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Maximum stellar iron core mass

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Abstract. An analytical method of estimating the mass of a stellar iron core, just prior to core collapse, is described in this paper. The method employed depends, in part, upon an estimate of the true relativistic mass increase experienced by electrons within a highly compressed iron core, just prior to core collapse, and is significantly different from a more typical Chandrasekhar mass limit approach. This technique produced a maximum stellar iron core mass value of 2.69×10^{30} kg (1.35 solar masses). This mass value is very near to the typical mass values found for neutron stars in a recent survey of actual neutron star masses. Although slightly lower and higher neutron star masses may also be found, lower mass neutron stars are believed to be formed as a result of enhanced iron core compression due to the weight of non-ferrous matter overlying the iron cores within large stars. And, higher mass neutron stars are likely to be formed as a result of fallback or accretion of additional matter after an initial collapse event involving an iron core having a mass no greater than 2.69×10^{30} kg.

Keywords. Maximum; stellar; star; iron; core; mass; collapse; neutron star.

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1. Introduction

It is generally thought that type II supernova events are preceded by a collapse in the central cores of relatively massive stars. The central core regions of these pre-nova stars are believed to have evolved into a high density/high temperature neutral plasma consisting primarily of iron nuclei (or near iron nuclei) and free (or relatively free) electrons. When these highly evolved core regions reach a certain critical size the gravitational forces tending to compress the stellar core into ever increasing densities eventually overcomes repulsive forces due principally to free electrons. When this happens, electrons within the central core of the star are able to combine with protons within the iron (or near iron) nuclei forming additional neutrons in place of almost all of the electrons and protons. The rapid drop in electron concentration (and thus electron pressure) within the central stellar core region, due to this process (sometimes also referred to as inverse beta decay), may precipitate a complete and rapid collapse of the central core region of a large highly evolved star. This process is believed to be responsible for an accompanying supernova event and to also lead to the production of a neutron star (or pulsar) at the center of the collapsed stellar core region. Additional details, as well as other somewhat similar explanations regarding this process and its associated effects, are available within the extant literature [1–11].

A important question related to this process is how massive a stellar iron core must be before it can spontaneously collapse into a neutron star due to gravitational compression. Typically, one resorts to using the Chandrasekhar mass limit approximation for white dwarfs [12,13] as a means of estimating this core mass maximum. In one of his first published papers covering this subject [12], Chandrasekhar estimated this maximum mass limit to be approximately 0.91 solar masses even though he was aware of an earlier publication indicating a white dwarf mass limit (corrected for relativistic electron mass increases) of about 1.1 solar masses [14]. However, by the time Chandrasekhar's book entitled *Introduction to the Study of Stellar Structure* was published [13], this maximum white dwarf mass limit was adjusted upward to about 1.4 solar masses.

The Chandrasekhar mass limit, for a specific element, can be calculated using the general formula: $M_{\mathrm{CH}} = 5.83 Y_{\mathrm{e}}^2 M_{\mathrm{S}}$ [11], where M_{CH} is the Chandrasekhar mass limit, Y_{e} (in the case of a pure isotope) the ratio of that neutral element's number of electrons to the sum of the protons and neutrons in its nucleus and M_{S} the mass of sun. For pure ${}_{26}\mathrm{Fe}^{56}$, $Y_{\mathrm{e}} = 0.464$. So, the corresponding Chandrasekhar mass limit for this isotope of iron is about $1.26 M_{\mathrm{S}}$.

Keeping the foregoing in mind, the purpose of the remaining part of this paper is to illustrate that at least one other approximation method (significantly different from the Stoner and Chandrasekhar method) will produce a result that is close to the Chandrasekhar mass limit of approximately $1.26 M_{\mathrm{S}}$ for the maximum iron core mass that can be reached just prior to a stellar core collapse.

2. Computational details

2.1 *Iron core assumption*

Of all the detailed assumptions listed below, the reader here is alerted to the fact that the computations that follow have been made by assuming that a stellar core, existing just prior to core collapse, consists primarily of highly compressed and very hot iron nuclei and electrons. Although nuclei near iron in atomic number, as well as smaller concentrations of other subatomic particles, may also exist in a stellar core just prior to core collapse, an analysis based only upon iron nuclei is thought to be capable of producing numerical results that will not be seriously influenced by the presence of these other substances.

