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$O(12)$ limit and complete classification of symmetry schemes in proton–neutron interacting boson model

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Abstract. It is shown that the proton–neutron interacting boson model (pnIBM) admits new symmetry limits with $O(12)$ algebra which break F spin but preserves the F_z quantum number M_F. The generators of $O(12)$ are derived and the quantum number v of $O(12)$ for a given boson number N is determined by identifying the corresponding quasi-spin algebra. The $O(12)$ algebra generates two symmetry schemes and for both of them, complete classification of the basis states and typical spectra are given. With the $O(12)$ algebra identified, complete classification of pnIBM symmetry limits with good M_F is established.

Keywords. Proton–neutron interacting boson model; pnIBM; symmetry limits; complete classification; F spin; F spin breaking; good M_F; $O(12)$ limit; $O(12) \supset O(6) \otimes O(2)$; $O(12) \supset O(2) \oplus O(10)$.

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1. Introduction

The most significant aspect of the interacting boson model (IBM) of even–even nuclei [1] is the dynamical symmetries of the model. In its most elementary version with scalar (s) and quadrupole (d) bosons, the model (called IBM-1) admits the well-established (they are now part of several text books [2]) vibrational $U(5)$, rotational $SU(3)$ and γ-unstable $O(6)$ symmetries starting with the $U(6)$ spectrum generating algebra (SGA) (the 6 in $U(6)$ corresponds to the sum of one degree of freedom from s bosons (with angular momentum $\ell = 0$) and five from d bosons ($\ell = 2$). Its extended version, with proton (π) and neutron (ν) degrees of freedom attached to s and d bosons, the proton–neutron interacting boson model (pnIBM or IBM-2) admits dynamical symmetries starting from its $U(12)$ SGA [1,3,4]. An important development here is the introduction of the so-called F spin; π and ν bosons are considered as two projections of a spin-$\frac{1}{2}$ boson. Going beyond IBM-1 and IBM-2 models, in the last five years, the dynamical symmetries of the isospin invariant IBM-3 [5] and spin–isospin invariant IBM-4 [6,7] models are also being studied as they are shown to have applications for nuclei near the proton drip line. Although the IBM-1 model was introduced nearly 20 years back, remarkably there is now a new interest in re-examining the symmetries of these models with developments and an interest in quantum chaos and phase transitions. In the context of the former, the importance of $\overline{O(6)}$ and $\overline{SU(3)}$ algebras

[8] is brought out and for the latter the so-called $E(5)$ and $X(5)$ symmetries are introduced [9]. Also new interpretations for the $SU(3)$ and $O(6)$ limits are proposed with higher-order interactions [10]. Then an immediate question that arises is whether there are new symmetries in pnIBM that are not recognized before. This question is answered in the affirmative in this paper by identifying and analyzing the $O(12)$ limit group chains of pnIBM.

The *sd* boson pnIBM with $\pi - \nu$ degrees of freedom is a standard model for analyzing the properties of heavy even–even nuclei with protons and neutrons occupying different oscillator shells [1,11]. The SGA of pnIBM is $U(12)$ as a single boson carries 12 degrees of freedom (6 from s and d and two from π and ν) in this model. Then there are the well-known $U(6) \otimes SU_F(2)$ with the $SU_F(2)$ algebra generating F spin and the $U_\pi(6) \oplus U_\nu(6)$ symmetry limits in this model; in addition there are also the $U_s(2) \oplus U_d(10)$ symmetry limits (see §4). The various group chains starting from $U(6) \otimes SU_F(2)$ and $U_\pi(6) \oplus U_\nu(6)$ are identified and studied in great detail in the past; see for example [1,3,4]. Let us point out that IBM-1 model corresponds to the $F = N/2$ states in pnIBM where N is the total number of bosons. The $F = N/2 - 1$ states are the so-called mixed symmetry states. In rotational $SU(3)$ nuclei they correspond to the now well-known scissors states that are seen in many nuclei [12]. It should be emphasized that there are many new experiments with the advent of the EUROBALL cluster detector in the last five years in identifying the mixed symmetry states of pnIBM in $O(6)$ type nuclei (for example in ^{196}Pt, ^{134,136}Ba and ^{94}Mo isotopes [13]). Though the focus is in identifying good F-spin states in $O(6)$ nuclei, it is well-known that F spin is broken in many situations [14]. In this paper, following the $O(18)$ and $O(36)$ symmetry limits of IBM-3 and IBM-4 models [5,6], it is identified that pnIBM admits a $O(12)$ limit with broken F spin but good F_z quantum number M_F. It should be mentioned that, although the existence of $O(12)$ limit is mentioned in the past in [15,16], in this paper for the first time the $O(12)$ symmetry chains, as they are closely related to $O(6)$ nuclei, are analyzed in any detail. Section 2 gives the generators and the quadratic Casimir operator of $O(12)$ by identifying the corresponding quasi-spin algebra; also discussed here is the closely related $O(10)$ algebra in d boson space. Two group chains are possible with $O(12)$ and §3 gives classification of states and typical spectra for both of them. In §4 complete classification of pnIBM symmetry limits with good M_F is discussed in detail. Finally, §5 gives concluding remarks.

2. *O*(12) symmetry in pnIBM

2.1 *Preliminaries*

The pnIBM, with proton–neutron degrees of freedom, can be described in general in terms of $\pi - \nu$ representation or the equivalent F-spin representation with the identification $|\pi\rangle = \left|F = \frac{1}{2}\,, M_F = \frac{1}{2}\right\rangle = \left|\frac{1}{2}\;\frac{1}{2}\right\rangle$ and $|\nu\rangle = \left|F = \frac{1}{2}, M_F = -\frac{1}{2}\right\rangle = \left|\frac{1}{2}\; -\frac{1}{2}\right\rangle$ [1]. In the F-spin representation, given the one boson creation and destruction operators $b^\dagger_{\ell,m_\ell;\,\frac{1}{2},m_f}$ and $\tilde{b}_{\ell,m_\ell;\,\frac{1}{2},m_f} = (-1)^{\ell+m_\ell+\frac{1}{2}+m_f}\; b_{\ell,-m_\ell;\,\frac{1}{2},-m_f}$, the 144 double tensors $\left(b^\dagger_{\ell,\frac{1}{2}}\tilde{b}_{\ell',\frac{1}{2}}\right)^{L_0,F_0}_{M_0,M_{F_0}}$ generate the $U(12)$ SGA; note that for us $\ell = 0(s)$ or $2(d)$. Similarly, in the $\pi - \nu$ representation, given the one boson creation and destruction operators $b^{\dagger}_{\ell,m_\ell;\,\rho}$ and $\tilde{b}_{\ell,m_\ell;\,\rho} = (-1)^{\ell+m_\ell}\, b_{\ell,-m_\ell;\,\rho}$; $\rho = \pi$ or ν, the 144 operators $\left(b^{\dagger}_{\ell,\rho}\tilde{b}_{\ell',\rho'}\right)^{L_0}_{M_0}$ generate the $U(12)$ algebra. These results follow directly from the following commutation relations

