, that is a cloud of massless strings. The case $m = 2/3$ leads to the string model due to Pant [4] which goes over to the Einstein–de Sitter flat $\Omega = 1$ model as $t \to \infty$.

In the case $2/3 < m < 1$, ρ, $\rho_P \ge 0$ for $t_1 < t < \infty$; $\lambda > 0$ in the initial stage but vanishes at a critical time $t = t_{\rm c}$ given by

$$t_{\rm c} = \left[\frac{(1-3m-k)}{(1-3m+k)}\right]^{1/2k} t_1 < t_1 \tag{25}$$

and $\lambda < 0$ for $t_{\rm c} < t < \infty$. Also $q > 0$ for $0 \le t < \infty$.

Thus the class of solutions (23) describes two types of string cosmologies. In one type the model starts with inflation and the matter density and string's tension density fall with cosmic expansion, whereas in the other type no inflation is observed and strings' tension density vanishes at a critical time and then changes sign signifying the phase transition, as in Letelier's models.

Case (iv). $\alpha = t^m$, $\beta \neq \gamma$. A class of solutions is

$$\alpha = t^m, \quad \beta = t^{(1-m+k)/2}, \quad \gamma = t^{(1-m+k)/2} + Ct^{(1-m-k)/2} \tag{26}$$

with

$$\lambda = \frac{k}{2t^2}\frac{(1+k-3m)t^k}{(t^k+C)}, \quad \rho = \frac{k}{2t^2}\frac{(1+k+m)t^k}{(t^k+C)}, \quad \rho_P = \frac{2mkt^k}{t^2(t^k+C)}, \tag{27}$$

C being an arbitrary constant. The solution obeys Takabayashi's equation of state. In the case $0 < m < 2/3$; ρ, ρ_P, λ are positive for $0 < t < \infty$, provided $C > 0$. For t approaching 0, the asymptotic solution is given by

$$\alpha = t^m, \quad \beta = t^{(1-m+k)/2}, \quad \gamma = Ct^{(1-m-k)/2} \tag{28}$$

which describes Kasner's class of anisotropic homogeneous vacuum space-times.

In the case $m = 0$ the solution provides a model of geometric strings whereas for $m = 2/3$, a dust-filled universe without strings results whose future asymptote is Einstein–de Sitter universe.

The cosmological parameters are as follows:

$$\theta = \frac{(1+k)t^k + C}{t(t^k + C)}, \quad R^3 = t^{(1+k)}(1 + Ct^{-k}),$$
$$H = \frac{(1+k)t^k + C}{3t(t^k + C)},$$
$$\sigma^2 = \frac{1}{6t^2}\left[\frac{(1-3m+k)^2 t^{2k} + 2(1-6m+9m^2+k-3mk)Ct^k + 6(1-2m+3m^2)C^2}{(t^k + C)^2}\right],$$
$$\Omega = \frac{3}{2}\frac{k(1+k+m)t^k(t^k + C)}{\{(1+k)t^k + C\}^2},$$
$$q = \frac{[(2-k)(1+k)t^{2k} + (4-3k)(k+1)Ct^k + 2C^2]}{\{(1+k)t^k + C\}^2}$$
$$l_\alpha = \left[t^m, t^{(1-m+k)/2}, t^{(1-m+k)/2} + Ct^{(1-m-k)/2}\right]. \tag{29}$$

At $t = 0$, $q = 2$; as $t \to \infty$, $q = [(2-k)/(1+k)] > 0$. For $0 < t < \infty$, $q > 0$. The model does not exhibit inflation. The singularity at $t = 0$ is cigar type.

Case (v). $\beta = t^m$, $\beta \neq \gamma$. The resulting class of solutions is

$$\alpha = t^{(1-m+k)/2}, \quad \beta = t^m, \quad \gamma = t^{(1+m-k+p)/4} + Ct^{(1+m-k-p)/4} \tag{30}$$

with

$$k = \pm\sqrt{(1+3m)(1-m)}, \quad p = \sqrt{4 - 2k(1+k-3m)}. \tag{31}$$

Also

$$\lambda = \frac{1}{8t^2}\left[\frac{4m(3m-2) - p(1+k-3m)}{(1+Ct^{-p/2})}\right],$$
$$\rho = \frac{1}{8t^2}\left[\frac{4m(1+k) + p(1+k+m)}{(1+Ct^{-p/2})}\right],$$
$$\rho_P = \frac{1}{4t^2}\left[\frac{2m(3+k-3m) + p(1+k-m)}{(1+Ct^{-p/2})}\right]. \tag{32}$$

This class of solutions obeys Takabayashi's equation of state. In the case $2/3 < m < 1$, ρ, $\rho_P, \lambda > 0$ for $0 < t < \infty$ provided $C > 0$. In the case $m = 2/3$, a stringless dust filled universe is obtained whose future asymptote is Einstein–de Sitter universe whereas for $m = 1$ a model of geometric strings results. As $t \to 0$, the class of solutions (30) asymptotically approaches to Kasner's class of vacuum solutions.