2.2 *Other assumptions*

It is assumed that it is possible to assemble a large spherical, non-rotating, cloud of pure iron particles, in an otherwise empty space, with a total mass equal to M. It is further assumed that this spherical mass of pure iron has an initial radius of R_0 and an initial uniform density of β_0, before gravitational compression is allowed to cause the entire sphere to rapidly decrease to a much smaller spherical volume having a final radius of R, and a final uniform critical density of β, where $R \ll R_0$ and $\beta \gg \beta_0$. It is also assumed that the final overall density, throughout this compressed mass of iron, is infinitesimally smaller than the critical central density that would be required to cause complete collapse of this mass of iron into a neutron star. The radius of this highly compressed theoretical sphere may be somewhat smaller than the actual radius of a real spherical mass of iron, just prior to core collapse, because an unstable real spherical mass of iron is likely to achieve the critical density only at its center just before it collapses. The actual mass density, at all radial distances from the center of this spherical mass of iron, would probably vary inversely with radius (or inversely with some other more complicated function of the radius). So, the overall radius of an unstable real spherical mass of iron would be somewhat greater than R, say R'. However, gravitational collapse can still be thought of as occurring initially from radius R' down to R without the formation of any additional neutrons (due to the combination of electrons and protons into neutrons) but with a corresponding increase in the density of the entire (but smaller) sphere. It is also reasonable to imagine that this new and unstable sphere can then spontaneously collapse further until the entire new compressed sphere has reached the radius of a corresponding neutron star. The main advantage in this approach is that the gravitational potential energy of a uniform spherical mass of any substance is easy to calculate from: $(3/5)GM^2/R$. This equation, in this form or in other equivalent forms, can be derived relatively easily and may also be found in various places within the existing literature (e.g., see Kitchin [15]).

It is also logical to assume that the collapse of a highly compressed sphere of iron, into a neutron star, cannot occur until electron masses plus bound proton masses (within iron nuclei) equal bound neutron masses. This is the condition implied above when a large mass (M) of iron has decreased in size from radius R_0 down to R. If this requisite condition is achieved, the conversion of electrons and protons into neutrons (i.e., inverse beta decay) is favorable. This conditional behavior can only occur when electron velocities reach approximately $0.92c$, where c is the speed of light. This statement can easily be verified using the special relativity (SR) mass velocity relationship as well as the known masses of protons and neutrons. Mass increases in iron nuclei (as well as in transmuted product nuclei), under the same temperature conditions, can be neglected because their thermal equilibrium velocities will tend to be non-relativistic.

If it is possible to find the mass of a spherically shaped quantity of iron that satisfies these conditions, that mass is also the maximum mass of iron that can exist just prior to its collapse into a neutron star. Any spherically shaped mass of iron exceeding this limit should spontaneously collapse into a neutron star. If this mass of iron happens to exist within the core of a larger star, a supernova (type II) explosion will rapidly follow the collapse event. And, this collapse event will occur as soon as the maximum iron core mass limit (M) is reached, no matter what the mass of the original star is that produced the iron core in the first place. Collapse of a large enough spherical mass of iron can occur even if there is no local star that produced the iron in the first place although this kind of condition is unlikely. However, the process described above and the mass of pure iron involved (M) is also the maximum iron core mass that any pre-nova star can contain just prior to a real core collapse. The remaining part of this paper illustrates how this maximum iron core mass limit may be estimated from the assumptions noted above.

3. Calculations and discussion

First, without regard to algebraic sign, one equates the change in the gravitational potential energy (expressed here as δE) of an extremely large spherical mass of pure iron, of initial radius R_0 (that collapses due to gravitational forces to a much smaller radius of R), to the energy increase associated with the increase in electron masses $[(\delta M)c^2]$ that occurs due to their substantial increase in velocity as the spherical mass of iron is compressed. Here, it is assumed that the change in the gravitational potential energy of the iron is converted almost entirely into increases in electron masses. In other words, it has been assumed that there are insignificant energy losses due to radiation and all other energy effects related to the gravitational collapse of the spherical mass of iron (such as mass increases in heavy nuclei) can be ignored as insignificant when compared to relativistic electron mass increases.