$$\begin{aligned}
&\left[\left(b^{\dagger}_{\ell_1,\frac{1}{2}}\tilde{b}_{\ell_2,\frac{1}{2}}\right)^{L_{12},F_{12}}_{M_{12},M_{F_{12}}} \quad \left(b^{\dagger}_{\ell_3,\frac{1}{2}}\tilde{b}_{\ell_4,\frac{1}{2}}\right)^{L_{34},F_{34}}_{M_{34},M_{F_{34}}}\right]_{-} \\
&= \sqrt{(2L_{12}+1)(2L_{34}+1)(2F_{12}+1)(2F_{34}+1)}\;(-1)^{1+\ell_1+\ell_4} \\
&\times \sum_{L_0\,,F_0} \left\langle L_{12}M_{12}\;\; L_{34}M_{34}\;\mid L_0M_0\right\rangle \left\langle F_{12}M_{F_{12}}\;\; F_{34}M_{F_{34}}\;\mid F_0M_{F_0}\right\rangle\;(-1)^{L_0+F_0} \\
&\times \left[\left\{\begin{matrix} L_{12} & L_{34} & L_0 \\ \ell_4 & \ell_1 & \ell_2 \end{matrix}\right\}\left\{\begin{matrix} F_{12} & F_{34} & F_0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}\right\}\left(b^{\dagger}_{\ell_1,\frac{1}{2}}\tilde{b}_{\ell_4,\frac{1}{2}}\right)^{L_0,F_0}_{M_0,M_{F_0}}\;\delta_{\ell_2\,\ell_3}\right. \\
&-(-1)^{\ell_1+\ell_2+\ell_3+\ell_4+L_{12}+L_{34}+L_0+F_{12}+F_{34}+F_0} \\
&\left.\times \left\{\begin{matrix} L_{12} & L_{34} & L_0 \\ \ell_3 & \ell_2 & \ell_1 \end{matrix}\right\}\left\{\begin{matrix} F_{12} & F_{34} & F_0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}\right\}\left(b^{\dagger}_{\ell_3,\frac{1}{2}}\tilde{b}_{\ell_2,\frac{1}{2}}\right)^{L_0,F_0}_{M_0,M_{F_0}}\;\delta_{\ell_1\,\ell_4}\right]
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
&\left[\left(b^{\dagger}_{\ell_1,\rho_1}\tilde{b}_{\ell_2,\rho_2}\right)^{L_{12}}_{M_{12}} \quad \left(b^{\dagger}_{\ell_3,\rho_3}\tilde{b}_{\ell_4,\rho_4}\right)^{L_{34}}_{M_{34}}\right]_{-} \\
&= \sqrt{(2L_{12}+1)(2L_{34}+1)}\;(-1)^{\ell_1+\ell_4}\sum_{L_0}\left\langle L_{12}M_{12}\;\; L_{34}M_{34}\;\mid L_0M_0\right\rangle\;(-1)^{L_0} \\
&\times \left[\left\{\begin{matrix} L_{12} & L_{34} & L_0 \\ \ell_4 & \ell_1 & \ell_2 \end{matrix}\right\}\left(b^{\dagger}_{\ell_1,\rho_1}\tilde{b}_{\ell_4,\rho_4}\right)^{L_0}_{M_0}\;\delta_{\ell_2\,\ell_3}\,\delta_{\rho_2,\rho_3}\right. \\
&-(-1)^{\ell_1+\ell_2+\ell_3+\ell_4+L_{12}+L_{34}+L_0}\;\left\{\begin{matrix} L_{12} & L_{34} & L_0 \\ \ell_3 & \ell_2 & \ell_1 \end{matrix}\right\} \\
&\left.\times\left(b^{\dagger}_{\ell_3,\rho_3}\tilde{b}_{\ell_2,\rho_2}\right)^{L_0}_{M_0}\;\delta_{\ell_1\,\ell_4}\,\delta_{\rho_1,\rho_4}\right].
\end{aligned} \tag{2}$$

Dynamical symmetry limits of pnIBM correspond to the group chains starting with $U(12)$ generating N and ending with $O_L(3) \otimes \left[SU_F(2) \supset O_{M_F}(2)\right]$ or $O_L(3) \otimes O_{M_F}(2)$ generating states with good (N, L, F, M_F) or only (N, L, M_F) respectively. Note that $N = (N_\pi + N_\nu)$ and $M_F = (N_\pi - N_\nu)/2$ where N_π and N_ν are proton and neutron boson numbers respectively. Before going further it is useful to write down the number, F spin and angular momentum (L) operators [16a]

$$\begin{aligned}
\hat{n} &= \hat{n}_s + \hat{n}_d = \hat{n}_\pi + \hat{n}_\nu \\
&= \sum_{\ell=0,2}\sqrt{2(2\ell+1)}\;\left(b^{\dagger}_{\ell,\frac{1}{2}}\tilde{b}_{\ell,\frac{1}{2}}\right)^{0,0}_{0,0}
\end{aligned}$$

$$= \sum_{\rho=\pi,\nu} \left(s^\dagger_\rho \tilde{s}_\rho\right) + \sqrt{5} \sum_{\rho=\pi,\nu} \left(d^\dagger_\rho \tilde{d}_\rho\right)^0$$

$$= \left[s^\dagger_\pi \tilde{s}_\pi + \sqrt{5}\left(d^\dagger_\pi \tilde{d}_\pi\right)^0\right] + \left[s^\dagger_\nu \tilde{s}_\nu + \sqrt{5}\left(d^\dagger_\nu \tilde{d}_\nu\right)^0\right]$$

$$L^1_\mu = \sqrt{20}\left(d^\dagger_{\frac{1}{2}} \tilde{d}_{\frac{1}{2}}\right)^{1,0}_{\mu,0} = \sqrt{10} \sum_{\rho=\pi,\nu} \left(d^\dagger_\rho \tilde{d}_\rho\right)^1_\mu ,$$

$$F^1_\mu = \frac{1}{\sqrt{2}} \sum_{\ell=0,2} \sqrt{(2\ell+1)} \left(b^\dagger_{\ell,\frac{1}{2}} \tilde{b}_{\ell,\frac{1}{2}}\right)^{0,1}_{0,\mu} ,$$

$$F^1_0 = \frac{1}{2}\left(\left[\hat{n}_{s;\pi} + \hat{n}_{d;\pi}\right] - \left[\hat{n}_{s;\nu} + \hat{n}_{d;\nu}\right]\right) ,$$

$$F^1_1 = -\frac{1}{\sqrt{2}}\left(\left[s^\dagger_\pi s_\nu\right] + \sum_m \left[d^\dagger_{m;\pi} d_{m;\nu}\right]\right) ,$$

$$F^1_{-1} = \frac{1}{\sqrt{2}}\left(\left[s^\dagger_\nu s_\pi\right] + \sum_m \left[d^\dagger_{m;\nu} d_{m;\pi}\right]\right) . \tag{3}$$

The proton s and d boson and neutron s and d boson number operators $\hat{n}_{s;\pi}$, $\hat{n}_{s;\nu}$, $\hat{n}_{d;\pi}$ and $\hat{n}_{d;\nu}$ are defined by the third equality in (3) and similarly, the decompositions of L into π and ν parts and F components into s and d parts follow immediately from (3).

At the primary level, as pointed out in the Introduction, identified by the first sub-algebra of $U(12)$, pnIBM has four symmetry limits [15]: (i) $U(6) \otimes SU_F(2)$; (ii) $U_\pi(6) \oplus U_\nu(6)$; (iii) $U_s(2) \oplus U_d(10)$; (iv) $O(12)$. With the condition that N, L and $M_F = (N_\pi - N_\nu)/2$ must be good quantum numbers, there will be no other chains in pnIBM except those related to (i)–(iv). Complete classification of group chains with good (N, L, M_F) in pnIBM will be discussed in detail in §4. In the present section the $O(12)$ algebra is studied in detail. As the $O(12)$ algebra is defined in *sd* boson space, it is more appropriate to start first with the corresponding $O(10)$ algebra in d boson space.