The cosmological parameters are as follows:

$$\theta = \frac{1}{4t}\left[\frac{(3+3m+k+p) + (3+3m+k-p)Ct^{-p/2}}{(1+Ct^{-p/2})}\right],$$
$$R^3 = t^{(3+3m+k+p)/4}[1 + Ct^{-p/2}],$$

$$\sigma^2 = \left[\frac{3(4-k^2+k-3mk)(1+C^2t^{-p})-3pk(1-C^2t^{-p}) \\ +(8-15m+12m^2+12k-36mk)Ct^{-p/2}}{36(1+Ct^{-p/2})^2t^2}\right],$$

$$q = \frac{[(3+3m+k+p)(9-3m-k-p)(1+Ct^{-p/2})^2 \\ -4p\{2p+(3-3m-k)(1+Ct^{-p/2})\}Ct^{-p/2}]}{[(3+3m+k+p)+(3+3m+k-p)Ct^{-p/2}]^2},$$

$$\Omega = \frac{6(1+Ct^{-p/2})[\{4m(1+k)+p(1+m+k)\} \\ +\{4m(1+k)-p(1+m+k)\}Ct^{-p/2}]}{[(3+3m+k+p)+(3+3m+k-p)Ct^{-p/2}]^2},$$

$$l_\alpha = [t^{(1-m+k)/2}, t^m, t^{(1+m-k+p)/4} + Ct^{(1+m-k-p)/4}]. \tag{33}$$

At $t = 0$,

$$q = \frac{[(3+3m+k+p)(9-3m-k-p)+4p(3m-3+k)]}{(3+3m+k-p)^2} > 0,$$

as $t \to \infty$,

$$q = \frac{9-(3m+k+p)}{(3+3m+k+p)} > 0.$$

This class of models has no inflationary epoch. Also the singularity at $t = 0$ is cigar type irrespective of the sign of k.

5. Conclusions

In the foregoing sections we obtained five one-parameter classes of solutions describing string cosmologies in Bianchi type I spatially homogeneous and anisotropic space-times. Broadly speaking these cosmologies are of three types. One type of models has single singularity in the interval $0 \le t < \infty$, viz., at $t = 0$, and obeys Takabayashi's equation of state. The second type consists of models with two distinct real singularities in which string disappears at a critical epoch signifying the transit from string phase to the anisotropic fluid, like that in Letelier's models. Lastly, the third type has two distinct real singularities with strings staying put and string's tension density and particle density falling with cosmic expansion. In models with two singularities the energy conditions are violated in the interval between the two singularities. Letelier's models, too, show this inconsistency. A desirable feature of a meaningful string model is the presence of an inflationary epoch in the very early universe. One of the classes of models presented in this paper display initial inflation.

It is found that all Bianchi type I perfect fluid models obeying a γ-law equation of state have Kasner's vacuum space-time as past asymptote as t approaches zero [7]. This is not a necessary property of Bianchi type I string cosmologies. However, we have been able to find two classes of models which approach Kasner's vaccum space-time as t approaches zero.

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References

[1] J Stachel, *Phys. Rev.* **D21**, 2171 (1980)
[2] P S Letelier, *Phys. Rev.* **D28**, 2414 (1983)
[3] A Banerjee, A K Sanyal and S Chakraborty, *Pramana – J. Phys.* **34**, 1 (1990)
[4] D N Pant (in preparation)
[5] G F R Ellis, S T C Siklos and J Wainwright, in *Dynamical systems in cosmology* edited by J Wainwright and G F R Ellis (Cambridge Univ. Press, 1997) p. 1
[6] K S Thorne, *Astrophys. J.* **148**, 51 (1967)
[7] J Wainwright, in *Dynamical systems in cosmology* edited by J Wainwright and G F R Ellis (Cambridge Univ. Press, 1997) p. 136