Energy losses due to ionization can be ignored because electrons are not removed from the initially stable iron atoms during the compression process. They are, however, forced closer and closer together and thus must experience substantial increases in momentum in order to avoid crashing into their respective atomic nuclei. This process will eventually crush the initially stable iron atoms into an extremely dense neutral plasma consisting of relatively free atomic nuclei and free electrons, all within a much smaller volume than that originally occupied by the iron atoms alone. In this final mixture of subatomic particles, the electrons are not bound to any particular atomic nucleus but can move freely throughout this exotic form of matter. A concomitant effect of this process is that the effective temperature of all of the matter undergoing these changes will increase substantially.

If it is also assumed that the highly compressed iron nucleus/electron plasma behaves approximately as an ideal gas, the mean kinetic energies of the free electrons and atomic nuclei will be equal. Therefore, because the atomic nuclei are so much more massive than individual electrons, their relativistic mass increases (due to their temperature-induced increases in velocity) will be negligible when compared to electron mass increases. So, it is a good approximation to equate the entire change in gravitational potential energy to electron mass increases alone.

In equation form

$$\delta E = (3/5)GM^2[(1/R) - (1/R_0)] = (\delta M)c^2 = kMc^2, \tag{1}$$

where G is the universal gravitational constant (6.6726×10^{-11} Nm^2/kg^2) and k is a constant (not Boltzmann's constant) that is equal to the ratio of $(\delta M)/M$.

Simplifying eq. (1) with the approximation that $R_0 \gg R$, and dividing both sides by M, leads to

$$(3/5)GM/R = kc^2. \tag{2}$$

Now, expressing M in terms of π (i.e., 3.14159...), β, and R:

$$M = (4/3)\beta\pi R^3. \tag{3}$$

Solving the equation above for R yields

$$R = [3M/(4\pi\beta)]^{1/3}. \tag{4}$$

Placing this value for R into eq. (2), and simplifying, produces

$$M = [5k/3G]^{3/2}[3/(4\pi\beta)]^{1/2}c^3. \tag{5}$$

Except for k and β, all of the other terms in eq. (5) are known. However, it is possible to estimate reasonable values for k and β. A value for k may be found by first calculating the mass increase that each electron must achieve so that the combined mass of each 'heavy' electron and one proton (at rest) will equal the mass of one neutron (at rest). It is believed that neglecting the binding energy of protons (i.e., the reduced mass of protons in an iron nucleus) is justified here because of the compensating effect of the reduced mass of bound neutrons. A further justification of this approach is related to the fact that we cannot know for sure what specific isotopes (or concentrations thereof) are involved in this reaction just prior to the core collapse condition. So, we cannot simply subtract isotopic masses of specific nuclei (presumably, inverse beta decay involving the conversion of iron nuclei into manganese nuclei would initiate the complete collapse of a stellar iron core into a neutron star) in order to find average electron masses that would exist just prior to the onset of an irreversible inverse beta decay process.

In any case, this kind of calculation indicates that each electron (note: there are 26 electrons associated with each iron nucleus) must gain about 0.0008398 a.m.u. in order to gain enough energy to combine with bound protons to form neutrons (see Appendix A). But, since kM must equal (δM), one may write the approximation that

$$\begin{aligned} kM = {} & (0.0008398\,\text{a.m.u./e})(1.661\times 10^{-24}\ \text{g/a.m.u.}) \\ & \times (26\ \text{e/atom})(M/55.85\ \text{g/mol})(6.022\times 10^{23}\,\text{atom/mol}). \end{aligned} \tag{6}$$

So, after canceling the M terms on each side of eq. (6), it is possible to write

$$k = (8.398)(1.661)(26/55.85)(6.022)(10^{-5}) = 3.910\times 10^{-4}. \tag{7}$$

The value of the iron core density (β), after initial compression but just prior to spontaneous collapse at the core center, can be estimated by first assuming that the minimum iron 'ion' radius is approximately equal to the first Bohr orbit radius of an iron nucleus containing only one electron moving at a velocity of approximately $0.92c$ (see Appendix B). If the first Bohr orbit becomes even smaller, due to additional gravitational compression, electron velocities will exceed $0.92c$ and the relativistic mass increase of the electrons will facilitate the electron/proton conversion process into neutrons. Once this process starts, complete collapse of the remaining iron core will result.