2.2 $O(10)$ *algebra in d boson space*

In d boson space the SGA is $U(10)$ and starting with it there are two chains: (i) $U(10) \supset [U(5) \supset O(5) \supset O_L(3)] \otimes \left[SU_F(2) \supset O_{M_F}(2)\right]$ where F spin is good; (ii) $U(10) \supset O(10) \supset [O(5) \supset O_L(3)] \otimes O_{M_F}(2)$ where only M_F is good. Here we are concerned with (ii), the $O(10)$ chain; chain (i) is considered in §4. It is known that $U(M)$ admits $O(M)$ as a sub-algebra and thus $U(10) \supset O(10)$ is always possible. But the question is whether there is a $O(10)$ that preserves L and M_F. The answer is in the affirmative and this is seen from the generators of $O(10)$ which are identified to be

$$O(10) : A^{L=1,3}_\mu = \left(d^\dagger_\pi \tilde{d}_\nu\right)^{1,3}_\mu , \; B^{L=1,3}_\mu = \left(d^\dagger_\nu \tilde{d}_\pi\right)^{1,3}_\mu ,$$

$$C^{L=0-4}_\mu = \left[\left(d^\dagger_\pi \tilde{d}_\pi\right)^L_\mu + (-1)^{1+L}\left(d^\dagger_\nu \tilde{d}_\nu\right)^L_\mu\right] . \tag{4}$$

It is seen from (2) that $\left[A^{L_1}_{\mu_1}\, A^{L_2}_{\mu_2}\right]_- = 0$, $\left[B^{L_1}_{\mu_1}\, B^{L_2}_{\mu_2}\right]_- = 0$, $\left[A^{L_1}_{\mu_1}\, B^{L_2}_{\mu_2}\right]_-$ is a sum of C^L's, $\left[A^{L_1}_{\mu_1}\, C^{L_2}_{\mu_2}\right]_-$ is a sum of A^L's, $\left[B^{L_1}_{\mu_1}\, C^{L_2}_{\mu_2}\right]_-$ is a sum of B^L's and finally $\left[C^{L_1}_{\mu_1}\, C^{L_2}_{\mu_2}\right]_-$ is a sum of C^L's. The C^1_μ generate $\vec{L}$, C^0 generates M_F and $C^{L=1,3}_\mu$ generate $O(5)$ in the chain $O(10) \supset [O(5) \supset O_L(3)] \otimes O_{M_F}(2)$. It is clear that, as $F^1_{d;\pm}$ operators are not in (4), the $O(10)$ chain breaks F spin. In order to understand the physical meaning of $O(10)$ and determine the $O(10)$ irreducible representations (irreps) contained in the symmetric irreps $\{N_d\}$ of $U(10)$ (N_d is the number of d bosons), following [17], the corresponding quasi-spin $SU_{S;d}(2)$ algebra is constructed. The generators $S_\pm(d)$ and $S_0(d)$ of $SU_{S;d}(2)$ and their commutation relations are

$$S_+(d) = \sqrt{5}\left(d^\dagger_\pi d^\dagger_\nu\right)^0, \;\; S_-(d) = \sqrt{5}\left(\tilde{d}_\pi \tilde{d}_\nu\right)^0, \;\; S_0(d) = \frac{\left(5+\hat{n}_d\right)}{2}$$
$$\left[S_+(d)\; S_-(d)\right]_- = -2S_0(d), \;\; \left[S_0(d)\; S_\pm(d)\right]_- = \pm S_\pm(d). \tag{5}$$

With $\{S(d)\}^2 = S_0(d)(S_0(d)-1) - S_+(d)S_-(d)$ and $S_0(d)$ defining $\left| S_d M_{S_d}\, \alpha' \right\rangle$ basis (α' labels states with the same S_d and M_{S_d} values in the d boson space) and the results $M_{S_d} = \frac{1}{2}(5+N_d)$, $S_d = M_{S_d}, M_{S_d}+1, \ldots$ and $\left\langle \{S(d)\}^2 \right\rangle^{S_d, M_{S_d}} = S_d(S_d-1)$ give, using $S_d = \frac{1}{2}(5+ \mathrm{v}_d)$

$$S_+(d)S_-(d) = 5\left(d^\dagger_\pi d^\dagger_\nu\right)^0 \left(\tilde{d}_\pi\tilde{d}_\nu\right)^0 = \sum_{L_0} (-1)^{L_0} \left(d^\dagger_\pi \tilde{d}_\pi\right)^{L_0} \cdot \left(d^\dagger_\nu \tilde{d}_\nu\right)^{L_0},$$
$$\left\langle S_+(d)S_-(d)\right\rangle^{S_d M_{S_d}} = \left\langle S_+(d)S_-(d)\right\rangle^{N_d \mathrm{v}_d} = \frac{1}{4}\left(N_d - \mathrm{v}_d\right)\left(N_d + \mathrm{v}_d + 8\right),$$
$$\mathrm{v}_d = N_d, N_d - 2, \ldots, 0 \text{ or } 1. \tag{6}$$

The relationship between $SU_{S;d}(2)$ and $O(10)$ is derived by examining the quadratic Casimir operators of $U(10)$ and $O(10)$,

$$C_2(U(10)) = \sum_k \left(d^\dagger_\pi \tilde{d}_\pi\right)^k \cdot \left(d^\dagger_\pi \tilde{d}_\pi\right)^k + \sum_k \left(d^\dagger_\nu \tilde{d}_\nu\right)^k \cdot \left(d^\dagger_\nu \tilde{d}_\nu\right)^k$$
$$+ \sum_k \left(d^\dagger_\pi \tilde{d}_\nu\right)^k \cdot \left(d^\dagger_\nu \tilde{d}_\pi\right)^k + \sum_k \left(d^\dagger_\nu \tilde{d}_\pi\right)^k \cdot \left(d^\dagger_\pi \tilde{d}_\nu\right)^k$$
$$C_2(O(10)) = 2\sum_{k=1,3} \left(d^\dagger_\pi \tilde{d}_\nu\right)^k \cdot \left(d^\dagger_\nu \tilde{d}_\pi\right)^k + 2\sum_{k=1,3} \left(d^\dagger_\nu \tilde{d}_\pi\right)^k \cdot \left(d^\dagger_\pi \tilde{d}_\nu\right)^k$$
$$+ \sum_L \left[\left(d^\dagger_\pi \tilde{d}_\pi\right)^L + (-1)^{1+L}\left(d^\dagger_\nu \tilde{d}_\nu\right)^L\right]$$
$$\cdot \left[\left(d^\dagger_\pi \tilde{d}_\pi\right)^L + (-1)^{1+L}\left(d^\dagger_\nu \tilde{d}_\nu\right)^L\right]. \tag{7}$$

Following [17] it can be recognized that the four terms in $C_2(U(10))$ give $[\hat{n}_{d;\pi}(\hat{n}_{d;\pi} - 1) + 5\hat{n}_{d;\pi}]$, $[\hat{n}_{d;\nu}(\hat{n}_{d;\nu} - 1) + 5\hat{n}_{d;\nu}]$, $[\hat{n}_{d;\pi}\hat{n}_{d;\nu} + 5\hat{n}_{d;\pi}]$ and $[\hat{n}_{d;\pi}\hat{n}_{d;\nu} + 5\hat{n}_{d;\nu}]$ respectively. Similarly $2\sum_{k=1,3} (d^\dagger_\pi \tilde{d}_\nu)^k \cdot (d^\dagger_\nu \tilde{d}_\pi)^k = \hat{n}_{d;\pi}\hat{n}_{d;\nu} + 4\hat{n}_{d;\pi} - S_+(d)S_-(d)$ gives

$$C_2(O(10)) = -4S_+(d)S_-(d) + \hat{n}_d(\hat{n}_d + 8). \tag{8}$$

Now applying (6) gives finally

$$\left| \begin{array}{ccc} U(10) & \supset & O(10) \\ N_d & & v_d \end{array} \right\rangle , \; v_d = N_d, N_d - 2, \ldots, 0 \text{ or } 1$$

$$\left\langle C_2(U(10)) \right\rangle^{N_d v_d} = N_d(N_d + 9) \,, \; \left\langle C_2(O(10)) \right\rangle^{N_d v_d} = v_d(v_d + 8). \tag{9}$$

Thus the pairs in the $O(10)$ limit are $\pi - \nu$ boson pairs and

$$|N_d \, v_d \, \alpha'\rangle = \left\{ \frac{(v_d + 4)!}{[(N_d - v_d)/2]! \; [(N_d + v_d + 8)/2]!} \right\}^{1/2}$$
$$\times \left[\sqrt{5} \left(d^\dagger_\pi d^\dagger_\nu \right)^0 \right]^{(N_d - v_d)/2} \; |v_d \, v_d \, \alpha'\rangle \, . \tag{10}$$

2.3 $O(12)$ *generators and the corresponding quasi-spin algebra*

In *sd* boson space, following the results in §2.2, it is natural to expect the appearance of $U(12) \supset O(12)$ algebra. From §2.2 it is clear that the 45 generators $A^{L=1,3}_\mu$, $B^{L=1,3}_\mu$ and $C^{L=0-4}_\mu$ of $O(10)$ in d boson space (see (4)) and the generator $D^0 = (s^\dagger_\pi \tilde{s}_\pi - s^\dagger_\nu \tilde{s}_\nu) = 2F^1_{s;0}$ of $O(2)$ in s boson space will be in the $O(12)$ algebra. Then the remaining 20 generators of $O(12)$ need to be identified. From the generators of $O(6)$ in $U(6)$ of IBM-1, it is easily seen that $E^2_\mu = \left[\left(s^\dagger_\pi \tilde{d}_\nu \right) + \alpha \left(d^\dagger_\pi \tilde{s}_\nu \right) \right]^2_\mu$ and $F^2_\mu = \left[\left(s^\dagger_\nu \tilde{d}_\pi \right) + \beta \left(d^\dagger_\nu \tilde{s}_\pi \right) \right]^2_\mu$ will be in the $O(12)$ algebra. The commutators $\left[A^L_\mu \, F^2_{\mu'} \right]_-$ and $\left[B^L_\mu \, E^2_{\mu'} \right]_-$ immediately give the remaining 10 generators $G^2_\mu = \left[\left(s^\dagger_\nu \tilde{d}_\nu \right) + \gamma \left(d^\dagger_\pi \tilde{s}_\pi \right) \right]^2_\mu$ and $H^2_\mu = \left[\left(s^\dagger_\pi \tilde{d}_\pi \right) + \delta \left(d^\dagger_\nu \tilde{s}_\nu \right) \right]^2_\mu$. By evaluating all the commutators, using (2), between the 66 generators $A^{L=1,3}_\mu$, $B^{L=1,3}_\mu$, $C^{L=0-4}_\mu$, D^0, E^2_μ, F^2_μ, G^2_μ and H^2_μ it is seen for example that $[A\,F]_-$ gives G, $[A\,H]_-$ gives E and $[E\,F]_-$ gives a sum of D^0 and C^L only if $\alpha = \beta = \gamma = \delta$ and $\alpha^2 = 1$. Applying these conditions it is seen that the following 66 operators generate the $O(12)$ algebra in pnIBM

$$\begin{aligned} O(12): A^{L=1,3}_\mu &= \left(d^\dagger_\pi \tilde{d}_\nu \right)^{1,3}_\mu , \; B^{L=1,3}_\mu = \left(d^\dagger_\nu \tilde{d}_\pi \right)^{1,3}_\mu , \\ C^{L=0-4}_\mu &= \left[\left(d^\dagger_\pi \tilde{d}_\pi \right)^L_\mu + (-1)^{1+L} \left(d^\dagger_\nu \tilde{d}_\nu \right)^L_\mu \right] , \\ D^0 &= (s^\dagger_\pi \tilde{s}_\pi - s^\dagger_\nu \tilde{s}_\nu) , \\ E^2_\mu &= \left[\left(s^\dagger_\pi \tilde{d}_\nu \right) + \alpha \left(d^\dagger_\pi \tilde{s}_\nu \right) \right]^2_\mu , \; F^2_\mu = \left[\left(s^\dagger_\nu \tilde{d}_\pi \right) + \alpha \left(d^\dagger_\nu \tilde{s}_\pi \right) \right]^2_\mu , \\ G^2_\mu &= \left[\left(s^\dagger_\nu \tilde{d}_\nu \right) + \alpha \left(d^\dagger_\pi \tilde{s}_\pi \right) \right]^2_\mu , \; H^2_\mu = \left[\left(s^\dagger_\pi \tilde{d}_\pi \right) + \alpha \left(d^\dagger_\nu \tilde{s}_\nu \right) \right]^2_\mu , \\ \alpha &= \pm 1 . \end{aligned} \tag{11}$$

Thus there are two $O(12)$ algebras, one with $\alpha = 1$ and other with $\alpha = -1$. Now we will construct the corresponding quasi-spin algebras.

Combining the $SU_{S;d}(2)$ quasi-spin algebra in d space and the corresponding algebra $SU_{S;s}(2)$ in s space defined by

$$S_+(s) = s^\dagger_\pi s^\dagger_\nu, \;\; S_-(s) = \tilde{s}_\pi \tilde{s}_\nu, \;\; S_0(s) = \frac{(1+\hat{n}_s)}{2}\,, \tag{12}$$

it is straightforward to introduce the quasi-spin $SU_S(2)$ algebra in the total sd space,

$$S_+ = S_+(s) + \beta S_+(d), \;\; S_- = S_-(s) + \beta S_-(d), \;\; S_0 = \frac{(6+\hat{n})}{2}, \;\; \beta = \pm 1. \tag{13}$$

The relationship between α in (11) and β in (13) is established ahead. With the quasi-spin algebra (13), we have $|N\mathfrak{v}\alpha'\rangle$ states exactly as in (6) and (9) and $\langle S_+S_-\rangle^{N,\mathfrak{v}} = (N-\mathfrak{v})(N+\mathfrak{v}+10)/4$. In order to see this, let us first define $C_2(O(12))$,

$$C_2(O(12)) = C_2(O(2)) + C_2(O(10)) + \alpha\,[E\cdot F + F\cdot E + G\cdot H + H\cdot G] \tag{14}$$

where $C_2(O(2))$ is

$$C_2(O(2)) = D^0D^0 = n_s^2 - 4S_+(s)S_-(s)\,, \tag{15}$$

$C_2(O(10))$ is defined by (8) and α is defined in (11). Recognizing that

$$\begin{aligned} E\cdot F + F\cdot E &= 2\left[S_+(s)S_-(d) + S_+(d)S_-(s)\right] \\ &\quad + \alpha\left[5n_s + n_d + 2n_{s;\pi}n_{d;\nu} + 2n_{d;\pi}n_{s;\nu}\right], \\ G\cdot H + H\cdot G &= 2\left[S_+(s)S_-(d) + S_+(d)S_-(s)\right] \\ &\quad + \alpha\left[5n_s + n_d + 2n_{s;\nu}n_{d;\nu} + 2n_{d;\pi}n_{s;\pi}\right] \end{aligned} \tag{16}$$

and using (8), (13) and (15) it is seen that $C_2(O(12))$ can be written in terms of S_+S_- only when $\alpha = -\beta$. Then finally, with $\alpha = -\beta$

$$C_2(O(12)) = -4S_+S_- + \hat{n}(\hat{n}+10). \tag{17}$$

Thus the $O(12)$ defined by the generators in (11) correspond to the quasi-spin algebra defined by (13) when $\alpha = -\beta$. With this we have, just as in (9) and (10),

$$\begin{aligned} &\left| \begin{matrix} U(12) & \supset & O(12) \\ N & & \mathfrak{v} \end{matrix} \right\rangle, \;\; \mathfrak{v} = N, N-2, \ldots, 0 \text{ or } 1 \\ &\langle C_2(O(12))\rangle^{N,\mathfrak{v}} = \mathfrak{v}(\mathfrak{v}+10) \\ &|N\,\mathfrak{v}\,\alpha'\rangle = \left\{\frac{(\mathfrak{v}+5)!}{[(N-\mathfrak{v})/2]!\;[(N+\mathfrak{v}+10)/2]!}\right\}^{1/2} \\ &\qquad \times \left[s^\dagger_\pi s^\dagger_\nu + \sqrt{5}\beta\left(d^\dagger_\pi d^\dagger_\nu\right)^0\right]^{(N-\mathfrak{v})/2} |\mathfrak{v}\,\mathfrak{v}\,\alpha'\rangle \end{aligned} \tag{18}$$

where $\beta = \pm 1$ and the α in the $O(12)$ generators in (11) is related to β by $\alpha = -\beta$.