The limiting first Bohr orbit relativistic radius, calculated in this way, is about 8.042×10^{-11} cm (see Appendix C). Note, this Bohr orbit calculation is not meant to imply that stable single electron ions exist within the hot iron cores of massive stars but only that this kind of calculation is capable of providing reasonable estimates of internuclear dimensions within a neutral, highly compressed, stellar plasma consisting primarily of heavy metal nuclei and electrons. For comparative purposes, the reader may also note that an iron nucleus alone has a radius of approximately 4.6×10^{-13} cm and that the ratio of the single electron ion radius and the nuclear radius is about 175 to one. If one neglects the effects of the other electrons in a normal (uncompressed) iron atom, this ratio is about 430 to one. In this case, a single electron, nearest to the atomic nucleus, will encircle the iron nucleus at velocities in the vicinity of $0.19c$. So, the relativistic electron mass, for all electrons within a normal 'uncompressed' iron atom, is very close to an electron's rest mass.

If one now assumes that each degenerate (i.e., highly compressed) iron 'atom' occupies a cubical volume with a side length of $[2(8.042\times 10^{-11}$ cm)], the volume occupied by each of these 'atoms' is about $[2(8.042\times 10^{-11}\ \text{cm})]^3$. This volume is roughly twice the spherical volume that would be occupied by a single iron nucleus and a single electron moving at $0.92c$ in the first Bohr orbit of a highly ionized iron ion. But, the actual volume occupied by each equivalent iron ‘atom’ must be somewhat larger than that required for a single iron nucleus and only one ‘captured’ electron because there are 26 electrons per iron nucleus to account for and the packing arrangement of the compressed iron nuclei is probably not perfect due to thermal vibrations. This is a very highly compressed condition that, in the real world, probably only occurs at the very center of a spherical mass of iron whose density varies from a maximum at the core’s center to a minimum at its ‘surface’.

The dimensional estimates made in the preceding text can be confirmed, at least approximately, by a quantum mechanics/‘particle in a box’ argument. This kind of calculation involves finding the dimensions of a box (or cube) that would be needed to confine an electron moving at a speed of $0.92c$. The calculated result is that a cubical box meeting these conditions would have a side length of about 9.036×10^{-11} cm (see Appendix D). The presence of a nucleus in this box, as well as 25 additional electrons, would also tend to make it somewhat larger. So, a final estimate of the box size needed to meet all the conditions needed to promote inverse beta decay might well be on the order of 16×10^{-11} cm, as estimated in the preceding paragraph.

In any case, as a result of the Bohr estimate assumptions made above, the calculated iron core density (β) is approximately 22.29×10^{9} kg/m^3 (see Appendix E). For comparative purposes, the approximate density of the ‘white dwarf’ Sirius B is about 3.0×10^{9} kg/m^3. The ratio of these two densities is a little greater than 7.4 indicating that the iron core mass density estimated herein is, at least, realistic. For additional comparative purposes, the reader may note that the average mass densities of neutron stars are believed to be in the vicinity of 2.0×10^{17} kg/m^3, or higher due to gravitational compression.

Using the values of k and β indicated in eq. (5), along with the other known constants, produces