3. Spectra in $O(12) \supset O(6) \otimes O(2)$ and $O(12) \supset O(2) \oplus O(10)$ limits

The $O(12)$ algebra admits $O(6) \otimes O(2)$ and $O(2) \oplus O(10)$ subalgebras with good M_F. In both cases one can write down the complete group chains with good (N, L, M_F). Hereafter these two chains are called $O(12) \supset O(6) \otimes O(2)$ and $O(12) \supset O(2) \oplus O(10)$ limits respectively of pnIBM. Let us point out that, in addition to M_F, the $O(12) \supset O(2) \oplus O(10)$ limit also preserves M_{F_s} and M_{F_d} and hence it is more restrictive.

3.1 $O(12) \supset O(6) \otimes O(2)$ *limit*

The group chain and irrep labels in the $O(12) \supset O(6) \otimes O(2)$ limit are given by

$$\left| \begin{array}{ccccccccccc} U(12) & \supset & O(12) & \supset & [& O(6) & \supset & O(5) & \supset & O_L(3) &] \otimes O_{M_F}(2) \\ \{N\} & & [\mathfrak{v}] & & & [\sigma_1 \sigma_2] & & [\mathfrak{v}_1 \mathfrak{v}_2] & & L & M_F \end{array} \right\rangle . \tag{19}$$

The $O(6)$ algebra is generated by the 15 generators $C_\mu^{L_0=1,3}$ and $G_\mu^2 + \beta' H_\mu^2$ and similarly the $O_{M_F}(2)$ is generated by $D^0 + \sqrt{5} C^0$ where C^{L_0}, D^0, G^2 and H^2 are defined in (11); the $O(5)$ and $O_L(3)$ algebras are generated by $C_\mu^{L_0=1,3}$ and C_μ^1 respectively. For the $O(6)$ algebra in (19) to be the same as the $O(6)$ in the $U(6) \otimes SU_F(2) \supset O(6) \otimes O_{M_F}(2)$ limit of pnIBM (as stated earlier, this limit was studied in detail in the past [1,3]), one needs the conditions $\alpha = 1$ in (11) and $\beta' = 1$ in $G_\mu^2 + \beta' H_\mu^2$ generators. With these conditions met, it is possible to compare the results in these two limits and derive (see ahead) the new structures implied by (19). Before the results for irrep labels are given, it should be pointed out that for a given nucleus N, L and M_F are always good quantum numbers. The $N \to \mathfrak{v}$ reduction problem was already solved in §2 (see eq. (18)) and the $\mathfrak{v} \to [\sigma_1 \sigma_2]\; M_F$ reductions are given in Appendix A; note that here table 1 with $r = 6$ will apply. The rule for $[\sigma_1 \sigma_2] \to [\mathfrak{v}_1 \mathfrak{v}_2]$ is well-known [4,18,19], $\sigma_1 \geq \mathfrak{v}_1 \geq \sigma_2 \geq \mathfrak{v}_2 \geq 0$. Finally $[\mathfrak{v}_1 \mathfrak{v}_2] \to L$ can be solved using (A4) and the general solution for $[\tau]_{O(5)} \to L$. For example $[0]_{O(5)} \to L = 0$, $[1]_{O(5)} \to L = 2$, $[2]_{O(5)} \to L = 2, 4$, $[11]_{O(5)} \to L = 1, 3$, $[3]_{O(5)} \to L = 0, 3, 4, 6$ and $[21]_{O(5)} \to L = 1, 2, 3, 4, 5$. Using these irrep reductions and writing the Hamiltonian as a linear combination of the quadratic Casimir operators of the groups in (19) one can construct the typical spectrum in the $O(12) \supset O(6) \otimes O(2)$ limit. The Hamiltonian and the energy formula in this limit are

$$\begin{aligned} H &= E_0(N, M_F) + a_1 C_2(O(12)) + a_2 C_2(O(6)) + a_3 C_2(O(5)) + a_4 C_2(O(3)) \\ & E\left(N, \mathfrak{v}, [\sigma_1 \sigma_2], [\mathfrak{v}_1 \mathfrak{v}_2], L, M_F\right) = E_0(N, M_F) + a_1 \mathfrak{v}(\mathfrak{v} + 10) \\ & + a_2 \left[\sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2)\right] \\ & + a_3 \left[\mathfrak{v}_1(\mathfrak{v}_1 + 3) + \mathfrak{v}_2(\mathfrak{v}_2 + 1)\right] + a_4 L(L+1). \end{aligned} \tag{20}$$

The operator form for $C_2(O(12))$ and the formula for its eigenvalues are given in §2. The corresponding results for $C_2(O(6))$, $C_2(O(5))$ and $C_2(O(3))$ are easy to write down [1,4,19]. In order to get IBM-1 like states to be the lowest, for a given $\mathfrak{v}$ we need $[\sigma_1 \sigma_2] = [\mathfrak{v}, 0]$ to be lowest and therefore $a_2 < 0$ in (20). In order to get the ground $L = 0, 2, 4 \ldots$ band correctly we need $a_3 > 0$ and $a_4 \sim 0$. With these restrictions it is seen that the condition

Table 1. $[\tau]_{O(2r)} \to [\tau_1\,\tau_2]_{O(r)}\,(M_F)_{O(2)}$ irrep reductions for $\tau \le 6$. The results in the table are verified using the dimension formulas [18] for $r = 5, 6$, $d([\tau_1\tau_2]_{O(5)}) = (2\tau_1+3)(2\tau_2+1)(\tau_1-\tau_2+1)(\tau_1+\tau_2+2)/6$, $d([\tau_1\tau_2]_{O(6)}) = (\tau_1+2)^2(\tau_2+1)^2(\tau_1-\tau_2+1)(\tau_1+\tau_2+3)/12$. Used also is the formula $d([\tau]_{O(2r)}) = \begin{pmatrix} \tau+2r-1 \\ \tau \end{pmatrix} - \begin{pmatrix} \tau+2r-3 \\ \tau-2 \end{pmatrix}$ for any r.

$[\tau]_{O(2r)}$	$[\tau_1\,\tau_2]_{O(r)}$	$(M_F)_{O(2)}$
[0]	[0]	0
[1]	[1]	$\pm\frac{1}{2}$
[2]	[2]	$\pm1, 0$
	[11]	0
	[0]	±1
[3]	[3]	$\pm\frac{3}{2}, \pm\frac{1}{2}$
	[21]	$\pm\frac{1}{2}$
	[1]	$\pm\frac{3}{2}, \pm\frac{1}{2}$
[4]	[4]	$\pm2, \pm1, 0$
	[31]	$\pm1, 0$
	[22]	0
	[2]	$\pm2, \pm1, (0)^2$
	[11]	±1
	[0]	$\pm2, 0$
[5]	[5]	$\pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$
	[41]	$\pm\frac{3}{2}, \pm\frac{1}{2}$
	[32]	$\pm\frac{1}{2}$
	[3]	$\pm\frac{5}{2}, \pm\frac{3}{2}, \left(\pm\frac{1}{2}\right)^2$
	[21]	$\pm\frac{3}{2}, \pm\frac{1}{2}$
	[1]	$\pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$
[6]	[6]	$\pm3, \pm2, \pm1, 0$
	[51]	$\pm2, \pm1, 0$
	[42]	$\pm1, 0$
	[33]	0
	[4]	$\pm3, \pm2, (\pm1)^2, (0)^2$
	[31]	$\pm2, \pm1, (0)^2$
	[22]	±1
	[2]	$\pm3, \pm2, (\pm1)^2, 0$
	[11]	$\pm2, 0$
	[0]	$\pm3, \pm1$

$a_1 > 0$ gives a spectrum similar to the spectrum in the $U(6) \otimes SU_F(2) \supset O(6) \otimes O_{M_F}(2)$ limit. As an example, for $N = 6$ and $M_F = -1$ (then $N_\pi = 2$, $N_\nu = 4$) the typical spectrum in the $O(12) \supset O(6) \otimes O(2)$ limit is shown in figure 1 and this should be compared with the $U(6) \otimes SU_F(2) \supset O(6) \otimes O_{M_F}(2)$ limit spectrum given in figure 4 of ref. [3]. Firstly the states with the $O(6)$ irreps [6] and [51] in figure 1 belong to $v = 6$ (i.e. $v = N$) and therefore it is not possible in general to separate them too far. Due to this, as seen from figure 1, the [51] states start appearing around 1.5 MeV excitation. Typically states with the irrep [6] are IBM-1 states and the [51] states are the mixed symmetry states. In the $U(6) \otimes SU_F(2) \supset O(6) \otimes O_{M_F}(2)$ limit the [6] and [51] $O(6)$ irreps belong to different $U(6)$ irreps and therefore in this limit it is possible to split them far by using the $U(6)$ Casimir operator (the Majorana operator [1,3]). With this in the $U(6) \otimes SU_F(2) \supset O(6) \otimes O_{M_F}(2)$ limit the mixed symmetry states are expected around 3 MeV excitation as found in many nuclei. Unlike this, in the $O(12) \supset O(6) \otimes O(2)$ limit they are expected to appear around 1.5 to 2 MeV as in figure 1. Therefore to find empirical examples for this symmetry limit one has to look for $O(6)$ type even–even nuclei with 1^+ states (see figure 1) appearing around 1.5 MeV. In fact there are many such nuclei [20] and in order to establish their structure one need to study their $B(E2)$'s. It should be added that the $[\sigma_1 \sigma_2] = [N]$ and $[N-1, 1]$ states with $v = N$ in the $O(12) \supset O(6) \otimes O(2)$ limit will have the same structure as in the $U(6) \otimes SU_F(2) \supset O(6) \otimes O_{M_F}(2)$ limit as the corresponding $U(6)$ irreps are uniquely determined. Additional signatures for the $O(12) \supset O(6) \otimes O(2)$ limit come from the $v = N - 2$ (in figure 1 they correspond to $v = 4$) states which should start appearing around 2.2–2.5 MeV excitation (around this, states with $v = N$ and $O(6)$ irrep $[N-2, 2]$ also will start

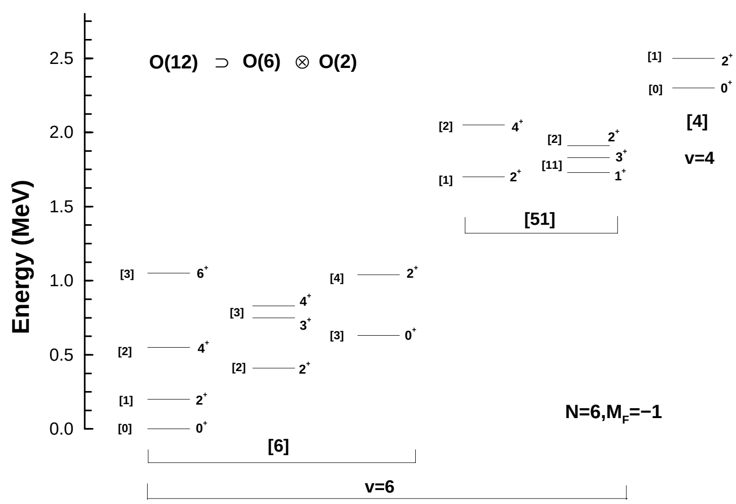

Figure 1. Typical energy spectrum in the $O(12) \supset O(6) \otimes O(2)$ limit of pnIBM for $N = 6$ bosons with $M_F = -1$ (i.e. $N_\pi = 2$, $N_\nu = 4$). The parameters in the energy formula (20) are chosen to be $a_1 = 30$ keV, $a_2 = -125$ keV, $a_3 = 35$ keV and $a_4 = 10$ keV. The $O(5)$ quantum numbers $[v_1 v_2]$ are shown to the left of each energy level and to the right shown are L^π values. Energy levels for $v = 6$ with $[\sigma_1 \sigma_2] = [6]$ and [51] and $v = 4$ with $[\sigma_1 \sigma_2] = [4]$ are shown in the figure.

appearing but they are not shown in the figure) and they will have one *sd* boson $\pi-\nu$ pair (see (18)). Therefore these states carry much more definite signatures of $O(12)$. For further understanding of the $O(12)\supset O(6)\otimes O(2)$ limit, it is necessary to study the structure of the eigenstates in this limit in terms of the amount of F-spin mixing they contain and also derive formulas for $B(E2)$'s and $B(M1)$'s between the states in this limit. They will be addressed in a future publication.

3.2 $O(12)\supset O(2)\oplus O(10)$ *limit*

The group chain and irrep labels in the $O(12)\supset O(2)\oplus O(10)$ limit are given by

$$\left|\begin{array}{ccccccccc} U(12) & \supset & O(12) & \supset & [\; O(10) & \supset & \{\; O(5) & \supset & O_L(3)\;\} \\ \{N\} & & [\upsilon] & & [\upsilon_d] & & [\upsilon_1\upsilon_2] & & L \end{array}\right.$$
$$\left.\begin{array}{ccccccccc} \otimes & O_{M_{F_d}}(2)\;] & \oplus & O_{M_{F_s}}(2) & \supset & O_L(3) & \otimes & O_{M_F}(2) \\ & M_{F_d} & & M_{F_s} & & L & & M_F=M_{F_s}+M_{F_d} \end{array}\right\rangle . \tag{21}$$

The generators of all the groups in (21) will follow from the results in §§2 and 3.1. The irrep labels in (21) are determined as follows. It is seen from (18) that $\upsilon=N,N-2,\ldots,0$ or 1. The $\upsilon\to(\upsilon_d,M_{F_s})$ is given by the rule (see [21]) $\upsilon=\upsilon_s+\upsilon_d+2k$, $k=0,1,2,\ldots$ where $\upsilon_s=2|M_{F_s}|$. The rules for $\upsilon_d\to[\upsilon_1\upsilon_2]\,M_{F_d}$ are given in Appendix A; note that here table 1 with $r=5$ will apply. The $[\upsilon_1\upsilon_2]\to L$ reductions are the same as in the case of $O(12)\supset O(6)\otimes O(2)$ limit. Now let us consider the Hamiltonian and the energy formula in the $O(12)\supset O(2)\oplus O(10)$ limit

$$\begin{aligned} H &= E_0(N,M_F)+b_1C_2(O(12))+b_2C_2(O(10))+b_3C_2(O(5)) \\ &\quad +b_4C_2(O(3))+b_5\left(F^1_{d;0}\right)^2 \\ E&\left(N,\upsilon,\upsilon_d,[\upsilon_1\upsilon_2],L,M_{F_d},M_F\right) \\ &= E_0(N,M_F)+b_1\upsilon(\upsilon+10)+b_2\upsilon_d(\upsilon_d+8) \\ &\quad +b_3\left[\upsilon_1(\upsilon_1+3)+\upsilon_2(\upsilon_2+1)\right] \\ &\quad +b_4L(L+1)+b_5(M_{F_d})^2. \end{aligned} \tag{22}$$