$$M = 2.69\times 10^{30}\ \text{kg} \ = \ 1.35M_{\text{S}}. \tag{8}$$

This value of M is very close to the Chandrasekhar’s mass limit of about $1.26M_{\text{S}}$ for an iron sphere consisting of pure ${}_{26}\text{Fe}^{56}$. However, the reader should also keep in mind that if an original pre-nova star is very large, a smaller iron core may be compressed into the collapse condition before it is able to reach mass M, due to the additional pressure exerted by the matter surrounding the core. So, the iron core mass M calculated is the maximum iron core mass. The fact that some neutron stars have masses greater than M is probably due to ‘fallback’ of some of the ejected nova event matter [16] or due to accretion of additional matter (possibly at the expense of a nearby companion star or other nearby source of matter) by the original neutron star after its initial formation. Furthermore, the accumulation of accreted mass, upon an initially small neutron star, is probably more likely if the original pre-nova star is very large and/or if it is close to at least one other companion star. If accretion and/or ‘fallback’ does not occur to any appreciable extent, the largest neutron stars should form from precursor stars just large enough to form a critical iron core mass of approximately $1.35M_{\text{S}}$. The list of fairly recently determined neutron star masses, compiled in table 1, tends to confirm the results obtained herein as well as the results obtained by Chandrasekhar long ago. However, the analytical approach described in this paper provides a different and interesting method of estimating maximum neutron star masses.

Table 1. Approximate neutron star masses [17].

Designation	Neutron star mass (in solar masses)
4U 1700-37	1.80 ± 0.40
U 1538-52	1.30 ± 0.30
Cen X-3	1.05 ± 0.50
Her X-1	1.50 ± 0.15
LMC X-1	1.40 ± 0.50
5MC X-1	1.05 ± 0.30
Vela X-1	1.45 ± 0.10
PSR 1802-07	1.40 ± 0.40
PSR 2303-46	1.15 ± 0.30
PSR 2303-46C	1.40 ± 0.30
PSR 1855-09	1.30 ± 0.20
PSR 1534-12	1.30 ± 0.05
PSR 1534-12C	1.35 ± 0.05
PSR 1913-16	1.45 ± 0.05
PSR 1913-16C	1.40 ± 0.05
PSR 2127+11C	1.35 ± 0.20
PSR 2127+11CC	1.37 ± 0.20
Average	1.35 ± 0.24

Appendix A: Rest masses of fundamental particles

Electron = 0.0005486 a.m.u.
Proton = 1.0072765 a.m.u.
Neutron = 1.0086649 a.m.u.

So $M_{\rm r} = M_{\rm n} - M_{\rm p} = 1.0086649$ a.m.u. – 1.007265 a.m.u. = 0.0013884 a.m.u. Therefore, the mass increase of the electron (δM) is $\delta M = 0.0013884$ a.m.u. $-$ 0.0005486 a.m.u. = 0.0008398 a.m.u. (ref. [18]).

Appendix B: Relativistic electron mass/velocity calculation

Using rest mass of electron = $M_0 = 0.0005486$ a.m.u. and relativistic electron mass = $M_{\rm r} = M_{\rm n} - M_{\rm p} = 0.001388$ amu:

$$V = [1.0 - (M_0/M_{\rm r})^2]^{1/2} c = 0.92c.$$

This value of V is the approximate velocity an electron must have so that its relativistic mass will be large enough to combine with protons to form neutrons.

Appendix C: Calculation of relativistic first Bohr orbit for iron nucleus and one electron

Using the first Bohr orbit radius formula to find the relativistic electron radius for a single iron nucleus and a single electron:

$$R_r = ([\varepsilon_0 n^2 h^2)/(Z\pi M_0 q^2)][M_0/M_r]$$
$$= ([0.5292 \times 10^{-8}\ \text{cm}/\text{Z})][M_0/M_r] = 8.042 \times 10^{-11}\ \text{cm}.$$

Appendix D: Particle in a box calculation

$$E_0 = M_0 c^2$$
$$E = M_r c^2$$
$$L = (nhc/2)[3/(E^2 - E_0^2)]^{1/2} = 9.036 \times 10^{-11}\ \text{cm}.$$

Appendix E: Estimation of iron core mass density

$$\beta = (55.85\ \text{a.m.u.})(1.661 \times 10^{-24}\ \text{g/a.m.u.})/[2(8.042 \times 10^{-11}\ \text{cm})]^3$$
$$= 2.229 \times 10^7\ \text{g/cm}^3 = 22.29 \times 10^9\ \text{kg/m}^3.$$

For comparative purposes, the average density of Sirius B (a white dwarf) is about 3.0×10^9 kg/m^3.

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