Assuming $b_1<0$, the $\upsilon=N$ states will be lowest in energy. Then choosing $b_2>0$ in principle a phonon spectrum can be obtained. For $\upsilon_d=0$ one has $L=0$ with $M_{F_d}=0$. Note that $M_{F_s}=\pm\frac{(\upsilon-\upsilon_d)}{2},\pm\frac{(\upsilon-\upsilon_d-2)}{2},\ldots,0$ or $\pm\frac{1}{2}$ and given M_{F_d} the M_{F_s} value must be $M_{F_s}=M_F-M_{F_d}$. For $\upsilon_d=1$ (one phonon excitation) one has $[\upsilon_1\upsilon_2]=[1]$, $L=2$ with $M_{F_d}=\pm\frac{1}{2}$. Similarly for $\upsilon_d=2$ one has $L=0,2,4$ with $M_{F_d}=\pm1$ and $L=(2,4),(1,3)$ with $M_{F_d}=0$. These results follow from table 1 as $[\upsilon_1\upsilon_2]=[0]\to L=0$, $[1]\to L=2$, $[2]\to L=2,4$ and $[11]\to L=1,3$. This construction extends to $\upsilon_d=3$ etc. Then there will be similar states with $\upsilon=N-2$ at energies higher than those of $\upsilon=N$ states and so on. With these, by choosing $b_5=0$ one has, a ground 0^+, 1-phonon excited 2^+ with $M_{F_d}=\pm\frac{1}{2}$, 2-phonon $(0^+,2^+,4^+)$ with $M_{F_d}=\pm1$ and $(2^+,4^+)$, $(1^+,3^+)$ with $M_{F_d}=0$.

Thus, because of the degeneracies due to good (M_{F_s}, M_{F_d}) in this symmetry limit, the spectrum is unrealistic. However, by adding a term $b_6\left[\frac{1}{2}\mathrm{v}_d - M_{F_d}\right]\left[\mathrm{v}_d + R_1 M_{F_d} + R_2\right]$ with b_6, R_1 and R_2 being some constants, in the energy formula (22), it is possible to lift the M_{F_d} degeneracies to give a spectrum that looks realistic. At present a two-body interaction with eigenvalues $\left[\frac{1}{2}\mathrm{v}_d - M_{F_d}\right]\left[\mathrm{v}_d + R_1 M_{F_d} + R_2\right]$ could not be constructed. In conclusion, it is probable that the $O(12) \supset O(2) \oplus O(10)$ limit will not be seen in real nuclei.

4. Complete classification of pnIBM symmetry limits with good M_F

With the $O(12)$ algebra identified and studied in §§2 and 3, it is natural to address the question of complete classification of the symmetry schemes (group–subgroup chains) in pnIBM. As already pointed out, they are associated with the four $U(12)$ subalgebras (i) $U(6) \otimes SU_F(2)$, (ii) $U_\pi(6) \oplus U_\nu(6)$, (iii) $U_s(2) \oplus U_d(10)$ and (iv) $O(12)$. In the $U(6) \otimes SU_F(2)$ limit, the $U(6)$ algebra is generated by $\left(b^\dagger_{\ell_1,\frac{1}{2}}\tilde{b}_{\ell_2,\frac{1}{2}}\right)^{L_0,0}_{M_0,0}$; $\ell_1, \ell_2 = 0, 2$ and $SU_F(2)$ by the F-spin operators F^1_μ in (3). All the group chains in this limit are well-known [1,3] and they correspond to the sub-algebras G's in $U(12) \supset [U(6) \supset G \supset \cdot \supset \quad O_L(3)] \otimes \left[SU_F(2) \supset O_{M_F}(2)\right]$; $G = U(5), SU(3), O(6)$. Obviously all these chains preserve (N, L, F, M_F) (note that we are not showing $O_L(3) \supset O_{M_L}(2)$ as L is an exact symmetry). In the $U_\pi(6) \oplus U_\nu(6)$ limit the $U_\rho(6)$ generators follow easily from (2) and they are $\left(b^\dagger_{\ell_1,\rho}\tilde{b}_{\ell_2,\rho}\right)^{L_0}_{M_0}$; $\ell_1, \ell_2 = 0, 2$ and $\rho = \pi, \nu$. The boson numbers N_ρ are generated by $U_\rho(6)$ and therefore the group chains in the $U_\pi(6) \oplus U_\nu(6)$ limit will always preserve M_F. The various group chains in this limit are obtained by writing down all the $U_\rho(6)$ subalgebras with good L_ρ ($\rho = \pi, \nu$), then coupling the $\pi - \nu$ algebras at some level and further reducing this coupled algebra to $O_L(3)$. All these group chains are well-known [1] and they are of the form $U(12) \supset [U_\pi(6) \supset \ldots G_\pi \supset \ldots] \oplus [U_\nu(6) \supset \ldots G_\nu \supset \ldots] \supset G_{\pi+\nu} \cdots \supset O_L(3)$. In summary, the $U(6) \otimes SU_F(2)$ symmetry limit group chains preserve (N, L, F, M_F) and the $U_\pi(6) \oplus U_\nu(6)$ symmetry limit group chains preserve only (N, L, M_F) and all these group chains are known before [21a].

In the $U_s(2) \oplus U_d(10)$ limit, the *sd* boson space is decomposed into *s* and *d* spaces so that not only N but both N_s (generated by $U_s(2)$) and N_d (generated by $U_d(10)$) are good quantum numbers. The $U_s(2)$ generates *s*-boson F-spin $F_s = N_s/2$. The $U_d(10)$ admits two subalgebras as pointed out in §2.2 and with this there are two group chains in the $U_s(2) \oplus U_d(10)$ limit

$$\left|\begin{array}{ccccccccccc} U(12) & \supset & U_s(2) & \oplus & [\ U_d(10) & \supset & \{\ U(5) & \supset & O(5) & \supset & O_L(3)\ \} \\ \{N\} & & \{N_s\}; F_s = N_s/2 & & \{N_d\} & & \{f_1, f_2\} & & [\mathrm{v}_1 \mathrm{v}_2] & & L \end{array}\right.$$
$$\left.\begin{array}{ccccccc} \otimes & SU_{F_d}(2) &] & \supset & SU_F(2) & \supset & O_{M_F}(2) \\ & F_d = (f_1 - f_2)/2 & & & \vec{F} = \vec{F}_s + \vec{F}_d & & M_F \end{array}\right\rangle \tag{23}$$

$$\left| \begin{array}{ccccccccc} U(12) & \supset & [\; U_s(2) & \supset & O_{M_{F_s}}(2)\;] & \oplus & [\; U_d(10) & \supset & O(10) \supset \\ \{N\} & & \{N_s\}, F_s = N_s/2 & & M_{F_s} & & \{N_d\} & & [\upsilon_d] \end{array} \right.$$

$$\left. \begin{array}{ccccccc} \{\; O(5) & \supset & O_L(3)\; \} & \otimes & O_{M_{F_d}}(2)\;] & \supset & O_{M_F}(2) \\ [\upsilon_1 \upsilon_2] & & L & & M_{F_d} & & M_F = M_{F_s} + M_{F_d} \end{array} \right\rangle . \tag{24}$$

In the first chain (23) F spin is good and by examining the irrep reductions (basis states) it is easily seen that it is the same as the $U(6) \otimes SU(2)$ limit with $U(6) \supset U(5)$ (see [3] and figure 2 in this reference). Note that in (23), $N = N_s + N_d$, $f_1 + f_2 = N_d$, $f_1 \geq f_2 \geq 0$. The second chain (24) (hereafter called $U(2) \oplus [U(10) \supset O(10)]$ limit) is a new group chain in pnIBM. For this chain, the irrep reductions $N_d \to \upsilon_d$ and $\upsilon_d \to [\upsilon_1 \upsilon_2]\, M_{F_d}$ follow from the results in §2 and Appendix A; results in table 1 with $r = 5$ will apply here. The $[\upsilon_1 \upsilon_2] \to L$ reductions are given in §3.1. It is straightforward to write down the Hamiltonian and energy formula in this limit. Just as in the case of $O(12) \supset O(2) \oplus O(10)$, it is easily seen that the spectrum in the present case also will be unrealistic (with degeneracies due to good M_{F_d}). Thus both $U(2) \oplus [U(10) \supset O(10)]$ and $O(12) \supset O(2) \oplus O(10)$ which preserve (M_{F_s}, M_{F_d}), may not be seen in real nuclei but they should be useful for chaos and phase transition studies (see ahead).

In summary, combining the symmetry schemes in the $U(6) \otimes SU_F(2)$, $U_\pi(6) \oplus U_\nu(6)$ and $U_s(2) \oplus U_d(10)$ limits with the two $O(12)$ symmetry limits analyzed in §3, one has the complete classification of symmetry schemes with good (N, L, M_F) in pnIBM.

5. Conclusions

Proton–neutron interacting boson model admits a new $O(12)$ symmetry limit which breaks F spin but preserves the F_z quantum number M_F. The $O(12)$ algebra is analyzed in detail, for the first time in this paper, by identifying the corresponding quasi-spin algebra. With $O(12)$ there are two symmetry limits in pnIBM, $O(12) \supset O(6) \otimes O(2)$ and $O(12) \supset O(2) \oplus O(10)$ limits. In both cases complete classification of the basis states and typical energy spectra are given. It is argued that some $O(6)$-type (γ soft) nuclei may exhibit the $O(12) \supset O(6) \otimes O(2)$ limit and two important signatures here are the appearance of $O(6)$ $[N-1, 1]$ states around 1.5 MeV excitation and the 0_3^+ (or 0_4^+) states around 2.5 MeV with a correlated $\pi - \nu$ boson pair. Search for empirical examples is under progress.

Searching for complete classification of pnIBM symmetry schemes, it is found that, within the $U_s(2) \oplus U_d(10)$ algebra of $U(12)$, there is a new $U(2) \oplus [U(10) \supset O(10)]$ limit. This may be relevant for $U(5)$ (vibrational) type nuclei. For the three new symmetry limits discussed in this paper, $O(12) \supset O(6) \otimes O(2)$, $O(12) \supset O(2) \oplus O(10)$ and $U(2) \oplus [U(10) \supset O(10)]$ given by (19), (21) and (24) respectively (note that they all preserve M_F and in general break the F spin), results for electromagnetic transition strengths ($B(E2)$'s and $B(M1)$'s) and structure of wave functions in terms of the amount of F-spin mixing they contain, will be presented elsewhere. The group theoretical problems needed for these are being solved.

With the $O(12)$ limit studied in §3, another important problem addressed and solved in this paper is the complete classification of pnIBM symmetry schemes with good M_F.

Let us point out that a major application of the complete classification is in the studies of quantum chaos and phase transitions in finite quantum systems where one can use pnIBM as a model. Such studies with great success are carried out using IBM-1 [8,9,22,23] and only recently a beginning is made in this direction using pnIBM [24].

Appendix A

The problem of $[\tau] \to [\tau_1\,\tau_2]\,M_F$ irrep reductions in the group–subgroup chain

$$\left| \begin{array}{ccccccc} U(2r) & \supset & O(2r) & \supset & O(r) & \otimes & O(2) \\ \{N\} & & [\tau] & & [\tau_1\,\tau_2] & & M_F \end{array} \right\rangle ; \quad \tau = N, N-2, N-4, \ldots, 0 \text{ or} 1 \tag{A1}$$

is solved by using the group chain

$$\begin{gathered} \left| \begin{array}{ccccccccc} U(2r) & \supset & U(r) & \otimes & SU(2) & \supset & O(r) & \otimes & O(2) \\ \{N\} & & \{f_1 f_2\} & & F & & [\tau_1\,\tau_2] & & M_F \end{array} \right\rangle , \\ f_1 + f_2 = N, \;\; f_1 \geq f_2 \geq 0, \;\; F = (f_1 - f_2)/2, \\ M_F = -F, (-F+1), \ldots, F. \end{gathered} \tag{A2}$$

The $\{f_1\,f_2\} \to [\tau_1\,\tau_2]$ reduction in (A2) is obtained by the well-known rules for the $U(r)$ and $O(r)$ Kronecker ($\otimes$) products [18] and the $U(r) \supset O(r)$ reductions for the symmetric $U(r)$ irreps

$$\{f_1\}_{U(r)} \otimes \{f_2\}_{U(r)} - \{f_1+1\}_{U(r)} \otimes \{f_2 - 1\}_{U(r)} = \{f_1, f_2\}_{U(r)} \tag{A3}$$

$$[\kappa]_{O(r)} \otimes [\ell]_{O(r)} = \sum_{p=0}^{\ell} \sum_{q=0}^{\ell-p} [\kappa - \ell + p + 2q, p]_{O(r)} \oplus, \; \ell \leq \kappa \tag{A4}$$

$$\{f\}_{U(r)} \longrightarrow [f]_{O(r)} \oplus [f-2]_{O(r)} \oplus \ldots \oplus [0]_{O(r)} \;\; \text{or} \;\; [1]_{O(r)} . \tag{A5}$$

By writing all allowed $\{f_1 f_2\}$ in (A2) for a given N and then applying (A3), (A5) and (A4) in that order will give $\{N\} \to [\tau_1\,\tau_2]\,M_F$ reductions. Starting with $N = 1, 3, 5 \ldots$ and by successive subtraction of the $\{N\} \to [\tau_1\,\tau_2]\,M_F$ reductions will give, via $N \to \tau$ in (A1), the $[\tau]_{O(2r)} \to [\tau_1\,\tau_2]_{O(r)}\,(M_F)_{O(2)}$ irrep reductions for odd N. Similarly, starting with $N = 0, 2, 4, \ldots$ will give the reductions for even N. This procedure is easily implemented on a computer. Table 1 gives the results for $\tau \leq 6$. From the table it is seen that, in general

$$\begin{aligned} [\tau]_{O(2r)} \longrightarrow [\tau_1\,\tau_2]_{O(r)}\,(M_F)_{O(2)} = \; & [\tau] \; \pm \left(\tfrac{\tau}{2}\right), \pm \left(\tfrac{\tau}{2} - 1\right), \ldots, 0 \;\text{ or }\; \pm \tfrac{1}{2} \\ & \oplus [\tau - 1, 1] \; \pm \left(\tfrac{\tau}{2} - 1\right), \pm \left(\tfrac{\tau}{2} - 2\right), \ldots, 0 \;\text{ or }\; \pm \tfrac{1}{2} \\ & \oplus [\tau - 2, 2] \; \pm \left(\tfrac{\tau}{2} - 2\right), \pm \left(\tfrac{\tau}{2} - 3\right), \ldots, 0 \;\text{ or }\; \pm \tfrac{1}{2} \\ & \oplus \ldots \\ & \oplus [\tau - 2] \; \pm \left(\tfrac{\tau}{2}\right), \pm \left(\tfrac{\tau}{2} - 1\right), \ldots \\ & \oplus \ldots . \end{aligned} \tag{A6}$$

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